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Functions Which Operate by Composition on the Real Part of a Banach Function Algebra

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Introduction

In this paper we study functions which operate on a Banach space which is the real part of a Banach function algebra. We say that A is a *Banach function algebra* if A is a Banach algebra lying in $C(X)$, the algebra of all complex-valued continuous functions on a compact Hausdorff space X which separates the points of X and contains constant functions. The history of this problem probably begins with J. Wermer's paper [8] in which he proved that the real part of a non-trivial function algebra is not closed under multiplication. $\operatorname{Re} A=\{u \in C_R(X): \exists f \in A, \operatorname{Re} f=u\}$, the space of the real part of a Banach function algebra A with the norm $N(\cdot)$ on a compact Hausdorff space X, is complete with the norm $N_R(\cdot)$ defined as follows. For each u in $\operatorname{Re} A$

$$N_R(u)=\inf\{N(f): f \in A, \operatorname{Re} f=u\} .$$

Suppose that h is a real-valued continuous function on a non-degenerate interval I, we say that *h operates by composition on* $\operatorname{Re} A$ if $h\circ u$ is in $\operatorname{Re} A$ whenever $u \in \operatorname{Re} A$ has range in I. J. Wermer's theorem is made a change in the wording that $t\mapsto t^2$ does not operate by composition on the real part of a non-trivial function algebra. Obviously each affine function has such a property for any Banach function algebra. It is natural to consider whether this result may be extended to any Banach function algebra. However, we easily find many counter examples for which the question does not hold, e.g.,

$$C^{(n)}[0, 1],\ A(\Gamma)=\left\{f \in C(\Gamma): \sum_{-\infty}^{\infty} |\hat{f}(n)|<\infty\right\} ,$$

where Γ is the unit circle in the complex plane and $\hat{f}(n)$ is the n-th Fourier coefficient, and so on.

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§1. Ultraseparability.

One of our objective is to find conditions under which the result holds good. For the case of a function algebra it is shown inductively by J. Wermer's theorem that any polynomial of degree more than 1 does not operate by composition on the real parts of a non-trivial function algebra. But because of difference between uniform norm and $N_R(\cdot)$ on $\mathrm{Re}\, A$ it is not clear in the context that any nonaffine continuous function does not operate by composition on the real part of a non-trivial function algebra.

Boundedness condition for h induced by A. Bernard [2] has an effect to attack the problem. We say that *h operates boundedly on* $\mathrm{Re}\, A$ if there exist an $\varepsilon>0$ and a $\delta>0$ such that $N_R(h\circ u)<\varepsilon$ for every u in $\mathrm{Re}\, A$ with $N_R(u)<\delta$. S. J. Sidney [7] and O. Hatori [5] proved the problem for the case of a function algebra by applying weak boundedness principles in a suitable sense. We study in this paper more general case through some kind of boundedness principle for h.

Suppose that E is a (real or complex) Banach space with the norm $N(\cdot)$. We define $l^\infty(N, E)$ the space of all bounded sequences in E and the norm $\tilde{N}(\cdot)$ on $l^\infty(N, E)$ by $\tilde{N}(\hat{f})=\sup\{N(f_n): \tilde{f}=(f_n)\}$. Then $l^\infty(N, E)$ is a Banach space with the norm $\tilde{N}(\cdot)$ and we denote $\tilde{E}=l^\infty(N, E)$ in this paper. If A is a Banach algebra we also define a Banach algebra $\tilde{A}$ by the same way as above.

We say that E is a *Banach function space on X* if E is a (real or complex) Banach space of (real or complex) continuous functions on a compact Hausdorff space X whose norm $N(\cdot)$ dominates the supremum norm $\|\cdot\|_X$ and E separates the points of X and contains 1. Let B be a Banach function space (algebra) with the norm $N(\cdot)$ on X, and let K be a compact subset of X. We denote $B|K=\{f\in C(K): \exists F\in B, F|K=f\}$ where $F|K$ is the restriction of F to K and $N^K(f)=\inf\{N(F): F|K=f, F\in B\}$, then $B|K$ is a Banach function space (algebra) with respect to the norm $N^K(\cdot)$. Let E be a Banach space (algebra) lying in $C(X)$ we naturally identify $\tilde{E}$ a Banach space (algebra) lying in $C(\tilde{X})$ where $\tilde{X}=\beta(N\times X)$ is the Stone-Čech compactification of the product space $N\times X$ of the discrete space of the positive integers and X (it is noted that it is in rare cases that the space $\beta(N\times X)$ is naturally homeomorphic to the direct product of N and X [4]). Obviously $\tilde{E}$ contains constants but $\tilde{E}$ may not separate the points of $\tilde{X}$. We say that a Banach function space (algebra) E is *ultraseparating on X* if $\tilde{E}$ separates the points of $\tilde{X}$. Suppose that A is a *Dirichlet algebra on X*, that is, A is a function algebra such that $\mathrm{Re}\,A$ is uniformly dense in $C_R(X)$, then A is ultraseparating on X. In general ultraseparability of a Banach function algebra does *not* depend on uniform density of the real part in $C_R(X)$. Suppose that A is a Banach function space (algebra) and that Y is a compact subset of X, then $(A|Y)^{\sim}=\tilde{A}|\tilde{Y}$ and $(\mathrm{Re}\,A)^{\sim}=\mathrm{Re}\,\tilde{A}$. If A is a ultraseparating Banach function space (algebra), then $A|Y$ and $\mathrm{Re}\,A$ are also ultraseparating [2]. If A is a ultraseparating Banach function algebra then $\tilde{A}$ is also ultraseparating [1]. Suppose that X_1 and X_2 are compact subsets of X, then $\tilde{X}_1 \cup \tilde{X}_2=(X_1 \cup X_2)^{\sim}$ (for a characterization of ultraseparability for Banach function algebras, see [1]).

S. J. Sidney [7] showed that $A=C(X)$ if A is a ultraseparating Banach function algebra such that a "highly nonaffine" function operates by composition on $\mathrm{Re}\,A$.

Throughout this paper $\|\cdot\|_Y$ is the uniform norm on $C(Y)$, the space of all complex-valued bounded continuous functions on a topological space Y. For a subset E of $C(Y)$, $\mathrm{cl}(E)$ is the uniform closure of E in $C(Y)$. For $f \in C(Y)$, $\tilde{f}$ denotes the element $(f, f, f, \cdots)$ of $C(\tilde{Y})$ if Y is a compact Hausdorff space.

§2. The main results.

THEOREM. *Suppose that A is a ultraseparating Banach function algebra on a compact Hausdorff space X and that h is a nonaffine continuous function on an interval. If h operates by composition on* $\mathrm{Re}\,A$, *then $A=C(X)$.*

COROLLARY 1. *Suppose that A is a ultraseparating Banach function algebra, especially a Dirichlet algebra, on a compact Hausdorff space X. Then for each compact subset of K either of the following are satisfied.*

1) $A|K=C(K)$.
2) *Only affine functions operate by composition on* $\mathrm{Re}\,A|K$.

PROOF. For each compact subset of K, $A|K$ is a ultraseparating Banach function algebra on K. Thus the conclusion follows from Theorem.

COROLLARY 2 ([5], [7]). *Suppose that A is a function algebra on X and h satisfies the same conditions as Theorem. If h operates by composition on* $\mathrm{Re}\,A$, *then $A=C(X)$.*

PROOF. If h operates by composition on $\mathrm{Re}\,A$, then $\mathrm{Re}\,A$ is uniformly dense in $C_R(X)$, that is A is a Dirichlet algebra on X by a theorem of de Leeuw and Katznelson [3]. Thus A is ultraseparating on X, so $A=$

$C(X)$. In order to prove Theorem we show the following Lemma.

LEMMA. *Suppose that F is a Banach space with the norm $N(\cdot)$ that is continuously embeded in $C_R(K)$ with norm 1 and that F separates the points of a compact Hausdorff space K and that F contains at least one positive function on K and that h is a nonaffine real valued continuous function on an interval $[a, b]$ such that c is a point of which h is nonaffine on any open neighborhood. Let $V=\{\tilde{u}\in C_R(\tilde{K})\colon \forall\tilde{v}\in\tilde{F}, \tilde{v}\tilde{u}\in \mathrm{cl}(\tilde{F})\}$ and V' be a subalgebra of V. Suppose that there is a u_F in F such that $a+\gamma<u_F<b-\gamma$ on K for a small $\gamma>0$ and put $J=\{u+u_F\colon u\in F, N(u)<\gamma\}$. Suppose that for any different $\tilde{x}$ and $\tilde{y}$ in $\tilde{K}$ one of the following condition is satisfied.*

1) *There exists a $\tilde{u}\in\tilde{F}$ with $\tilde{N}(\tilde{u})<\gamma$ such that $\tilde{u}(\tilde{x})=0$, $\tilde{u}(\tilde{y})\neq 0$ and $(\tilde{u}+\tilde{u}_F)(\tilde{y})=c$.*

2) *V' separates $\tilde{x}$ and $\tilde{y}$.*

If h operates boundedly on a dense subset of J with respect to the topology induced by $N(\cdot)$, then $F=C_R(K)$.

PROOF. Let g be a positive function on F. For sufficiently small δ let λ_δ be a non-negative continuously differentiable function on R supported in $(-\delta, \delta)$ with integral 1. Let $H_\delta(x, y)$ denote

$$H_\delta(x, y)=\int h(x-yt)\lambda_\delta dt\ .$$

Since h is defined on $[a, b]$ we define $H_\delta(x, y)$ on $\{(x, y)\in R^2\colon a+\delta\|g\|_K\leqq x\leqq b-\delta\|g\|_K, |y|\leqq\|g\|_K\}$. Let $S=\{(x, y)\in R^2\colon a<x<b, 0<y<\|g\|_K\}$. $H_\delta(x, y)$ is continuously differentiable and converges uniformly to $h(x)$ on any compact subset of S as δ tends to 0. V is a uniformly closed subalgebra of $C_R(\tilde{K})$ and $V\tilde{v}=\{\tilde{u}\tilde{v}\colon \tilde{u}\in V\}$ is in $\mathrm{cl}(\tilde{F})$ for each $\tilde{v}$ in $\mathrm{cl}(\tilde{F})$. Let $d(J)$ be a dense subset of J on which h operates boundedly. For any $\tilde{u}=(u_n)$ in $\tilde{J}=\{\tilde{u}\colon \tilde{u}=(u_n), u_n\in J\}$ we have $h\circ\tilde{u}$ is in $\mathrm{cl}(\tilde{F})$. For if u_n is in $d(J)$ for every n, then $\{h\circ u_n\}$ is bounded sequence by the boundedness condition for h we have $(h\circ u_n)=h\circ\tilde{u}$ is in $\tilde{F}$ so in $\mathrm{cl}(\tilde{F})$. In general for every $\tilde{u}=(u_n)$ in $\tilde{J}$ and for any $\varepsilon>0$ there exists a $u_{n(\varepsilon)}$ in $d(J)$ such that $N(u_n-u_{n(\varepsilon)})<\varepsilon$ so $\|u_n-u_{n(\varepsilon)}\|_K<\varepsilon$ and $h\circ(u_{n(\varepsilon)})$ is in $\tilde{F}$, then by the uniform continuity of h, $\|h\circ\tilde{u}-h\circ(u_{n(\varepsilon)})\|_{\tilde{K}}$ tends to 0 as ε tends to 0. So we have $h\circ\tilde{u}$ is in $\mathrm{cl}(\tilde{F})$. Thus for every $\tilde{u}$ in $\tilde{J}$ such that $a<\tilde{u}<b$ and for a small t $h\circ(\tilde{u}-\tilde{g}t)$ is in $\mathrm{cl}(\tilde{F})$. So $H_\delta(\tilde{u}, \tilde{g})$ is in $\mathrm{cl}(\tilde{F})$ for a small δ. Similarly $H_\delta(\tilde{u}+\Delta\tilde{v}, \tilde{g})$ is in $\mathrm{cl}(\tilde{F})$ for a small δ and a small Δ and a $\tilde{v}$ in $\mathrm{cl}(\tilde{F})$. Since $\{(H_\delta(\tilde{u}+\Delta\tilde{v}, \tilde{g})-H_\delta(\tilde{u}, \tilde{g})\}/\Delta$ tends to $(\partial/\partial x)H_\delta(\tilde{u}, \tilde{g})\tilde{v}$ as Δ tends to 0, $(\partial/\partial x)H_\delta(\tilde{u}, \tilde{g})\tilde{v}$ is in $\mathrm{cl}(\tilde{F})$ for a $\tilde{v}$ in $\mathrm{cl}(\tilde{F})$ and a small δ.

Let $\tilde{x}$ and $\tilde{y}$ be in $\tilde{K}$ and 2) is not satisfied. Then there exists a $\tilde{u}$ in $\tilde{F}$ with $\tilde{N}(\tilde{u})<\gamma$ such that $\tilde{u}(\tilde{x})=0$ and $\tilde{u}(\tilde{y})\neq 0$ and $(\tilde{u}+\tilde{u}_F)(\tilde{y})=c$. We shall find $\tilde{w}$ in $\tilde{J}$ and a small δ such that $(\partial/\partial x)H_\delta(\tilde{w},\bar{g})(\tilde{x})\neq(\partial/\partial x)H_\delta(\tilde{w},\bar{g})(\tilde{y})$. If $(\partial/\partial x)H_\delta(\tilde{u}+\tilde{u}_F,\bar{g})(\tilde{x})\neq(\partial/\partial x)H_\delta(\tilde{u}+\tilde{u}_F,\bar{g})(\tilde{y})$, put $\tilde{w}=\tilde{u}+\tilde{u}_F$. Suppose that $(\partial/\partial x)H_\delta(\tilde{u}+\tilde{u}_F,\bar{g})(\tilde{x})=(\partial/\partial x)H_\delta(\tilde{u}+\tilde{u}_F,\bar{g})(\tilde{y})$. For any open neighborhood θ of $(\tilde{u}+\tilde{u}_F)(\tilde{y})$, there exists a small δ' such that $t\mapsto H_{\delta'}(t,\bar{g}(\tilde{y}))$ is not collinear on θ. So we can choose a small s and a small δ' such that $(1+\alpha)\tilde{u}\in\tilde{F}$ with $\tilde{N}(\tilde{u})<\gamma/(1+\alpha)$ for $\alpha=s/\tilde{u}(\tilde{y})$ and $(\partial/\partial x)H_{\delta'}(\tilde{u}+\tilde{u}_F,\bar{g})(\tilde{y})\neq(\partial/\partial x)H_{\delta'}(\tilde{u}+\tilde{u}_F+s,\bar{g})(\tilde{y})$. Thus $(\partial/\partial x)H_{\delta'}(\tilde{w},\bar{g})(\tilde{x})\neq(\partial/\partial x)H_{\delta'}(\tilde{w},\bar{g})(\tilde{y})$ for $\tilde{w}=(1+\alpha)\tilde{u}+\tilde{u}_F$. In either case we can choose $\tilde{w}$ in $\tilde{J}$ and a $\delta>0$ such that $(\partial/\partial x)H_\delta(\tilde{w},\bar{g})(\tilde{x})\neq(\partial/\partial x)H_\delta(\tilde{w},\bar{g})(\tilde{y})$. Thus for all different $\tilde{x}$ and $\tilde{y}$ in $\tilde{K}$ there exists a $\tilde{v}$ in V or V' which separates $\tilde{x}$ and $\tilde{y}$ and infact V contains V' so V separates the points of $\tilde{K}$ and V contains 1 so $V=C_R(\tilde{K})$. Let $V\bar{g}=\{v\bar{g}: v\in V\}$. By the definition of V, $V\bar{g}\subset\mathrm{cl}(\tilde{F})$. So $C_R(\tilde{K})\bar{g}=\{v\bar{g}: v\in C_R(\tilde{K})\}\subset\mathrm{cl}(\tilde{F})$. Thus $C_R(\tilde{K})=\mathrm{cl}(\tilde{F})$ since $\bar{g}>0$ on $\tilde{K}$.

§3. Proof of Theorem.

Without loss of generality we may assume that the domain of h is the interval $[-1,1]$ and h is not affine on any open neighborhood of 0. For any x in X we will choose a compact neighborhood which is also an interpolation set for A. Then by compactness of X there exist $x_1, x_2, \cdots, x_n$ in X and subsets $Y_1, Y_2, \cdots, Y_n$ of X such that $X=\bigcup_i Y_i$ and for each $i=1,2,\cdots,n$, Y_i is a compact neighborhood of x_i which is also an interpolation set for A. Then $\tilde{X}=\bigcup_i\tilde{Y}_i$ and $\tilde{A}|\tilde{Y}_i=C(\tilde{Y}_i)$ so $\tilde{A}$ is uniformly dense in $C(\tilde{X})$. Thus $A=C(X)$ by Bernard's lemma [2; Lemma 4.5].

Let x_0 be a fixed point in X. We construct a desired compact neighborhood Y of x_0 as follows. Let $B_0=\{u\in\mathrm{Re}\,A: u(x_0)=0,\ -1\leqq u\leqq 1\}$, then B_0 is nonempty and closed with respect to the norm $N_R(\cdot)$. For every positive integer n we denote $B_n=\{u\in B_0: N_R(h\circ u)<n\}$. Then $B_0=\bigcup_n B_n$ and for some number m $\bar{B}_m$, the closure in B_0, contains an open set by the Baire category theorem. Thus there are a u_0 in B_m and a small $\varepsilon>0$ such that $|u_0|<1-\varepsilon$ and $U=\{u\in B_0: N_R(u-u_0)<\varepsilon\}$ is contained in $\bar{B}_m$. $(\tilde{A})^{\check{}}$ is ultraseparating since A is ultraseparating [1]. Suppose that $\sigma(x,y)=\sup\{|f(x)|: f\in A,\ f(y)=0,\ N(f)\leqq 1\}$ for x and y in $(\tilde{X})^{\check{}}$. Then $2M=\inf\{\sigma(x,y): x,y\in(\tilde{X})^{\check{}},\ x\neq y\}$ is greater than 0 (see [1]). Let $\varepsilon'=(\varepsilon M^2)/9$ and $Y=\{y\in X: |u_0(y)|\leqq\varepsilon'\}$. Y is a compact neighborhood of x_0 and will be shown an interpolation set for A.

Let $A_0=\{f\in A: f(x_0)=0\}$ and $D=[(A_0|Y)^{\check{}},1]$ be the uniformly closed subalgebra of $C(\tilde{Y})$ which is generated by $(A_0|Y)^{\check{}}$ and constants. Let $\tilde{Y}_0$ be a quotient space reduced by $\tilde{Y}$ identified the points which are not separated by $(A_0|Y)^{\sim}$. In fact there is only one point in $\tilde{Y}_0$ which is identified more than one point in $\tilde{Y}$. Let $W \subset \tilde{Y}$ be a set of which the points and $(1, x_0) \in N \times Y \subset \tilde{Y}$ are not separated by $(A_0|Y)^{\sim}$. Since $A|Y$ is ultraseparating it is easy to show that $(A_0|Y)^{\sim}$ separates the points of $\tilde{Y}_0$. So we may assume that D is a function algebra on $\tilde{Y}_0$ where we denote by $\tilde{y}_0 \in \tilde{Y}_0$ a point which corresponds to W. Since $[D, (c_n)] = \mathrm{cl}((A|Y)^{\sim})$ and an algebra $\{(c_n)\}$ separates the points of W. Suppose that it follows that $D = C(\tilde{Y}_0)$. Then $\mathrm{cl}((A|Y)^{\sim}) = C(\tilde{Y})$. Thus $A|Y = C(Y)$ by the Bernard's lemma.

For any points in $\tilde{Y}_0$ which is not $\tilde{y}_0$ we construct a compact neighborhood of the point. Suppose that it follows that the compact neighborhoods are interpolation sets for D. Then $D = C(\tilde{Y}_0)$ since D is a function algebra on $\tilde{Y}_0$. Let $\tilde{z}$ be in $\tilde{Y}_0$ which is not $\tilde{y}_0$ and fix it. So there exists a $\tilde{g}$ in $(\mathrm{Re}\, A_0|Y)^{\sim}$ such that $\tilde{g}(\tilde{z}) = 1$. We denote $G = \{\tilde{x} \in \tilde{Y}_0 : \tilde{g}(\tilde{x}) \geqq 1/2\}$ and G is a compact neighborhood of $\tilde{z}$ and G will be shown a interpolation set for D by applying Lemma.

Let $F = (\mathrm{Re}\, A_0)^{\sim}|G$, $K = G$, $V' = \{\tilde{u} \in C_R(\tilde{G}) : \exists \tilde{f} = (f_n) \in (\mathrm{Re}\, \tilde{A})^{\sim}, \tilde{f}|\tilde{G} = \tilde{u}, f_n$ is constant on $\{k\} \times X$ for each positive integer n and $k\}$, $u_F = \bar{\tilde{u}}_0|G$ and $\gamma = \varepsilon$. Then the hypotheses of Lemma hold. By definition V' is clearly a subalgebra of V. Suppose that $\tilde{x}$ and $\tilde{y}$ are different points of $\tilde{G}$ and that V' does not separate $\tilde{x}$ and $\tilde{y}$. Let $v = ((v_{nk})_k)_n \in (\tilde{A})^{\sim}$ such that $v(\tilde{x}) = 0$, $v(\tilde{y}) = 1$ and $(\tilde{N})^{\sim}(v) \leqq 1/M$, then $v' = v - ((v_{nk}(x_0))_k)_n$ is in $(\tilde{A}_0)^{\sim}$ and since $((v_{nk}(x_0))_k)_n \in V'$ does not separate $\tilde{x}$ and $\tilde{y}$, $v'(\tilde{y}) - v'(\tilde{x}) = 1$ and $(\tilde{N})^{\sim}(v') \leqq 2/M$. Without loss of generality we may assume $|v'(\tilde{y})| \geqq 1/2$. Let $\tilde{u}' = \mathrm{Re}(\varepsilon M^2/4) v' v e^{i\theta}$ where $v' v e^{i\theta}(\tilde{y})$ is positive. Then $\tilde{u}'$ is in $(\mathrm{Re}\, \tilde{A}_0)^{\sim}$, $\tilde{u}'(\tilde{x}) = 0$, $\tilde{u}'(\tilde{y}) \geqq \varepsilon M^2/8$ and $(\tilde{N})^{\sim}_R(u') \leqq (\tilde{N})^{\sim}((\varepsilon M^2/4) v' v e^{i\theta}) \leqq (\varepsilon M^2/4) \cdot (1/M) \cdot (2/M) \leqq \varepsilon/2 = \gamma/2 < \gamma$. While $|\bar{\tilde{u}}_0(w)| \leqq \varepsilon M^2/9$ on $(\tilde{Y})^{\sim}$ so on $\tilde{G}$. So we can select real ξ such that $|\xi| < 1$ and $(\bar{\tilde{u}}_0 + \xi \tilde{u}')(\tilde{y}) = 0$ and $(\tilde{N})^{\sim}_R(\xi \tilde{u}') < \gamma$. Thus $\tilde{u} = \xi \tilde{u}'|G$ is a desired function. So by Lemma $(\mathrm{Re}\, A_0)^{\sim}|G = C_R(G)$ thus $[\tilde{A}_0|G, 1] = C(G)$ by Bernard's extension theorem of Hoffman and Wermer [2] and then $D|G = C(G)$, that is, G is interpolation set for D.

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