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Periodic Solution of Classical Hamiltonian Systems

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Introduction

Let $x=(x_1, x_2, \cdots, x_n)$, $y=(y_1, y_2, \cdots, y_n)\in \boldsymbol{R}^n$ and $H=H(x, y)$: $\boldsymbol{R}^{2n}\to \boldsymbol{R}$ be a smooth (C^∞) function.

We consider a Hamiltonian system of n degrees of freedom

$$(1)\qquad \dot{x}_i=H_{y_i}\ ,\quad \dot{y}_i=-H_{x_i}\ ;\quad i=1, 2, \cdots, n\ .$$

P. Rabinowitz [3] proved that "if an energy surface

$$H^{-1}(e)=\{(x, y)\in \boldsymbol{R}^{2n};\ H(x, y)=e\}$$

is star-shaped, then there exists at least one periodic solution of (1) on it".

"star-shaped" implies "diffeomorphic to the sphere S^{2n-1}", but it is not known whether the condition "star-shaped" can be replaced by "diffeo. to S^{2n-1}" or not. This is a generalized Hamiltonian version of the Seifert Conjecture (Has any sufficiently smooth flow on S^3 periodic orbit ?) in the theory of dynamical systems.

Classically, the system (1) is derived from the Lagrangian system and, in the time-independent case, the Hamiltonian H is the sum of the kinetic energy and the potential. So we define

DEFINITION. A Hamiltonian $H=H(x, y)$ is called *classical* if it has the form

$$(2)\qquad H=\sum_{i,j=1}^{n} a^{ij}(x)y_i y_j+U(x)\ ,$$

where a^{ij}, U: $\boldsymbol{R}^n\to \boldsymbol{R}$ are smooth functions and for any $x\in \boldsymbol{R}^n$, the matrix $(a^{ij}(x))$ is symmetric and positive definite.

The system (1) with classical Hamiltonian H is called a *classical Hamiltonian system.*

In this paper we have

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THEOREM. *For classical Hamiltonian systems, there exists at least one periodic solution on every compact regular energy surface.*

An energy surface is called *regular* if there are no critical points of H on it.

A. Weinstein [8] conjectured that, if a compact energy surface S of H is *contact* type (defined in [8]) and $H^1(S; \boldsymbol{R})=0$, then there may be a periodic solution on it. It is based on the fact that such type of energy surface is common to all the situation in which the existence of periodic solutions has been proved by variational methods.

Our method is a variational one and corresponds to Example 3 in [8] replacing (i) by $(i)'$: $\pi(S)$ is a compact manifold with boundary.

By his method, $(i)'$ does not seem to imply contactness.

The Hamiltonian system (1) with the classical Hamiltonian (2) is equivalent to the following Lagrangian system

$$(3) \qquad \frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i}=\frac{\partial}{\partial x_i}(T-U)\ ; \quad i=1, 2, \cdots, n$$

where $T=T(x, \dot{x})=\sum a_{ij}(x)\dot{x}_i\dot{x}_j$, $4(a_{ij})=(a^{ij})^{-1}$.

We fix a regular value e of H and consider the compact energy surface $H^{-1}(e)$. Using the words of the Lagrangian system (3), a solution $(x(t), y(t))$ of (1) on $H^{-1}(e)$ corresponds to the solution $x(t)$ of (3) with

$$T(x, \dot{x})+U(x)=e\ .$$

Since $T\geqq 0$, the solution $x(t)$ lies in the set

$$M=\{x\in \boldsymbol{R}^n;\ U(x)\leqq e\}\ .$$

Because e is also a regular value of U and $H^{-1}(e)$ is compact, M is a compact smooth manifold with smooth boundary $B=\partial M=\{x;\ U(x)=e\}$. H. Seifert [5] proved our theorem for the case $M\approx D^n$, the n-disk.

For simplicity we only treat Hamiltonian systems on $\boldsymbol{R}^{2n}$. But Hamiltonian systems can be considered on T^*P, where P is a smooth Riemannian manifold with the canonical symplectic structure [7]. Then we have: Let H: $T^*P\to \boldsymbol{R}$ be a Hamiltonian with the form

$$H(x, \xi)=\frac{1}{2}|\xi|^2+U(x)\ , \quad (x, \xi)\in T^*P$$

where $|\ \ |$ is the norm derived from the metric on P and U: $P\to \boldsymbol{R}$ is a smooth function. Then there exists at least one periodic solution of the Hamiltonian system on every compact regular energy surface.

In fact, if the set $\{x\in P;\ U(x)\leqq e\}$ has the boundary, the proof is almost parallel to one given in the following sections. If the set $\{U\leqq e\}$ has no boundary (and is not empty), it means the set coincide with P (assume that P is connected). In this case P must be compact and $\mathrm{Max}\{U(x);\ x\in P\}<e$. So there is a closed geodesic on P w. r. t. the Jacobi metric (4), which is the periodic solution of the system.

§ 1. Neighborhood of the boundary.

Although the equation (3) is the Euler equation of the variational problem for the Lagrangian $T-U$, we employ the principle of least action of Maupertuis-Jacobi. We state the principle using the words of geometry as follows.

We consider a Riemannian metric

$$ds^2=(e-U(x))a_{ij}(x)dx_i dx_j\ , \tag{4}$$

called Jacobi-metric for e. This metric ds is positive in $\mathrm{Int}\, M=M-B$ and degenerates on B.

Then a solution $x(t)$ of (3) with $T(x,\dot{x})+U(x)=e$ (along $x(t)$, $T(x(t),\dot{x}(t))+U(x(t))$ is a constant) is a geodesic w. r. t. ds after proper time change [5]. ([5] treated the analytic case. For C^3 case, see [4].)

For $b\in B$, we denote by $x_b(t)$ the solution of (3) with $x(0)=b$ and $\dot{x}(0)=0$ (so $T(x(0),\dot{x}(0))+U(x(0))=U(b)=e$). If this solution reaches B at a finite time $t_1>0$, then $\dot{x}_b(t_1)=0$ and $x_{b_1}(t)$, where $b_1=x_b(t_1)\in B$, also reaches B at $t=t_1$ and $x_{b_1}(t_1)=b$ by the reversibility of the system (3). Hence this solution is a periodic solution of (3). (Periodic solutions are not necessarily such type, there may be periodic solutions all the time in $\mathrm{Int}\, M$.)

Therefore we seek a smooth curve $\gamma=\gamma(s)$: $[0,1]\to M$ with $\gamma(0)$, $\gamma(1)\in B$ and $\gamma(s)$, $0<s<1$, being a geodesic w. r. t. ds.

Now we introduce a coordinate system near B.

Since B is an $(n-1)$-manifold, $b\in B$ is locally represented by $n-1$ coordinates

$$z^1,\ z^2,\ \cdots,\ z^{n-1}\ .$$

The n-th coordinate of $x_b(t)$, $t>0$ small, is determined to be the arc length w. r. t. ds of the curve $x_b(\tau)$, $0\leqq\tau\leqq t$. We denote by z^n this n-th coordinate. Since B is compact, for small δ, z^n can be taken in common for $0<z^n<\delta$, differently from $z^1, z^2, \cdots, z^{n-1}$. z^n plays the role of y_n in [5] with the relation $y_n=(z^n)^{2/3}$.

The curve $\gamma=\gamma(s)$ represented by

$$(5)\qquad \begin{aligned}&z^1=\text{const.}\,,\quad z^2=\text{const.},\ \cdots,\quad z^{n-1}=\text{const.}\,,\\ &z^n(s)=s\end{aligned}$$

is a geodesic w. r. t. ds for small $s>0$.

We denote by M_δ the set $\text{Int}\,M-\{0<z^n<\delta\}$ and put $B_\delta=\partial M_\delta=\{z^n=\delta\}$ for small $\delta>0$. M_δ is diffeomorphic to M.

In [5], it is proved that any curve (5) intersects with B_δ perpendicularly. So, if g_{ij} is the component of the metric ds w. r. t. $z^1, \cdots, z^n$, then we have

$$(6)\qquad g_{in}=g^{in}=0 \quad \text{if} \quad 1\leqq i<n\,,$$

$$(7)\qquad g_{nn}=g^{nn}=1\,.$$

Therefore a geodesic $\gamma=\gamma(s)\colon [0, 1]\to M_\delta$ starting from and reaching B_δ orthogonally gives a desired solution, connecting it with the curve (5).

We close this section by the following argument.

We change the metric ds on $0<z^n<\delta$ by multiplying a smooth function $\chi=\chi(z^n)>0$. The new metric is denoted by $d\tilde{s}$.

Then we have

LEMMA 1. *A curve*

$$\begin{aligned}&z^1=\mathit{const.}\,,\quad z^2=\mathit{const.},\ \cdots,\quad z^{n-1}=\mathit{const.}\,,\\ &z^n(s)=u(s)\quad \textit{is properly given}\end{aligned}$$

is a geodesic w. r. t. $d\tilde{s}$.

PROOF. Let $\tilde{g}_{ij}=\chi(z^n)g_{ij}$ be the component of $d\tilde{s}$ and $\tilde{\Gamma}^i_{jk}$ the Christoffel symbol w. r. t. $\tilde{g}_{ij}$.

For $1\leqq i<n$,

$$\begin{aligned}\ddot{z}^i+\tilde{\Gamma}^i_{jk}\dot{z}^j\dot{z}^k&=\tilde{\Gamma}^i_{nn}\dot{u}\dot{u}\,,\\ \tilde{\Gamma}^i_{nn}&=\frac{1}{2}\tilde{g}^{ih}\left\{2\frac{\partial}{\partial z^n}\tilde{g}_{hn}-\frac{\partial}{\partial z^h}\tilde{g}_{nn}\right\}\\ &=\frac{1}{2}\sum_{h\neq n}\tilde{g}^{ih}\left(0-\frac{\partial}{\partial z^h}\chi(z^n)\right)\quad \text{(by (6) and (7))}\\ &=0\,.\end{aligned}$$

For $i=n$,

$$\ddot{z}^n+\tilde{\Gamma}^n_{jk}\dot{z}^j\dot{z}^k=\ddot{u}+\tilde{\Gamma}^n_{nn}\dot{u}\dot{u}\,,$$

$$\begin{aligned}\tilde{\Gamma}^n_{nn} &= \frac{1}{2}\tilde{g}^{nk}\left\{2\frac{\partial}{\partial z^n}\tilde{g}_{kn} - \frac{\partial}{\partial z^k}\tilde{g}_{nn}\right\} \\ &= \frac{1}{2}\tilde{g}^{nn}\frac{\partial}{\partial z^n}\tilde{g}_{nn} \\ &= \frac{1}{2}\chi^{-1}(u)\dot{\chi}(u) .\end{aligned}$$

Let $u=u(s)$ be the solution of

$$\ddot{u}+\frac{1}{2}\chi^{-1}(u)\dot{\chi}(u)\dot{u}\dot{u}=0$$

with $0<u(0)<\delta$ and $\dot{u}(0)=1$. Then the curve represented in Lemma 1 is a geodesic w. r. t. $d\tilde{s}$. Q.E.D.

§ 2. Path spaces.

In general, let $\Omega(X; A_0, A_1)$ be the set of continuous curves $\omega=\omega(t)$: $[0, 1]\to X$ with $\omega(i)\in A_i$ $(i=0, 1)$, endowed with compact open topology $(A_i\subset X)$.

Put $Y=\Omega(M; B, B)$ and identify $b\in B$ with the constant curve whose image is b, so B is regarded as a subset of Y.

Then we have

LEMMA 2. *$H_0(Y, B; Z)\neq 0$ or $\pi_k(Y, B)\neq 0$ for some $k\geqq 1$.*

PROOF. (i) If B is not arcwise connected, then $H_0(Y, B)\neq 0$.

In fact, if $H_0(Y, B)=0$, then $i_{Y*}: H_0(B)\to H_0(Y)$ is onto, where $i_Y: B\subset Y$. Let $\omega: [0, 1]\to M$ be a path whose end points belong to different arc components of B. $\omega\in Y$ but the element of $H_0(Y)$ containing ω does not belong to $\operatorname{Im} i_{Y*}$.

(ii) If B is arcwise connected and Y is not so, then $H_0(Y, B)\neq 0$. This is given by the following exact sequence

$$\begin{array}{ccccccc}\tilde{H}_0(B) & \longrightarrow & \tilde{H}_0(Y) & \longrightarrow & H_0(Y, B) & \longrightarrow & 0 .\\ \| & & \nparallel & & & & \\ 0 & & 0 & & & & \end{array}$$

(iii) If B and Y are both arcwise connected, then

$$\pi_k(Y, B)\neq 0 \quad \text{for some} \quad k\geqq 1 .$$

We choose a base point $p \in B$ and assume $\pi_k(Y, B)=0$ for all $k \geqq 1$.

We put $Y'=\Omega(B; B, B)$ then $B \subset Y' \overset{j}{\subset} Y$. Since $B \simeq Y'$, we have by the assumption

$$\pi_k(Y, B) \simeq \pi_k(Y, Y')=0 \quad \text{for} \quad k \geqq 1 .$$

Let $\pi: Y \to B \times B$ be the fibration defined by $\pi(\omega)=(\omega(0), \omega(1))$ and put $F=\pi^{-1}(p, p)=\Omega M$, the loop space.

Also put $\pi'=\pi_{|Y'}: Y' \to B \times B$ and $F'=\pi'^{-1}(p, p)=\Omega B$.

Then we have a commutative diagram of fibration

$$\begin{array}{ccccc} \Omega B & \longrightarrow & Y' & \longrightarrow & B \times B \\ \cap \Omega i & & \cap j & & \| \\ \Omega M & \longrightarrow & Y & \longrightarrow & B \times B \end{array}$$

where $\Omega i: \Omega B \subset \Omega M$ is induced from the inclusion $i: B \subset M$.

This diagram derives the following commutative diagram of long exact sequence of homotopy groups of fibration $(k \geqq 1)$

$$\begin{array}{ccccccccc} \pi_k(Y') & \longrightarrow & \pi_k(B \times B) & \longrightarrow & \pi_{k-1}(\Omega B) & \longrightarrow & \pi_{k-1}(Y') & \longrightarrow & \pi_{k-1}(B \times B) \\ \downarrow j_* & & \| & & \downarrow (\Omega i)_* & & \downarrow j_* & & \| \\ \pi_k(Y) & \longrightarrow & \pi_k(B \times B) & \longrightarrow & \pi_{k-1}(\Omega M) & \longrightarrow & \pi_{k-1}(Y) & \longrightarrow & \pi_{k-1}(B \times B) \end{array} .$$

Since $\pi_k(Y, Y')=0$ for $k \geqq 1$, we have $j_*: \pi_k(Y')=\pi_k(Y)$ for $k \geqq 1$ (recall that we assumed Y and $Y' \simeq B$ are arcwise connected).

Hence by the five lemma and the naturality, the following diagram is obtained $(k \geqq 1)$

$$\begin{array}{ccc} \pi_{k-1}(\Omega B) & \overset{(\Omega i)_*}{\cong} & \pi_{k-1}(\Omega M) \\ \wr\| & \circlearrowleft & \wr\| \\ \pi_k(B) & \xrightarrow{i_*} & \pi_k(M) \end{array} .$$

Therefore $i_*: \pi_k(B) \cong \pi_k(M)$ for $k \geqq 1$.

B and M are arcwise connected and CW complexes (compact smooth manifolds), so $i: B \subset M$ is a homotopy equivalence [7.6.24 in 6].

But on the other hand, $H_n(M, B; Z/2) \neq 0$ [6.3.8 in 6].

This is a contradiction. Q.E.D.

Now we put

$$\Lambda_s=\{\lambda: [0, 1] \to M_s; \quad \text{piecewise smooth with } \lambda(0), \lambda(1) \in B_s\}$$

with the distance as §16 in [2].

For $\lambda \in \Lambda_\delta$, we define

$$E(\lambda)=\int_0^1 |\dot{\lambda}(t)|^2 dt$$

where $|\dot{\lambda}(t)|$ is the norm w.r.t. ds, that is

$$|\dot{\lambda}(t)|^2=(e-U(\lambda(t)))T(\lambda(t), \dot{\lambda}(t)) .$$

The distance in Λ_δ is defined so that $E\colon \Lambda_\delta \to [0, \infty)$ is continuous [§16 in 2].

As Theorem 17.1 in [2], we can prove $\Lambda_\delta \simeq \Omega(M_\delta; B_\delta, B_\delta) \approx \Omega(M; B, B)$, so we have $H_0(\Lambda_\delta, B_\delta) \neq 0$ or $\pi_k(\Lambda_\delta, B_\delta) \neq 0$ for some $k \geqq 1$ from Lemma 2.

§ 3. Mini-max principle.

We choose $\delta_1 > 0$ so small that

(8) we can use the coordinate z^n for $0 < z^n \leqq 3\delta_1$,

(9) for any $v=(v^1, \cdots, v^{n-1})$ and $z=(z^1, \cdots, z^{n-1}, z^n)$, $z'=(z^1, \cdots, z^{n-1}, z'^n)$ with z^n, $z'^n \in (0, 2\delta_1]$, we have

$$\sum_{i,j}^{n-1} a_{ij}(z) v^i v^j \leqq 2 \sum_{i,j}^{n-1} a_{ij}(z') v^i v^j$$

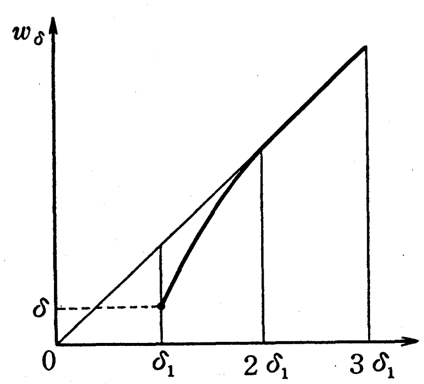
(10) $U(z)=U(z^1, \cdots, z^n)$ is monotone decreasing in $z^n \in (0, 2\delta_1]$ (recall that e is a regular value of U and $B=\{U=e\}$ is compact).

For $\delta \in (0, \delta_1]$, let

$$w_\delta \colon [\delta_1, 3\delta_1] \longrightarrow [\delta, 3\delta_1]$$

be a smooth function with

(11) $$w_\delta(\rho)=\rho \quad \text{if} \quad \rho \in [2\delta_1, 3\delta_1] ,$$



(12) $$w_\delta(\delta_1)=\delta\ ,$$

(13) $$1\leqq \dot{w}_\delta(\rho)\leqq 2 \quad \text{if} \quad \rho\in[\delta_1, 2\delta_1]\ .$$

Using this w_δ, we define a diffeomorphism

$$\psi_\delta\colon M_{\delta_1} \longrightarrow M_\delta$$

which is the identity on $M_{2\delta_1}$ and on $\delta_1\leqq z^n\leqq 2\delta_1$, sends

$$(z^1, \cdots, z^{n-1}, z^n) \quad \text{to} \quad (z^1, \cdots, z^{n-1}, w_\delta(z^n))\ .$$

And define a homeomorphism $\Psi_\delta\colon \Lambda_{\delta_1}\to\Lambda_\delta$ by $\lambda_1\in\Lambda_{\delta_1}\mapsto\psi_\delta\circ\lambda_1\in\Lambda_\delta$.
Then we have

(14) $$E(\Psi_\delta(\lambda_1))\leqq 4E(\lambda_1) \quad \text{for any} \quad \lambda_1\in\Lambda_{\delta_1}\ .$$

In fact, put $\lambda(t)=\Psi_\delta(\lambda_1)(t)=\psi_\delta(\lambda_1(t))$.
It is sufficient to prove that $|\dot{\lambda}(t)|^2\leqq 4|\dot{\lambda}_1(t)|^2$.
If $\lambda_1(t)\in M_{2\delta_1}$, $|\dot{\lambda}_1(t)|=|\dot{\lambda}(t)|$.
If $\lambda_1(t)\in M_{\delta_1}-M_{2\delta_1}$, let $z^1(t), \cdots, z^{n-1}(t), z^n_1(t)$ be the coordinate of it. Then $\lambda(t)$ has the coordinates

$$z^1(t), \cdots, z^{n-1}(t), z^n(t)=w_\delta(z^n_1(t))\ ,$$

and

$$\begin{aligned}
|\dot{\lambda}(t)|^2 &= \sum_{i,j}^{n-1} g_{ij}(\lambda(t))\dot{z}^i(t)\dot{z}^j(t)+(\dot{z}^n(t))^2 && \text{(by (6) and (7))}\\
&= \sum^{n-1}(e-U(\lambda(t)))a_{ij}(\lambda(t))\dot{z}^i(t)\dot{z}^j(t)+(\dot{w}_\delta(z^n_1(t))\dot{z}^n_1(t))^2\\
&\leqq \sum^{n-1}(e-U(\lambda_1(t)))2a_{ij}(\lambda_1(t))\dot{z}^i(t)\dot{z}^j(t) && \text{(by (9) and (10))}\\
&\quad +4(\dot{z}^n_1(t))^2 && \text{(by (13))}\\
&= 2\sum^{n-1} g_{ij}(\lambda_1(t))\dot{z}^i(t)\dot{z}^j(t)+4(\dot{z}^n_1(t))^2\\
&\leqq 4|\dot{\lambda}_1(t)|^2\ .
\end{aligned}$$

This proves (14).

Now let $a_{\delta_1}\in H_0(\Lambda_{\delta_1}, B_{\delta_1})$ or $\alpha_{\delta_1}\in\pi_k(\Lambda_{\delta_1}, B_{\delta_1})$ for some $k\geqq 1$ be the nontrivial element.

$$\Psi_{\delta*}\colon H_0(\Lambda_{\delta_1}, B_{\delta_1}) \longrightarrow H_0(\Lambda_\delta, B_\delta) \quad \text{or} \quad \Psi_{\delta*}\colon \pi_k(\Lambda_{\delta_1}, B_{\delta_1}) \longrightarrow \pi_k(\Lambda_\delta, B_\delta)$$

are both isomorphism and we put

$$a_\delta=\Psi_{\delta*}a_{\delta_1} \quad \text{or} \quad \alpha_\delta=\Psi_{\delta*}\alpha_{\delta_1}\ .$$

For the homology case, $\alpha_\delta \neq 0$ in $H_0(\Lambda_\delta, B_\delta)$ means α_δ is an arc-component of Λ_δ other than B_δ.

Then we define

$$c_\delta = \inf_{\lambda \in \alpha_\delta} E(\lambda) . \tag{15}$$

For the homotopy case, a representative $f \in \alpha_\delta$ is a continuous function $(D^k, S^{k-1}) \to (\Lambda_\delta, B_\delta)$. So $\operatorname{Im} f = f(D^k)$ is a compact subset of Λ_δ, hence $E(\operatorname{Im} f)$ attains a maximal value.

In this case we define

$$c_\delta = \inf_{f \in \alpha_\delta} \operatorname{Max} E(\operatorname{Im} f) . \tag{16}$$

Then we have

LEMMA 3. *There exists $K \geqq 1$ such that*

$$c_\delta + 1 \leqq K \quad \textit{for any} \quad \delta \in (0, \delta_1] .$$

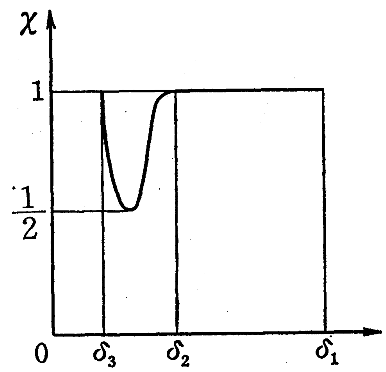
PROOF. For the homotopy case, by the definition of c_{δ_1}, there is $f_1 \in \alpha_{\delta_1}$ with $\operatorname{Max} E(\operatorname{Im} f_1) \leqq c_{\delta_1} + 1$. Then $\Psi_\delta \circ f_1 \in \Psi_{\delta *} \alpha_{\delta_1} = \alpha_\delta$. So

$$\begin{aligned} c_\delta &\leqq \operatorname{Max} E(\operatorname{Im} \Psi_\delta \circ f_1) \\ &\leqq 4 \operatorname{Max} E(\operatorname{Im} f_1) \quad \text{(by (14))} \\ &\leqq 4(c_{\delta_1} + 1) . \end{aligned}$$

Therefore, if we put $K = 4c_{\delta_1} + 5$, the lemma is proved. For the homology case, also putting $K = 4c_{\delta_1} + 5$ gives the lemma. Q.E.D.

§ 4. Curve shortening procedure.

Let $0 < \delta_3 < \delta_2 < \delta_1$ (the smallness of δ_2 is determined in the next section) and $\chi: [\delta_3, \delta_1] \to [1/2, 1]$ be a smooth function satisfying



(17) $$\chi(\delta)=1 \quad \text{for} \quad \delta \in [\delta_2, \delta_1]\,,$$

(18) $$\chi(\delta_3)=1$$

(19) $|\dot{\chi}(\delta_3)|$ is sufficiently large so that M_{δ_3} is geodesic convex w.r.t. $d\tilde{s}$, where $d\tilde{s}$ is defined as in §1 and $d\tilde{s}=ds$ on M_{δ_2}. (This can be done as §6 in [5].)

We denote by $|\cdot|_\chi$ the norm w.r.t. $d\tilde{s}$ and put

$$\tilde{E}(\lambda)=\int_0^1 |\dot{\lambda}(t)|_\chi^2 dt \quad \text{for} \quad \lambda \in \Lambda_{\delta_3}\,.$$

Then $\tilde{E}\colon \Lambda_{\delta_3} \to [0, \infty)$ is a continuous function and

(20) $$\frac{1}{4}E(\lambda) \leqq \tilde{E}(\lambda) \leqq E(\lambda) \quad \text{for} \quad \lambda \in \Lambda_{\delta_3}\,.$$

Let $\tilde{d}(\ ,\ )$ be the Riemannian distance on M_{δ_3} w.r.t. $d\tilde{s}$ and choose $\eta>0$ so that

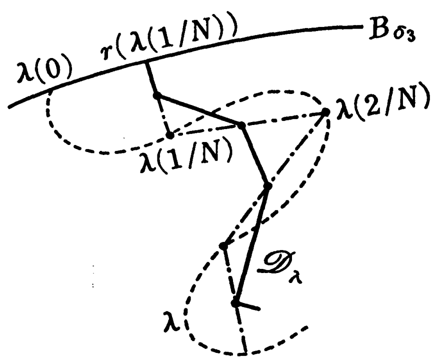
(21) two points $x, y \in M_{\delta_3}$ with $\tilde{d}(x, y) \leqq \eta$ is uniquely combined by the shortest geodesic in M_{δ_3},

(22) for $x \in M_{\delta_3}$ with $\tilde{d}(x, B_{\delta_3}) \leqq \eta$, there is the unique point $r(x) \in B_{\delta_3}$ satisfying $\tilde{d}(x, r(x))=\tilde{d}(x, B_{\delta_3})$.

We put $\tilde{\Lambda}^a=\{\lambda \in \Lambda_{\delta_3};\ \tilde{E}(\lambda) \leqq a\}$ for $a \geqq 0$ and let N be an integer greater than $(K/\eta)^2$. Then for $\lambda \in \tilde{\Lambda}^K$ and $0 \leqq t_1 \leqq t_2 \leqq 1$ with $t_2 - t_1 \leqq 1/N$,

$$\tilde{d}(\lambda(t_1), \lambda(t_2)) \leqq \int_{t_1}^{t_2} |\dot{\lambda}(t)|_\chi dt \leqq (t_2-t_1)^{1/2} E(\lambda) \leqq (1/N)^{1/2} K \leqq \eta\,.$$

Now we define a deformation $\mathscr{D}\colon \tilde{\Lambda}^K \to \tilde{\Lambda}^K$ which is employed in [5] (see also Appendix of [1]), but our procedure is slightly different from



it at end points.

For $\lambda \in \tilde{\Lambda}^K$, we join $r(\lambda(1/N))$, $\lambda(1/N)$, $\lambda(2/N)$, $\cdots$, $\lambda(1-1/N)$ and $r(\lambda(1-1/N))$ successively by the shortest geodesics, then mark the centers of the geodesics and join the centers by the shortest geodesics in order.

We denote by $\mathscr{D}\lambda$ the new curve so constructed. Then $\mathscr{D}: \tilde{\Lambda}^K \to \tilde{\Lambda}^K$ is continuous and

(23) $$\mathscr{D} \text{ is } \tilde{E}\text{-decreasing },$$

(24) $\mathscr{D} \simeq i_d: \tilde{\Lambda}^K \to \tilde{\Lambda}^K$ and the homotopy between $\mathscr{D}$ and i_d is also $\tilde{E}$-decreasing (we denote by i_d the identity map).

(25) $\tilde{E}(\mathscr{D}\lambda) = \tilde{E}(\lambda) > 0$, $\lambda \in \tilde{\Lambda}^K$, implies that $\lambda: [0, 1] \to M_{\delta_3}$ is a geodesic w.r.t. $d\tilde{s}$ and both end points intersect B_{δ_3} perpendicularly (cf. [A.1.2. in 1]).

Now we put

(26) $$\tilde{c} = \inf_{f \in \alpha_{\delta_3}} \operatorname{Max} \tilde{E}(\operatorname{Im} f) .$$

By (20), we have

(27) $$\frac{1}{4} c_{\delta_3} \leqq \tilde{c} \leqq c_{\delta_3} .$$

We prove $\tilde{c} > 0$ in the next section. (We only consider the homotopy case. The homology case can be treated in a similar fashion.)

By the definition (26) of $\tilde{c}$, for any natural number j, there is $f_j \in \alpha_{\delta_3}$ with

$$\begin{aligned} \tilde{c} \leqq \operatorname{Max} \tilde{E}(\operatorname{Im} f_j) &\leqq \tilde{c} + 1/j \\ &\leqq c_{\delta_3} + 1 \quad \text{(by (27))} \\ &\leqq K \quad \text{(by Lemma 3)} . \end{aligned}$$

So $\operatorname{Im} f_j \subset \tilde{\Lambda}^K$, hence $\mathscr{D} \circ f_j \in \alpha_{\delta_3}$ by (24). Therefore

$$\tilde{c} \leqq \operatorname{Max} \tilde{E}(\operatorname{Im}(\mathscr{D} \circ f_j)) \leqq \operatorname{Max} \tilde{E}(\operatorname{Im} f_j) \leqq \tilde{c} + 1/j .$$

Let $\mathscr{D}\lambda_j$, where $\lambda_j \in \operatorname{Im} f_j$, be the element of $\operatorname{Im}(\mathscr{D} \circ f_j)$ which attains the maximal value. Then

$$\tilde{c} \leqq \tilde{E}(\mathscr{D}\lambda_j) \leqq \tilde{E}(\lambda_j) \leqq \operatorname{Max} \tilde{E}(\operatorname{Im} f_j) \leqq \tilde{c} + 1/j .$$

Thus we have a sequence $\{\lambda_j\}_{j=1,2,\ldots}$ in $\tilde{\Lambda}^K$ with

$$\lim_{j\to\infty} \tilde{E}(\mathscr{D}\lambda_j)=\lim_{j\to\infty} E(\lambda_j)=\tilde{c}\ .$$

Then, as A.1.3 in [1], there exists $\lambda_\infty \in \tilde{\Lambda}^K$ with

$$\tilde{E}(\lambda_\infty)=\tilde{E}(\mathscr{D}\lambda_\infty)=\tilde{c}\ .$$

By (25), λ_∞: $[0, 1]\to M_{\delta_3}$ is a geodesic w.r.t $d\tilde{s}$ starting from and reaching B_{δ_3} orthogonally.

$d\tilde{s}=ds$ on M_{δ_2} and according to Lemma 1, a curve

$$z^1=\text{const.},\ z^2=\text{const.},\ \cdots,\ z^{n-1}=\text{const.}\ ,$$
$$\delta_3\leqq z^n\leqq\delta_2$$

is a geodesic w.r.t. both ds and $d\tilde{s}$. So this λ_∞ seems to give the desired geodesic. But we have not avoided the possibility that after entering M_{δ_2}, λ_∞ intersects B_{δ_2} not orthogonally.

Hence we use a little trick in the next section.

§ 5. Proof of the Theorem.

We assume that there are no periodic solutions on the regular energy surface $H^{-1}(e)$ and derive a contradiction. In particular the solution $x_b(t)$ cannot reach B for all $b\in B$, that is $x_b(t)\in \operatorname{Int} M$ for $0<t<\infty$. By Satz 3 in §5 of [5], the arc length of $x_b(t)$, $0<t<\infty$, w.r.t. ds is also ∞.

For $b\in B$, let $\sigma_b=\sigma_b(s)$: $[0, 1]\to M$ be the curve which is reparameterized from $x_b(t)$ so that $\sigma_b(0)=b$ and $\sigma_b(s)$, $0<s\leqq 1$, is a geodesic w.r.t. ds and whose arc length w.r.t. ds is equal to $K^{1/2}+\delta_1$.

By the assumption at the beginning of this section, $\sigma_b(s)$ never reach B and B is compact, hence we can choose $\delta_2\in(0, \delta_1)$ so small that σ_b all the time lies in M_{δ_2} after once it enters in M_{δ_1} for any $b\in B$.

The arc length of the curve $\sigma_b(s)$, $s_1\leqq s\leqq 1$, where $s_1>0$ is the first time at which $\sigma_b(s_1)\in B_{\delta_1}$, is $K^{1/2}$.

We choose δ_3 arbitrarily in $(0, \delta_2)$.

Let $\lambda_\infty=\lambda_\infty(s)$: $[0, 1]\to M_{\delta_3}$ be the geodesic w.r.t. $d\tilde{s}$ constructed in §4. The part of λ_∞

$$z^1=\text{const.}\ ,\quad z^2=\text{const.},\ \cdots\ ,\quad z^{n-1}=\text{const.}\ ,$$
$$\delta_3\leqq z^n\leqq\delta_2\quad (s\in[0, \tilde{s}_2]\quad \text{for some}\quad \tilde{s}_2>0)$$

is coincide with the part $\sigma_b(s)$, $s\in[s_3, s_2]$, for some $b\in B$ and some small $0<s_3<s_2$ with $\sigma_b(s_3)\in B_{\delta_3}$ and $\sigma_b(s_2)\in B_{\delta_2}$, by Lemma 1.

Since $\lambda_\infty\in\tilde{\Lambda}^K$, the arc length of the curve $\lambda_\infty(s)$, $\tilde{s}_2\leqq s\leqq 1$, w.r.t. $d\tilde{s}$ is less than $K^{1/2}$.

$d\tilde{s}=ds$ on M_{δ_2}, so λ_∞ coincide with σ_b as long as $\lambda_\infty(s)\in M_{\delta_2}$. By the determination of δ_2, σ_b, after it enters M_{δ_2} for the first time, lies in M_{δ_2} as long as the arc length of $\sigma_{b|[s_2,s]}$ is less than $K^{1/2}$. Hence $\lambda_\infty(1)\in M_{\delta_2}$. This is the contradiction proving the Theorem.

Finally we prove $\tilde{c}>0$.

We consider $\tilde{\Lambda}^\varepsilon$, where $\varepsilon=\delta_1^2$. Then for $\lambda\in\tilde{\Lambda}^\varepsilon$,

$$\begin{aligned} E(\lambda) &\leqq 4\tilde{E}(\lambda) \quad \text{(by (20))} \\ &\leqq 4\delta_1^2 . \end{aligned}$$

Hence the arc length of λ w.r.t. $ds\leqq 2\delta_1$.

Let $z^n(t)$ be the n-th coordinate of $\lambda(t)$. Then

$$\begin{aligned} \int_0^1 |\dot{z}^n(t)|dt &\leqq \int_0^1 |\dot{\lambda}(t)|dt \quad \text{(by (6))} \\ &\leqq 2\delta_1 . \end{aligned}$$

Since $z^n(0)=z^n(1)=\delta_3$, we have

$$\operatorname*{Max}_{0\leqq t\leqq 1} z^n(t)\leqq \delta_3+\delta_1<2\delta_1 .$$

Therefore, by (8), the curve $\lambda\in\tilde{\Lambda}^\varepsilon$ is "projected" to B_{δ_3} by $z^n(t)\mapsto\delta_3$ homotopically (preserving other coordinates $z^1, \cdots, z^{n-1}$), and after that the curve on B_{δ_3} is contracted to the center of it homotopically.

Thus we have

(28) the inclusion $i^\varepsilon\colon B_{\delta_3}\subset\Lambda^\varepsilon$ is a homotopy equivalence.

Then we have

$$\tilde{c}\geqq\varepsilon>0 . \tag{29}$$

In fact, consider $j^\varepsilon_*\colon \pi_k(\Lambda_{\delta_3}, B_{\delta_3})\to\pi_k(\Lambda_{\delta_3}, \tilde{\Lambda}^\varepsilon)$, where $j^\varepsilon\colon(\Lambda_{\delta_3}, B_{\delta_3})\subset(\Lambda_{\delta_3}, \tilde{\Lambda}^\varepsilon)$. By (28), j^ε_* is an isomorphism.

If $\tilde{c}<\varepsilon$, then there is an $f\in\alpha_{\delta_3}$ with $\operatorname{Max}\tilde{E}(\operatorname{Im} f)\leqq\varepsilon$.

So $\operatorname{Im}(j^\varepsilon\circ f)\subset\tilde{\Lambda}^\varepsilon$, hence $j^\varepsilon_*\alpha_{\delta_3}$, which is represented by $j^\varepsilon\circ f$, is the trivial element of $\pi_k(\Lambda_{\delta_3}, \tilde{\Lambda}^\varepsilon)$.

Since j^ε_* is an isomorphism and α_{δ_3} has been taken to be nontrivial in $\pi_k(\Lambda_{\delta_3}, B_{\delta_3})$, this is a contradiction.

This completes the proof of (29).

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