TOKYO J. MATH.
VOL. 6, NO. 2, 1983

On the Strong Purity of the Sublattice-Lattice of a Finite Distributive Lattice

C. C. CHEN and K. M. KOH

National University of Singapore and Sophia University
(Communicated by M. Morimoto)

Introduction

For a lattice L, let Sub(L) be the set of sublattices of L, inclusive of the empty set. The set Sub(L) forms a *poset* under set inclusion. Indeed, it is known (see Grätzer [7], for instance) that the poset $\langle \mathrm{Sub}(L); \subseteq\rangle$ forms an atomistic and algebraic lattice in which A is an *atom* in Sub(L) iff $A=\{a\}$ for some a in L, B is a *dual atom* in Sub(L) iff B is a proper maximal sublattice of L, and for all A, B in Sub(L), the *meet* $A\wedge B$ in Sub(L) is the set-intersection $A\cap B$ in L and the *join* $A\vee B$ in Sub(L) is the sublattice of L generated by $A\cup B$.

Motivated by the study on the lattice of subsemilattices of a semilattice by Sevrin [11], Filippov [6] undertook the first and intensive investigation on the structure of the lattice Sub(L). While easier proofs of some of Filippov's results can be found in Rival [10] and Koh [9], some of his results have been extended recently in Chen, Koh and Teo [5].

Let $L(FD)$ be the class of *finite distributive lattices*. In this paper we proceed to study the structure of Sub(L) of L, $L\in L(FD)$, by employing the notion of the *Frattini* sublattice of L. Following Birkhoff [1], the *Frattini* sublattice $\Phi(L)$ of a lattice L is the intersection of all proper maximal sublattices of L. Thus, the element $\Phi(L)$ in the lattice Sub(L) is the meet of all dual atoms in Sub(L). Denote by $\mathrm{Sub}^*(L)$ the interval $[\Phi(L), L]$ and by $\mathrm{Sub}_*(L)$ the interval $[\varnothing, \Phi(L)]$ in Sub(L). The lattice Sub(L) is said to be *pure* if $\mathrm{Sub}_*(L)$ forms a Boolean sublattice of Sub(L), and *doubly pure* if, in addition, $\mathrm{Sub}_*(L)$ also forms a Boolean sublattice of Sub(L). A *pure* lattice Sub(L) is said to be *strongly pure* if every atom in $\mathrm{Sub}(L)-\mathrm{Sub}_*(L)$ is contained in (less than) a unique

Received June 28, 1982
Revised June 7, 1983

atom in Sub*(L). In [3], Chen, Koh and Lee gave a sufficient condition on L, $L\in L(FD)$, whereby Sub(L) is pure, and they determined completely the structure of L, $L\in L(FD)$, such that Sub(L) is doubly pure. In this paper, we characterize lattices L, $L\in L(FD)$, such that the lattice Sub(L) is strongly pure.

§ 1. Preliminaries.

In this section we introduce some notation and terminology and state some known results which will be needed in the sequel.

Let L be a lattice. An element a in L is said to be *join reducible* if $a=b\vee c$ for some b, c in $L-\{a\}$. *Meet reducible* elements are defined dually. We write

$$L(\vee)=\{a\in L\mid a \text{ is join reducible}\}\,,$$
$$L(\wedge)=\{a\in L\mid a \text{ is meet reducible}\}\,,$$
$$J(L)=L-L(\vee)\,,$$
$$M(L)=L-L(\wedge)\,, \quad \text{and}$$
$$\mathrm{Irr}(L)=J(L)\cap M(L)=L-L(\vee)\cup L(\wedge)\,.$$

Note that $x\vee y\in L(\vee)$ if $x, y\in L(\vee)$ and $x\wedge y\in L(\wedge)$ if $x, y\in L(\wedge)$. Let a, b be in L. We say that b *covers* a or a is *covered by* b, in notation $b\succ a$ or $a\prec b$, if $a<b$ and $a<x<b$ for no x in L. Assume both the least element 0 and the greatest element 1 exist in L. An element a of L is called an *atom* (resp., a *dual atom*) if $a\succ 0$ (resp., $a\prec 1$). For a, b in L with $a<b$, the *closed interval* $\{x\in L\mid a\leqq x\leqq b\}$ is denoted by $[a, b]$ and the *open interval* $\{x\in L\mid a<x<b\}$ is denoted by (a, b). For a subset X of L, the sublattice of L generated by X is denoted by $\langle X\rangle$.

A non-empty sublattice N of L is called a *prime* sublattice of L if $L-N$ is either empty or a sublattice of L. A prime sublattice N of L is called a *minimal prime* sublattice of L if N contains no prime sublattice of L other than itself. The set of all minimal prime sublattices of L is denoted by mp(L).

The following provides a useful characterization of minimal prime sublattices of L, $L\in L(FD)$.

LEMMA 1[4]. *Let $L\in L(FD)$ and $N\subset L$. Then $N\in \mathrm{mp}(L)$ iff one of the following holds*:

(i) $N=\{a\}$ *where* $a\in \mathrm{Irr}(L)$,

(ii) $N=[a, b]$ *where* $a\in L(\wedge)-L(\vee)$, $b\in L(\vee)-L(\wedge)$, *and* $(a, b)\subseteq L(\vee)\cap L(\wedge)$.

For L in $L(FD)$, a relation between $\Phi(L)$ and the family $\mathrm{mp}(L)$ exists and is given below.

LEMMA 2[4]. *Let* $L \in L(FD)$. *Then* $L - \Phi(L) = \cup(N \mid N \in \mathrm{mp}(L))$.

Apparently, $\Phi(L) = \varnothing$ if L is a chain. The converse is not true in general. It is true provided that L is of *finite length*. This is due to the following more general result. Note that $L(\vee)$ and $L(\wedge)$ form *join-subsemilattice* and *meet-subsemilattice* of L respectively.

LEMMA 3[8]. *Let L be a lattice. If c is the greatest element in* $L(\vee)$, *then* $c \in \Phi(L)$. *Dually, if d is the least element in* $L(\wedge)$, *then* $d \in \Phi(L)$.

For a, b in L, we write $a \| b$ if a is *incomparable* with b. The following result provides ways to generate elements in $\Phi(L)$ if $\Phi(L) \neq \varnothing$.

LEMMA 4[2]. *Let* $L \in L(FD)$. *If* $a \in \Phi(L)$, $b \in M(L)$, *and* $a \| b$, *then* $a \vee b \in \Phi(L)$. *Dually, if* $a \in \Phi(L)$, $b \in J(L)$, *and* $a \| b$, *then* $a \wedge b \in \Phi(L)$.

We now introduce a special class of minimal prime sublattices which play a prominent role in our main result. A minimal prime sublattice N of L, $L \in L(FD)$, is called a *solid* sublattice of L if (i) $\Phi(L) \cup N \in \mathrm{Sub}(L)$ and (ii) $\Phi(L) \cup K \notin \mathrm{Sub}(L)$ for any non-empty proper subset K of N. Clearly, for $x \in L$, the singleton $\{x\}$ is solid iff $x \in \mathrm{Irr}(L)$. The set of all solid sublattices of L is denoted by $\mathrm{sd}(L)$. Of course, $\mathrm{sd}(L) \subseteq \mathrm{mp}(L)$.

Recall that the lattice $\mathrm{Sub}(L)$ is *pure* if $\mathrm{Sub}^*(L) \equiv [\Phi(L), L]$ forms a Boolean sublattice of $\mathrm{Sub}(L)$. In [3], Chen, Koh and Lee gave a sufficient condition on L, $L(FD)$, expressed in terms of solid sublattices of L, whereby $\mathrm{Sub}(L)$ is pure. $\dot{\cup}(X \mid X \in C)$ denotes $\cup(X \mid X \in C)$ where C is a collection of pairwise disjoint sets. Its proof is based on the following two results.

LEMMA 5[3]. *Let* $L \in (FD)$. *If* $L - \Phi(L) = \dot{\cup}(N \mid N \in C)$ *where* $C \subseteq \mathrm{sd}(L)$, *then for any* $B \subseteq C$, $\Phi(L) \cup \bigcup(N \mid N \in B) \in \mathrm{Sub}(L)$.

LEMMA 6[3]. *Let* $L \in L(FD)$ *such that* $L - \Phi(L) = \dot{\cup}(N \mid N \in C)$ *where* $C \subseteq \mathrm{sd}(L)$. *If* $A \in \mathrm{Sub}^*(L)$, *then* $A = \Phi(L) \cup \bigcup(N \mid N \in B)$ *for some* $B \subseteq C$.

The following result now follows from Lemmas 5 and 6.

LEMMA 7[3]. *Let* $L \in L(FD)$. *If* $L - \Phi(L) = \dot{\cup}(N \mid N \in C)$ *where* $C \subseteq \mathrm{sd}(L)$, *then the lattice* $\mathrm{Sub}(L)$ *is pure.*

REMARK. The converse of Lemma 7 is not true as was noted in [3]. It is still not true even if L is finite, distributive, and *planar*. The

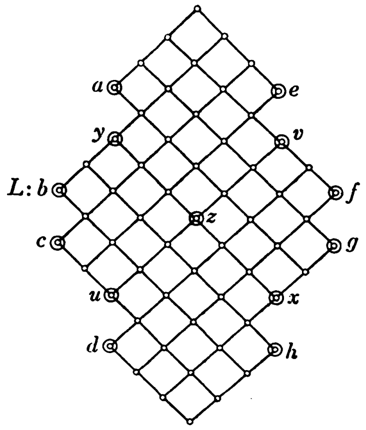

FIGURE 1

lattice of Figure 1, which is the smallest distributive and planar lattice we (with S. C. Lee) can find, provides such a counter example.

For the lattice L of Figure 1, we have $\Phi(L)=L-\mathrm{Irr}(L)\cup[x, y]\cup[u, v]$ and $\mathrm{Sub}^*(L)\cong\underset{\sim}{2}^{10}$ (in general, $\underset{\sim}{2}^n$ denotes the Boolean lattice of n atoms), in which the ten atoms are $\Phi(L)\cup\{a\}$, $\Phi(L)\cup\{b\}, \cdots, \Phi(L)\cup\{h\}$, $\Phi(L)\cup([x, y]-\{z\})$ and $\Phi(L)\cup([u, v]-\{z\})$. Thus $\mathrm{Sub}(L)$ is pure. On the other hand, every solid sublattice of L is a singleton, $z\notin\Phi(L)$, and z is contained in no solid sublattice of L.

§2. Some further results.

To ease the proof of our main result in section 3, we first prove some new results in this section.

The results contained in Lemmas 5-7 require that $L-\Phi(L)$ be expressed as the *disjoint union* of some members of $\mathrm{sd}(L)$. The following result says that every two distinct solid sublattices of L are automatically *disjoint*.

LEMMA 8. *Let* $L\in L(FD)$, $N\in\mathrm{sd}(L)$, *and* $M\in\mathrm{mp}(L)$ *be such that* $\Phi(L)\cup M\in\mathrm{Sub}(L)$. *If* $N\neq M$, *then* $N\cap M=\varnothing$.

PROOF. If $N\cap M\neq\varnothing$, let $x\in N\cap M$. Then $\Phi(L)\cup\{x\}\subseteq\Phi(L)\cup N\in\mathrm{Sub}(L)$ and so $\Phi(L)<\langle\Phi(L)\cup\{x\}\rangle\leqq\Phi(L)\cup N$ in $\mathrm{Sub}(L)$. As $N\in\mathrm{sd}(L)$, $\langle\Phi(L)\cup\{x\}\rangle=\Phi(L)\cup N$. Since $\Phi(L)\cup N=\langle\Phi(L)\cup\{x\}\rangle\subseteq\langle\Phi(L)\cup M\rangle=\Phi(L)\cup M$ by assumption, it follows that $N\subseteq M$. The fact that $M\in\mathrm{mp}(L)$ and N is prime implies that $N=M$. Hence $N\cap M=\varnothing$ if $N\neq M$.

COROLLARY. *Let* $L\in L(FD)$. *If* $N_1, N_2\in\mathrm{sd}(L)$ *and* $N_1\neq N_2$, *then* $N_1\cap N_2=\varnothing$.

LEMMA 9. *Let* $L \in L(FD)$ *and* $a \notin \Phi(L)$.

(i) *If* $a \in \mathrm{Irr}(L)$, *then* $\langle \Phi(L) \cup \{a\} \rangle = \Phi(L) \cup \{a\}$;

(ii) *If* $a \notin \mathrm{Irr}(L)$, *then*

$$\{y \wedge (x \vee a) \mid x,\ y \in \Phi(L)\} = \langle \Phi(L) \cup \{a\} \rangle = \{y \vee (x \wedge a) \mid x,\ y \in \Phi(L)\} .$$

PROOF. (i) The fact that $\Phi(L) \cup \{a\} \in \mathrm{Sub}(L)$ where $a \in \mathrm{Irr}(L)$ follows from Lemma 4.

(ii) Let $K = \{y \wedge (x \vee a) \mid x,\ y \in \Phi(L)\}$. Clearly, $K \subseteq \langle \Phi(L) \cup \{a\} \rangle$. We now prove the reverse inclusion. Since $x = x \wedge (x \vee a) \in K$ for each $x \in \Phi(L)$, we have $\Phi(L) \subseteq K$. We claim that $a \in K$. Since $a \notin \mathrm{Irr}(L)$, L is certainly not a chain. Thus $L(\vee) \neq \varnothing$ and $L(\wedge) \neq \varnothing$. Let $u = \min(x \mid x \in L(\wedge))$ and $v = \max(x \mid x \in L(\vee))$. By Lemma 3, $\{u, v\} \subseteq \Phi(L)$. It is clear that $u < a < v$. Thus $a = v \wedge (u \vee a) \in K$ by definition. Hence $\Phi(L) \cup \{a\} \subseteq K$. We next show that K is a sublattice of L. Thus, let $y_1 \wedge (x_1 \vee a)$ and $y_2 \wedge (x_2 \vee a)$ be in K where $\{x_1, x_2, y_1, y_2\} \subseteq \Phi(L)$. Observe that

$$\begin{aligned}
&[y_1 \wedge (x_1 \vee a)] \vee [y_2 \wedge (x_2 \vee a)] \\
&\quad = ([y_1 \wedge (x_1 \vee a)] \vee y_2) \wedge ([y_1 \wedge (x_1 \vee a)] \vee (x_2 \vee a)) \\
&\quad = (y_1 \vee y_2) \wedge (x_1 \vee y_2 \vee a) \wedge (y_1 \vee x_2 \vee a) \wedge (x_1 \vee x_2 \vee a) \\
&\quad = (y_1 \vee y_2) \wedge ([x_1 \vee y_2) \wedge (y_1 \vee x_2) \wedge (x_1 \vee x_2)] \vee a) \in K ,
\end{aligned}$$

as $\{y_1 \vee y_2,\ (x_1 \vee y_2) \wedge (y_1 \vee x_2) \wedge (x_1 \vee x_2)\} \subseteq \Phi(L)$. Also,

$$[y_1 \wedge (x_1 \vee a)] \wedge [y_2 \wedge (x_2 \vee a)] = (y_1 \wedge y_2) \wedge [(x_1 \wedge x_2) \vee a] \in K .$$

Hence K forms a sublattice of L and we have $\langle \Phi(L) \cup \{a\} \rangle = K$. A dual argument shows that $\langle \Phi(L) \cup \{a\} \rangle = \{y \vee (x \wedge a) \mid x,\ y \in \Phi(L)\}$. The proof of Lemma 9 is thus complete.

LEMMA 10. *Let* $L \in L(FD)$ *and* $A \in \mathrm{Sub}(L)$. *If* $A \succ \Phi(L)$ *in* $\mathrm{Sub}(L)$, *then*

(i) $A - \Phi(L) \in \mathrm{Sub}(L)$ *and*

(ii) $A - \Phi(L) \leqq N$ *in* $\mathrm{Sub}(L)$ *for some* $N \in \mathrm{mp}(L)$.

PROOF. (i) Let $a, b \in A - \Phi(L)$, $a \neq b$. Clearly, $\{a \vee b,\ a \wedge b\} \subseteq A$. We shall show that $\{a \vee b,\ a \wedge b\} \subseteq A - \Phi(L)$. Let $B = \langle \Phi(L) \cup \{a\} \rangle$. Then $\Phi(L) < B \leqq A$ in $\mathrm{Sub}(L)$. The assumption that $A \succ \Phi(L)$ forces $B = A$. Thus $b \in A = \langle \Phi(L) \cup \{a\} \rangle$ and by Lemma 9, $b = y \wedge (x \vee a)$ for some x, y in $\Phi(L)$. Since $b \notin \Phi(L)$, $b \in N$ for some $N \in \mathrm{mp}(L)$ by Lemma 2. Now $y \wedge (x \vee a) = b \in N$ and $y \notin N$ imply $x \vee a \in N$, which in turn implies $a \in N$ as $x \notin N$. Hence $\{a \vee b,\ a \wedge b\} \subseteq N$ and so $\{a \vee b,\ a \wedge b\} \subseteq A - \Phi(L)$.

(ii) Let a be an element in $A - \Phi(L)$. Then $a \in N$ for some $N \in \mathrm{mp}(L)$

by Lemma 2. We shall show that $A-\Phi(L)\subseteqq N$. Let $x\in A-\Phi(L)$. Then $\Phi(L)<\langle\Phi(L)\cup\{x\}\rangle\leqq A$ in Sub(L). The fact that $\Phi(L)\prec A$ implies $A=\langle\Phi(L)\cup\{x\}\rangle$. By Lemma 9, $u\wedge(v\vee x)=a\in N$ for some u, v in $\Phi(L)$. Hence $x\in N$ as $u, v\notin N$. This shows that $A-\Phi(L)\leqq N$ in Sub(L) by (i).

§3. Main result.

For a lattice L, the lattice Sub(L) is said to be *strongly pure* if (1) Sub(L) is pure and (2) for each *atom* $\{a\}$ in Sub(L)$-$Sub$_*$(L) there is exactly one *atom* A of Sub*(L) such that $a\in A$. Note that the uniqueness of such an atom A is automatically derived, because if $a\in A$ and $a\in B$ for two atoms A and B of Sub*(L) then $a\in A\cap B=\Phi(L)$, contrary to $a\notin\Phi(L)$. We are now in a position to give characterizations of L, $L\in L(FD)$, such that Sub(L) is strongly pure.

THEOREM. *Let* $L\in L(FD)$. *The following are equivalent*:
(i) Sub(L) *is strongly pure,*
(ii) $L-\Phi(L)=\dot\cup(N\mid N\in \mathrm{sd}(L))$,
(iii) $\mathrm{mp}(L)=\mathrm{sd}(L)$.

PROOF. (i)$\Rightarrow$(ii). Since Sub(L) is strongly pure, Sub*(L)$\cong\underset{\sim}{2}^n$ for some positive integer n. Let $\{A_i\mid i=1, 2, \cdots, n\}$ be the set of atoms in Sub*(L). By condition (2) of the definition of strong purity, for each a in $L-\Phi(L)$, there exists a unique A_i, $i=1, 2, \cdots, n$ such that $a\in A_i$. Evidently, $L=\cup(A_i\mid i=1, 2, \cdots, n)$. Let $N_i=A_i-\Phi(L)$ for each $i=1, 2, \cdots, n$. By Lemma 10, each N_i is a sublattice of L. Observe that

$$L-\Phi(L)=\cup(A_i\mid i=1, 2, \cdots, n)-\Phi(L)$$
$$=\cup(A_i-\Phi(L)\mid i=1, 2, \cdots, n)=\cup(N_i\mid i=1, 2, \cdots, n)\,.$$

We now prove the following:

Claim. Each sublattice N_i is prime in L.

Assume that N_r is *not* prime for some $r=1, 2, \cdots, n$. Then there exist x, y in $L-N_r$ such that $x\vee y\in N_r$ or $x\wedge y\in N_r$ (say the former).

Case (i). $x\notin\Phi(L)$ and $y\notin\Phi(L)$.

Since $x, y\in L-\Phi(L)=\cup(N_i\mid i=1, 2, \cdots, n)$ and $x, y\in L-N_r$, there exist $j, k=1, 2, \cdots, n$, $j\neq r$ and $k\neq r$ such that $x\in N_j\subseteq A_j$ and $y\in N_k\subseteq A_k$. Clearly, $x\vee y\in\langle A_j\cup A_k\rangle$ in L which means $\{x\vee y\}\leqq A_j\vee A_k$ in Sub(L). As $x\vee y\in N_r$, we also have $\{x\vee y\}\leqq N_r\leqq A_r$ in Sub(L). Since Sub*(L) is a Boolean lattice, it follows that

$$\{x\vee y\}\leqq A_r\wedge(A_j\vee A_k)=(A_r\wedge A_j)\vee(A_r\wedge A_k)=\Phi(L)\ ,$$

which implies $x \vee y \in \Phi(L) \cap N_r$, a contradiction.

Case (ii). $x \in \Phi(L)$ and $y \notin \Phi(L)$.

Since $y \notin \Phi(L)$, $y \in N_k \subseteq A_k$ for some $k \neq r$. As $x \in \Phi(L) \subseteq A_k$, we have $x \vee y \in A_k$ or $\{x \vee y\} \leqq A_k$ in Sub(L). But then $\{x \vee y\} \leqq A_r \wedge A_k = \Phi(L)$, which means that $x \vee y \in \Phi(L) \cap N_r$, a contradiction.

The case that $\{x, y\} \subseteq \Phi(L)$ is clearly impossible. Hence we conclude that each sublattice N_i must be prime in L, as required.

Now by Lemma 10, each $N_i = A_i - \Phi(L)$ is contained in some N, $N \in \mathrm{mp}(L)$. Since N_i is prime and $N \in \mathrm{mp}(L)$, we must have $N_i = N$, which shows that each N_i is itself a minimal prime sublattice.

Finally, we show that each N_i is solid. Apparently, $\Phi(L) \cup N_i = A_i \in \mathrm{Sub}(L)$. If $\Phi(L) \cup K \in \mathrm{Sub}(L)$ for some K with $\varnothing \subset K \subset N_i$, then $\Phi(L) < \Phi(L) \cup K < A_i$ in Sub(L) which contradicts the fact that $A_i \succ \Phi(L)$ in Sub(L). Hence $N_i \in \mathrm{sd}(L)$ for each $i = 1, 2, \cdots, n$. Now by Lemma 2 and the corollary to Lemma 8, we conclude that $L - \Phi(L) = \dot{\cup}(N \mid N \in \mathrm{sd}(L))$.

(ii)$\Rightarrow$(iii). It suffices to show that $\mathrm{mp}(L) \subseteq \mathrm{sd}(L)$. Thus, let $M \in \mathrm{mp}(L)$.

Claim. $\Phi(L) \cup M \in \mathrm{Sub}(L)$.

Let $x \in \Phi(L)$ and $y \in M$. If $x \vee y \notin \Phi(L)$, then by the assumption, $x \vee y \in L - \Phi(L) = \dot{\cup}(N \mid N \in \mathrm{sd}(L))$ and thus $x \vee y \in N$ for some $N \in \mathrm{sd}(L)$. Since $x \notin N$, we must have $y \in N$. Observe that $\Phi(L) < \langle \Phi(L) \cup \{x \vee y\} \rangle \leqq \Phi(L) \cup N$ in Sub(L) and hence $\Phi(L) \cup N = \langle \Phi(L) \cup \{x \vee y\} \rangle$ since $N \in \mathrm{sd}(L)$. As $y \in N \subseteq \Phi(L) \cup N = \langle \Phi(L) \cup \{x \vee y\} \rangle$, we have by Lemma 9, $u \wedge (w \vee (x \vee y)) = y \in M$ for some u, w in $\Phi(L)$. Since $u, w \notin M$ and $M \in \mathrm{mp}(L)$, it follows that $x \vee y \in M$. Dually, we have $x \wedge y \in \Phi(L) \cup M$. Hence $\Phi(L) \cup M \in \mathrm{Sub}(L)$, as claimed.

We now show that $M \in \mathrm{sd}(L)$. By Lemma 2 and the given assumption, $M \subseteq L - \Phi(L) = \dot{\cup}(N \mid N \in \mathrm{sd}(L))$. Thus $M \cap N \neq \varnothing$ for some $N \in \mathrm{sd}(L)$. Since $\Phi(L) \cup M \in \mathrm{Sub}(L)$, it follows that $M = N \in \mathrm{sd}(L)$ by Lemma 8. Hence $\mathrm{mp}(L) \subseteq \mathrm{sd}(L)$, as required.

(iii)$\Rightarrow$(i). By Lemma 2, the corollary to Lemma 8, and the given assumption, we have $L - \Phi(L) = \cup(N \mid N \in \mathrm{mp}(L)) = \dot{\cup}(N \mid N \in \mathrm{sd}(L))$. Thus by Lemma 7, the lattice Sub(L) must be pure. To show that Sub(L) is strongly pure, it remains to show that every atom in $\mathrm{Sub}(L) - \mathrm{Sub}_*(L)$ is contained in exactly one atom of $\mathrm{Sub}^*(L)$. Since $L - \Phi(L) = \dot{\cup}(N \mid N \in \mathrm{sd}(L))$, by Lemma 6, a sublattice A of L is an atom in $\mathrm{Sub}^*(L)$ iff $A = \Phi(L) \cup N$ for some $N \in \mathrm{sd}(L)$. Now, let $\{a\}$ be an atom in $\mathrm{Sub}(L) - \mathrm{Sub}_*(L)$. Then $a \in L - \Phi(L) = \dot{\cup}(N \mid N \in \mathrm{sd}(L))$ and so $a \in N$ for a unique $N \in \mathrm{sd}(L)$. Thus, $\{a\}$ is contained in exactly one atom, namely $\Phi(L) \cup N$, of $\mathrm{Sub}^*(L)$.

The proof of the theorem is thus complete.

ACKNOWLEDGEMENT. This paper was written during April-May, 1982 while the second author was visiting Sophia University, Tokyo, Japan. He wishes to express his sincere thanks to Sophia University for the support under Sophia Lecturing-Research Grants.

References

[1] G. BIRKHOFF, Lattice Theory, 3rd ed., Amer. Math. Soc., Providence, R. I., 1967.
[2] C. C. CHEN and K. M. KOH, An algorithm for determining $\Phi(L)$ in finite distributive lattices, Algebra Universalis, **8** (1978), 151–158.
[3] C. C. CHEN, K. M. KOH and S. C. LEE, On the purity of the lattice of sublattices of a finite distributive lattice, Algebra Universalis, to appear.
[4] C. C. CHEN, K. M. KOH and S. K. TAN, Frattini sublattices of distributive lattices, Algebra Universalis, **3** (1973), 294–303.
[5] C. C. CHEN, K. M. KOH and K. L. TEO, On sublattice-lattice of a lattice, Algebra Universalis, to appear.
[6] N. D. FILIPPOV, Projection of lattices, Mat. Sb. **70** (112) (1966), 36–54; Amer. Math. Soc. Transl., vol. **2**, no. 2 (1970), 37–58.
[7] G. GRÄTZER, General Lattice Theory, Academic Press, 1978.
[8] K. M. KOH, On the Frattini sublattice of a lattice, Algebra Universalis, **1** (1971), 104–116.
[9] K. M. KOH, On sublattices of a lattice, Nanta Math., vol. **6**, no. 1 (1973), 68–79.
[10] I. RIVAL, Projective images of modular (distributive, complemented) lattices are modular (distributive, complemented), Algebra Universalis, **2** (1972), 395.
[11] L. N. SEVRIN, Basic problems in the theory of projections of semi-lattices, Mat. Sb. **66** (108) (1965), 568–597; Amer. Math. Soc. Transl., vol. **96**, no. 2 (1970), 1–35.

Present Address:
DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
SINGAPORE