TOKYO J. MATH.
VOL. 6, NO. 2, 1983

A Note on Rings with Finite Local Cohomology

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(Communicated by T. Mitsui)

Introduction

Let A be a Noetherian local ring of $\dim A=n$ and $\mathfrak{m}$ the maximal ideal of A. Let $H_{\mathfrak{m}}^i(\cdot)$ stand for the i^{th} local cohomology functor relative to $\mathfrak{m}$. Then we say that A has *finite local cohomology*, if the A-module $H_{\mathfrak{m}}^i(A)$ is finitely generated for every $i\neq n$.*) In this note we shall characterize rings with finite local cohomology in terms of d-sequences. Recall that a sequence $x_1, x_2, \cdots, x_r$ of elements in A is called a *d-sequence* if the equality

$$(x_1, \cdots, x_{i-1})\colon x_j=(x_1, \cdots, x_{i-1})\colon x_ix_j$$

holds whenever $1\leqq i\leqq j\leqq r$ ([5]). With this definition our result is stated as follows:

THEOREM. *The following conditions are equivalent.*

(1) *A has finite local cohomology.*

(2) *There exists an integer $N>0$ such that every system of parameters of A contained in $\mathfrak{m}^N$ is a d-sequence.*

When this is the case, $\mathfrak{m}^N\cdot H_{\mathfrak{m}}^i(A)=(0)$ for all $i\neq n$.

Our theorem is a natural extension of Huneke's characterization of Buchsbaum rings. Recall that a Noetherian local ring A is called *Buchsbaum* if the difference

$$l_A(A/\mathfrak{q})-e_{\mathfrak{q}}(A)$$

is an invariant of A not depending on the choice of a parameter ideal $\mathfrak{q}$ of A, where $l_A(A/\mathfrak{q})$ and $e_{\mathfrak{q}}(A)$ denote the length of the A-module $A/\mathfrak{q}$ and the multiplicity of A relative to $\mathfrak{q}$, respectively ([10]). Buchsbaum rings have, as is well-known (cf. [6]), finite local cohomology, and

Received October 12, 1982.

*) In [8], rings with finite local cohomology are called *generalized Cohen-Macaulay*.

Huneke [5] showed that a given local ring A is Buchsbaum if and only if every system of parameters for A is a d-sequence.

In a certain special situation a Noetherian local ring A is Buchsbaum once it has finite local cohomology. More explicitly, let $p>0$ be a prime number and assume that A has characteristic p. Let $F: A \to A$ denote the Frobenius endomorphism of A, i.e., $F(a)=a^p$ for each $a \in A$ and let B stand for A when A is regarded, via F, as an algebra over itself. Then we say that A is *F-pure* if for every A-module M, the map

$$F_M: M \longrightarrow B \otimes_A M$$

defined by $F_M(x)=1\otimes x$ for each $x \in M$ is a monomorphism ([4]). With this terminology, as a consequence of Theorem, we have the following

COROLLARY. *Let A be a Noetherian local ring of characteristic p, a prime number and assume that A is F-pure. Then A is Buchsbaum if and only if A has finite local cohomology.*

This result is known by Schenzel [7] in *graded* case. However his proof depends on a characterization of Buchsbaum rings in terms of dualizing complexes and is essentially appealing to the surjectivity criterion obtained by [9] and [11]. Our proof is much more elementary and the result obviously contains his assertion.

We will prove Theorem and its corollary in sections 2 and 3, respectively.

Throughout this paper let A denote a Noetherian local ring of $\dim A=n$ and $\mathfrak{m}$ the maximal ideal of A.

§1. Proof of Theorem.

First of all we note

LEMMA 1.1. *Let $x_1, x_2, \cdots, x_r$ be elements of A and assume that $x_1, x_2, \cdots, x_r$ is a d-sequence. Then $x_2, x_3, \cdots, x_r$ also forms a d-sequence in A/x_1A.*

This follows immediately from the definition of d-sequences.

LEMMA 1.2. *Let a be an element of $\mathfrak{m}$ and assume that the length $l_A([(0): a]_A)$ is finite. Then there exists an exact sequence*

$$0 \longrightarrow H^0_{\mathfrak{m}}(aA) \longrightarrow H^0_{\mathfrak{m}}(A) \longrightarrow H^0_{\mathfrak{m}}(A/aA) \longrightarrow H^1_{\mathfrak{m}}(A) \xrightarrow{a} H^1_{\mathfrak{m}}(A) \longrightarrow H^1_{\mathfrak{m}}(A/aA)$$
$$\longrightarrow \cdots \longrightarrow H^{i-1}_{\mathfrak{m}}(A/aA) \longrightarrow H^i_{\mathfrak{m}}(A) \xrightarrow{a} H^i_{\mathfrak{m}}(A) \longrightarrow H^i_{\mathfrak{m}}(A/aA) \longrightarrow \cdots$$

of local cohomology modules.

PROOF. Let $W=[(0):a]_A$. First of all, split the exact sequence

$$0 \longrightarrow W \longrightarrow A \xrightarrow{a} A \longrightarrow A/aA \longrightarrow 0$$

into short exact sequences

$$(1) \qquad 0 \longrightarrow W \longrightarrow A \longrightarrow aA \longrightarrow 0$$

$$(2) \qquad 0 \longrightarrow aA \longrightarrow A \longrightarrow A/aA \longrightarrow 0\,,$$

and apply the functors $H^i_{\mathfrak{m}}(\cdot)$ to the sequence (1) (resp. (2)). Then we get isomorphisms

$$H^i_{\mathfrak{m}}(A) \cong H^i_{\mathfrak{m}}(aA)$$

for all $i>0$ (resp. a long exact sequence

$$(3) \qquad \begin{aligned} 0 &\longrightarrow H^0_{\mathfrak{m}}(aA) \longrightarrow H^0_{\mathfrak{m}}(A) \longrightarrow H^0_{\mathfrak{m}}(A/aA) \longrightarrow H^1_{\mathfrak{m}}(aA) \\ &\longrightarrow H^1_{\mathfrak{m}}(A) \longrightarrow H^1_{\mathfrak{m}}(A/aA) \longrightarrow \cdots \longrightarrow H^{i-1}_{\mathfrak{m}}(A/aA) \\ &\longrightarrow H^i_{\mathfrak{m}}(aA) \longrightarrow H^i_{\mathfrak{m}}(A) \longrightarrow H^i_{\mathfrak{m}}(A/aA) \longrightarrow \cdots \end{aligned}$$

of local cohomology modules.) Replace $H^i_{\mathfrak{m}}(aA)$ by $H^i_{\mathfrak{m}}(A)$ for $i>0$ in the sequence (3) and we shall have the required exact sequence at once.

PROOF OF THEOREM. (1)$\Rightarrow$(2): By [8, (3.3)] we may choose an integer $N>0$ so that for every system $a_1, a_2, \cdots, a_n$ of parameters of A contained in $\mathfrak{m}^N$ and for every integer $1 \leqq i \leqq n$, the equality

$$(\#) \qquad (a_1, \cdots, a_{i-1}): a_i = (a_1, \cdots, a_{i-1}): \mathfrak{m}^N$$

holds. Now take a system $a_1, a_2, \cdots, a_n$ of parameters of A so that a_k is in $\mathfrak{m}^N$ for all $1 \leqq k \leqq n$ and let $1 \leqq i \leqq j \leqq n$ be integers. Then as both the systems $a_1, \cdots, a_{i-1}, a_j$ and $a_1, \cdots, a_{i-1}, a_i a_j$ are contained in $\mathfrak{m}^N$ and may be extended to systems of parameters of A, we get by (#) that

$$(a_1, \cdots, a_{i-1}): a_j = (a_1, \cdots, a_{i-1}): \mathfrak{m}^N = (a_1, \cdots, a_{i-1}): a_i a_j$$

whence $a_1, a_2, \cdots, a_n$ is, by definition, a d-sequence.

(2)$\Rightarrow$(1): It is enough to show that $\mathfrak{m}^N \cdot H^i_{\mathfrak{m}}(A) = (0)$ for all $i \neq n$. We may assume that $n = \dim A > 0$. Let us fix an element a of $\mathfrak{m}^N$ so that $\dim A/aA = n-1$.

CLAIM. $[(0): a]_A = H^0_{\mathfrak{m}}(A)$.

In fact, choose a system $a_1 = a, a_2, \cdots, a_n$ of parameters in $\mathfrak{m}^N$. Then as $a_1, a_2, \cdots, a_n$ is by the assumption (2) a d-sequence, we see that

$$\text{(a)}\quad [(0): a]_A=[(0): a^2]_A \quad\text{and}\quad \text{(b)}\quad [(0): a_i]_A=[(0): aa_i]_A$$

$(2\leqq i\leqq n)$. Let $x\in H^0_{\mathfrak{m}}(A)$. Then as $a^s x=0$ for some $s>0$, we see by (a) that $ax=0$. Conversely let $x\in[(0): a]_A$. Then since $(aa_i)x=0$, we get by (b) that $a_ix=0$ for all $1\leqq i\leqq n$. Therefore $(a_1, a_2, \cdots, a_n)x=(0)$ whence $x\in H^0_{\mathfrak{m}}(A)$. Thus we conclude that $[(0): a]_A=H^0_{\mathfrak{m}}(A)$.

It follows from this claim that $\mathfrak{m}^N\cdot H^0_{\mathfrak{m}}(A)=(0)$, because the ideal $\mathfrak{m}^N$ can be generated by the elements a such that $\dim A/aA=n-1$. In particular we get our implication for $n=1$. Now let $n\geqq 2$ and assume that our assertion is true for $n-1$. Let $1\leqq i\leqq n-1$ be an integer. Then because every system of parameters for A/aA contained in $\mathfrak{m}^N$ forms a d-sequence in A/aA (cf. Lemma 1.1), we have by the hypothesis of induction on n that $\mathfrak{m}^N\cdot H^{i-1}_{\mathfrak{m}}(A/aA)=(0)$. Hence

$$(\#\#)\qquad \mathfrak{m}^N\cdot[(0): a]_{H^i_{\mathfrak{m}}(A)}=(0)$$

as the A-module $[(0): a]_{H^i_{\mathfrak{m}}(A)}$ is a homomorphic image of $H^{i-1}_{\mathfrak{m}}(A/aA)$ (cf. Lemma 1.2). Notice that the equality (##) holds for any element a of $\mathfrak{m}^N$ with $\dim A/aA=n-1$. Let $x\in H^i_{\mathfrak{m}}(A)$ and choose an integer $s>0$ so that $a^s x=0$. Then applying the equality (##) to a^s instead of a, we immediately get that $\mathfrak{m}^N x=(0)$. Thus $\mathfrak{m}^N\cdot H^i_{\mathfrak{m}}(A)=(0)$ as required.

Let $S(I)$ (resp. $R(I)=\bigoplus_{s\geq 0} I^s$) denote, for a given ideal I of A, the symmetric algebra of the A-module I (resp. the Rees algebra of I). Notice that there is a canonical epimorphism

$$h_I\colon S(I)\longrightarrow R(I)$$

of A-algebras.

COROLLARY 1.3. *Suppose that A has finite local cohomology. Then there is an integer $N>0$ such that the canonical map*

$$h_I\colon S(I)\longrightarrow R(I)$$

is an isomorphism for any ideal I of A which is generated by a subsystem of parameters of A contained in $\mathfrak{m}^N$.

PROOF. Choose an integer $N>0$ for which the condition (2) of Theorem is fulfilled. Let $a_1, a_2, \cdots, a_r$ be a subsystem of parameters of A contained in $\mathfrak{m}^N$, and put $I=(a_1, a_2, \cdots, a_r)$. Then we get, immediately by [3, 2.5], that the canonical map $h_I\colon S(I)\to R(I)$ is an isomorphism since $a_1, a_2, \cdots, a_r$ forms a d-sequence.

Let $\mathrm{N}(A)$ denote, in case A has finite local cohomology, the smallest integer $N>0$ for which the condition (2) of Theorem is fulfilled.

EXAMPLE 1.4. Let $N>0$ be an integer. Then there exists a Noetherian local domain A satisfying the following conditions:

(1) $\dim A=2$.

(2) The A-module $H^1_{\mathfrak{m}}(A)$ is finitely generated.

(3) $\mathrm{N}(A)=N$.

PROOF. Let $S=k[X, Y, Z, W]$ be a polynomial ring over an infinite field k, and choose a graded prime ideal P of S with height 2 so that

$$H^1_M(S/P)\cong S/M^N$$

as S-modules, where $M=(X, Y, Z, W)S$ (cf., e.g., [1]). We put $A=S_M/PS_M$ and $\mathfrak{m}=MA$. Then $\dim A=2$ and

$$(\#)\qquad H^1_{\mathfrak{m}}(A)\cong A/\mathfrak{m}^N$$

clearly. Let a, b be a system of parameters of A contained in $\mathfrak{m}^N$. Then as $a\cdot H^1_{\mathfrak{m}}(A)=(0)$, we get by Lemma 1.2 an isomorphism $H^0_{\mathfrak{m}}(A/aA)\cong H^1_{\mathfrak{m}}(A)$ of local cohomology modules, whence we find that

$$\mathfrak{m}^N\cdot H^0_{\mathfrak{m}}(A/aA)=(0)\ .$$

On the other hand, recalling that $(A/aA)/H^0_{\mathfrak{m}}(A/aA)$ is a one-dimensional Cohen-Macaulay A-module and b is a parameter for $(A/aA)/H^0_{\mathfrak{m}}(A/aA)$, we have that

$$[(0)\colon b]_{A/aA}\subset H^0_{\mathfrak{m}}(A/aA)\ .$$

Therefore $\mathfrak{m}^N\cdot[(0)\colon b]_{A/aA}=(0)$ and consequently

$$aA\colon b=aA\colon \mathfrak{m}^N\ .$$

Thus by virtue of Proof of Theorem (cf. Proof of $[(1)\Rightarrow(2)]$), we see that every system of parameters of A contained in $\mathfrak{m}^N$ forms a d-sequence, whence we find that $\mathrm{N}(A)\leqq N$. The opposite inequality $\mathrm{N}(A)\geqq N$ follows from the last assertion in Theorem, because $(0)\colon H^1_{\mathfrak{m}}(A)=\mathfrak{m}^N$ by (#). Thus $\mathrm{N}(A)=N$, which guarantees that the ring A is a required example.

§2. Proof of Corollary.

We note

PROPOSITION 2.1 ([5, (1.7)]). *The following conditions are equivalent.*

(1) *A is a Buchsbaum ring.*

(2) *Every system of parameters of A is a d-sequence.*

Let $f: R \to S$ be a homomorphism of commutative rings. Then f is said to be *pure* if for every R-module M, the map

$$f_M: M \longrightarrow S \otimes_R M$$

defined by $f_M(x) = 1 \otimes x$ for each $x \in M$ is a monomorphism.

LEMMA 2.2. *Let $f: R \to S$ be a pure homomorphism of commutative rings. Then*

$$IS \cap R = I$$

for every ideal I of R.

This follows from the fact that the canonical map $f_{R/I}: R/I \to S \otimes_R R/I = S/IS$ is a monomorphism.

PROOF OF COROLLARY. We have only to prove the *if* part. Let $a_1, a_2, \cdots, a_n$ be a system of parameters of A and we will show that $a_1, a_2, \cdots, a_n$ is a d-sequence. First of all, let $N>0$ be an integer for which the condition (2) of Theorem is fulfilled and choose an integer $e>0$ so that $p^e \geqq N$. Let $1 \leqq i \leqq j \leqq n$ be integers and $x \in (a_1, \cdots, a_{i-1}): a_i a_j$. Then since $(a_i^{p^e} a_j^{p^e}) \cdot x^{p^e}$ is in $(a_1^{p^e}, \cdots, a_{i-1}^{p^e})$ and since $a_1^{p^e}, a_2^{p^e}, \cdots, a_n^{p^e}$ is a d-sequence (recall that by our choice of N and e, $a_k^{p^e} \in \mathfrak{m}^N$ for all $1 \leqq k \leqq n$), we get that

$$a_j^{p^e} \cdot x^{p^e} \in (a_1^{p^e}, \cdots, a_{i-1}^{p^e}) .$$

Therefore applying Lemma 2.2 to the situation where $R=S=A$, $f=F^e$, and $I=(a_1, \cdots, a_{i-1})$, we find that

$$a_j x \in (a_1, \cdots, a_{i-1})S \cap R = (a_1, \cdots, a_{i-1})$$

whence

$$(a_1, \cdots, a_{i-1}): a_j = (a_1, \cdots, a_{i-1}): a_i a_j .$$

Thus $a_1, a_2, \cdots, a_n$ is a d-sequence and so A is, by Proposition 2.1, a Buchsbaum ring.

COROLLARY 2.3. *Let* $\dim A = 2$ *and assume that* (1) *A is a homomorphic image of a Cohen-Macaulay ring and* (2) *A is an integral domain of positive characteristic. Then A is Buchsbaum if A is F-pure.*

PROOF. As A is an integral domain of $\dim A=2$, the ring $A_{\mathfrak{p}}$ must be a Cohen-Macaulay local ring of $\dim A_{\mathfrak{p}}=2-\dim A/\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of A ($\mathfrak{p}\neq\mathfrak{m}$). Therefore by the assumption (1) and [8, (2.5) and (3.8)], A must have finite local cohomology. Hence the assertion follows from Corollary.

EXAMPLE 2.4. Let $R=k[\![X_1, X_2, \cdots, X_{2n}]\!]$ ($n\geqq 2$) be a formal power series ring over a perfect field k of characteristic $p>0$. We put

$$A=R/(X_1, \cdots, X_n)\cap(X_{n+1}, \cdots, X_{2n}) .$$

Then

(1) A is a Buchsbaum ring of $\dim A=n$.

(2) $H^1_{\mathfrak{m}}(A)=A/\mathfrak{m}$ and $H^i_{\mathfrak{m}}(A)=(0)$ ($i\neq 1, n$).

(3) A is F-pure.

PROOF. (1) and (2) See [6, p. 469, Beispiel].

(3) Let F be the Frobenius endomorphism of R and let S denote R when R is considered to be an algebra, via F, over itself. Then S is a finitely generated free R-module with basis $\{X_1^{c_1}X_2^{c_2}\cdots X_{2n}^{c_{2n}} | 0\leqq c_i<p$ for all $1\leqq i\leqq 2n\}$. Let $G\colon S\to R$ be the R-linear map defined by

$$\begin{aligned} G(X_1^{c_1}X_2^{c_2}\cdots X_{2n}^{c_{2n}})&=1 \quad (c_i=0 \text{ for all } 1\leqq i\leqq 2n)\\ &=0 \quad (\text{otherwise}) \end{aligned}$$

for each $X_1^{c_1}X_2^{c_2}\cdots X_{2n}^{c_{2n}}$ with $0\leqq c_i<p$. Then as $G\cdot F=1_R$, in order to see that A is F-pure it is enough to check that the ideal $I=(X_1, \cdots, X_n)\cap(X_{n+1}, \cdots, X_{2n})$ of R is stable under the action of G, i.e., $G(I)\subset I$. This is routine and we omit it.

EXAMPLE 2.5. Let k be a perfect field of characteristic $p>0$ and K/k a finite extension of fields with degree $m\geqq 2$. Let $R=K[\![X_1, X_2, \cdots, X_n]\!]$ ($n\geqq 2$) be a formal power series ring and put

$$A=\{f\in R | f(0, 0, \cdots, 0)\in k\} .$$

Then

(1) A is a Buchsbaum complete local domain of $\dim A=n$.

(2) $H^1_{\mathfrak{m}}(A)=(A/\mathfrak{m})^{m-1}$ and $H^i_{\mathfrak{m}}(A)=(0)$ ($i\neq 1, n$).

(3) A is F-pure.

PROOF. (1) and (2), see [2, (5.6)].

(3) Let S denote R which is regarded as an algebra over itself by the Frobenius endomorphism F. Then S is a free R-module with basis

$\{X_1^{c_1}X_2^{c_2}\cdots X_n^{c_n} \mid 0\leqq c_i<p$ for all $1\leqq i\leqq n\}$. Let $G: S\to R$ be the R-linear map defined by

$$G(X_1^{c_1}X_2^{c_2}\cdots X_n^{c_n})=1 \quad (c_i=0 \text{ for all } 1\leqq i\leqq n)$$
$$=0 \quad (\text{otherwise})$$

for each $X_1^{c_1}X_2^{c_2}\cdots X_n^{c_n}$ with $0\leqq c_i<p$. Then A is clearly stable under the action of G, whence it must be F-pure (notice that $G\cdot F=1_R$).

We close this paper with the following

REMARK 2.6. (1) In case $\dim A>2$, the conclusion of Corollary 2.3 is not true in general. For instance, take $n=2$ in the example A of Example 2.5 and let $B=A[\![Y_1, Y_2, \cdots, Y_r]\!]$ $(r\geqq 1)$ be a formal power series ring. Then B is an F-pure ring of $\dim B=r+2$ and satisfies both the conditions (1) and (2) of Corollary 2.3. However B is not Buchsbaum (cf. [8, (4.6)]).

(2) The converse of Corollary 2.3 is not true, i.e., all two-dimensional Buchsbaum local domains of positive characteristic are not F-pure. For example, let $k[\![s, t]\!]$ be a formal power series ring over a field k of positive characteristic and put $A=k[\![s^2, s^3, t, st]\!]$ in $k[\![s, t]\!]$. Then A is Buchsbaum but not F-pure.

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