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On Homeomorphisms with Pseudo-Orbit Tracing Property

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Let $f: X \hookleftarrow$ be a homeomorphism of a compact metric space onto itself and Ω be the non-wandering set of f. The following is a main result of this paper.

THEOREM 1. *If (X, f) has the pseudo-orbit tracing property, then so does (Ω, f).*

This is a problem proposed by A. Morimoto [5].

Let d be a metric function of X. A sequence of points $\{x_i\}_{i\in(a,b)}$ $(-\infty \leqq a < b \leqq \infty)$ is called a *δ-pseudo-orbit* (abbrev. p.o.) of f if $d(f(x_i), x_{i+1}) < \delta$ for $i \in (a, b-1)$. A sequence $\{x_i\}_{i\in(a,b)}$ is called to be *ε-traced* by $x \in X$ if $d(f^i(x), x_i) < \varepsilon$ holds for $i \in (a, b)$. We say that (X, f) has the *pseudo-orbit tracing property* (abbrev. P.O.T.P.) if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-p.o. of f can be ε-traced by some point $x \in X$. Given $x, y \in X$ and $\alpha > 0$, x is *α-related* to y (written $x \overset{\alpha}{\sim} y$) if there are α-pseudo-orbits of f such that $x_0 = x, x_1, \cdots, x_k = y$ and $y_0 = y, y_1, \cdots, y_l = x$. If $x \overset{\alpha}{\sim} y$ for every $\alpha > 0$, then x is *related* to y (written $x \sim y$). The *chain recurrent set* of f, R is $\{x \in X: x \sim x\}$.

Recall that $\Omega = \{x \in X$: for every neighborhood U of x, $f^n(U) \cap U \neq \emptyset$ for some $n \geqq 1\}$. Clearly $\Omega \subset R$ and both sets are f-invariant and closed (a set E will be called f-invariant when $f(E) = E$). Assume that (X, f) has the P.O.T.P., then $\Omega = R$. For, if $x \in R$ then for every $\alpha > 0$ there is $\alpha' > 0$ with property of the P.O.T.P.; i.e., for every α'-p.o. $\{x_i\}$ such that $x_0 = x, x_1, \cdots, x_k = x$, there is $y \in X$ with $d(f^i(y), x_i) < \alpha$ $(0 \leqq i \leqq k)$. Hence $U_\alpha(x) \cap f^{-k}(U_\alpha(x)) \neq \emptyset$ where $U_\alpha(x) = \{y \in X: d(x, y) < \alpha\}$, and so $x \in \Omega$.

We proceed with a sequence of lemmas leading to the proof of Theorem 1. For the following (L. 1), (L. 2) and (L. 3), it is assumed that (X, f) has the P.O.T.P.. Denote by per(f) the set of all periodic points of f.

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(L. 1) *For* $\alpha>0$, *every* $x\in\Omega$ *is* α*-related to* $f(x)$.

Proof. If $x\in \mathrm{per}(f)$ then the statement is true. For the case $x\notin \mathrm{per}(f)$, take γ with $0<\gamma<\alpha$ such that $d(z, x)<\gamma$ implies $\max\{d(f(z), f(x)), d(f^2(z), f^2(x))\}<\alpha$. Since $x\in\Omega$, there is $l\geqq 3$ with $f^{-l}(U_\gamma(x))\cap U_\gamma(x)\neq\emptyset$, and so $f^l(z)\in U_\gamma(x)$ for some $z\in U_\gamma(x)$. Hence $\{f(x), f^2(z), \cdots, f^{l-1}(z), x\}$ is an α-p.o..

(L. 2) *Let* $\alpha>0$ *and* $\{x^i\}_{i\geqq 1}\subset\Omega$ *be a sequence with* $\lim_{i\to\infty} x^i=x^0$. *If* $x^i \overset{\alpha}{\sim} x^{i+1}$ *for* $i\geqq 1$, *then* $x^i\overset{\alpha}{\sim} x^0$ *for all* $i\geqq 1$.

Proof. Let $J>0$ be an integer such that $d(x^j, x^0)<\alpha/2$ for all $j>J$. Since $R=\Omega$, we can construct α-pseudo-orbits from x^J to x^0 and from x^0 to x^J, respectively. Therefore we have $x^J\overset{\alpha}{\sim}x^0$.

Take and fix $\varepsilon>0$. Let $\delta=\delta(\varepsilon)>0$ be the number with property of the P.O.T.P.. Then we can split Ω into equivalence classes A_λ under the δ-relation; i.e., $\Omega=\bigcup_\lambda A_\lambda$. Each A_λ is f-invariant and closed (by (L. 1) and (L. 2)). From the following (L. 3), it follows that $\{A_\lambda\}$ is finite; i.e. $\Omega=\bigcup_1^k A_i$.

(L. 3) *Each* A_λ *is open in* Ω.

Proof. Take $x\in A_\lambda$. For every $y\in A_\lambda$ there is in Ω a δ-p.o. $\{x_0=x, x_1, \cdots, x_p=y\}$. We write $U_\alpha(x)=\{z\in\Omega: d(z, x)<\alpha\}$. Choose γ with $0<\gamma<\delta/3$ such that $f(U_\gamma(x_0))\subset U_\delta(x_1)$. Then for every $x_0'\in U_\gamma(x_0)$, $\{x_0', x_1, \cdots, x_p\}$ is a δ-p.o. in Ω. On the other hand, let $\{y_0=y, y_1, \cdots, y_l=x\}=\mathcal{O}$ be a δ-p.o. in Ω. If $f(y_{l-1})\in\overline{U_\gamma(x_0)}\cap\Omega$ ($\bar{E}$ means the closure of E), then $(\mathcal{O}\backslash\{y_l\})\cup\{x_0'\}=\{y_0, y_1, \cdots, y_{l-1}, x_0'\}$ is a δ-p.o. since $d(f(y_{l-1}), x_0')\leqq 2\gamma<\delta$, and hence $y\overset{\delta}{\sim}x_0'$. If $f(y_{l-1})\notin\overline{U_\gamma(x_0)}\cap\Omega$, then there is $z\in\overline{U_\gamma(x_0)}\cap\Omega$ with $d(f(y_{l-1}), \overline{U_\gamma(x_0)}\cap\Omega)=d(f(y_{l-1}), z)<\delta$. And so $d(x_0', z)\leqq 2\gamma$. Since $z\in\Omega=R$, we have $z\sim z$; i.e., there is a γ-p.o. $\{z_0=z, z_1, \cdots, z_{b+1}=z\}$ in Ω. Since $d(f(z_b), x_0')\leqq d(f(z_b), z)+d(z, x_0')\leqq 3\gamma<\delta$, the sequence $(\mathcal{O}\backslash\{y_l\})\cup\{z_0, \cdots, z_b, x_0'\}$ $(=\{y_0, \cdots, y_{l-1}, z_0, \cdots, x_0'\})$ is a δ-p.o. from y_0 to x_0'. Therefore $x_0'\in A_\lambda$ and so $U_\gamma(x_0)\subset A_\lambda$.

Proof of Theorem 1. Let ε and δ be as above. Since each A_i is open and closed, we have $d(A_i, A_j)=\inf\{d(a, b): a\in A_i, b\in A_j\}>0$ if $i\neq j$. Put $\delta_1=\min\{d(A_i, A_j): i\neq j\}$. For α with $0<\alpha<\min\{\delta, \delta_1\}$, let $\{x_i\}$ be an α-p.o. in Ω. It will be enough to prove that a ε-tracing point of $\{x_i\}$ is chosen in Ω. By using (L. 1) we see that $\{x_i\}$ is contained in some A_j. Take $x_a, x_b\in\{x_i\}$ $(a<b)$. Then we get $x_a\overset{\delta}{\sim}x_b$, so that there are $k_1, k_2>0$ and a (k_1+k_2)-cyclic δ-p.o. $\{z_i\}_{i\in Z}$ such that $x_a=z_{(k_1+k_2)i}$ and $x_b=z_{k_1+(k_1+k_2)i}$ for all $i\in \boldsymbol{Z}$. Put $k=k_1+k_2$. Since (X, f) has the P.O.T.P., there is $y_{a,b}\in X$

such that $d(f^i(y_{a,b}), z_i)<\varepsilon$ for $i\in \boldsymbol{Z}$ and so $d(f^{ki+j}(y_{a,b}), z_j)<\varepsilon$ $(i\in \boldsymbol{Z},\ 0\leqq j<k)$. If $D=\overline{\{f^{ki}(y_{a,b}): i\in \boldsymbol{Z}\}}$ is discrete, then there is $l>0$ such that $f^l(y_{a,b})=y_{a,b}$. Hence $y_{a,b}\in \Omega$. If D is not discrete, then there is a subsequence $\{f^{ki'}(y_{a,b})\}$ with $f^{ki'}(y_{a,b})\to y'_{a,b}\in X$ as $i'\to\infty$. Obviously $d(y'_{a,b}, x_a)\leqq\varepsilon$ and $d(f^j(y'_{a,b}), z_j)\leqq\varepsilon$ for $j\in \boldsymbol{Z}$. We shall see that $y'_{a,b}\in\Omega$. For $\alpha'>0$, we can take $J>0$ such that $d(f^{kj'}(y_{a,b}), y'_{a,b})<\alpha'$ and $d(f^{kj'+1}(y_{a,b}), f(y'_{a,b}))<\alpha'$ for $j'>J$. From this we see that $y'_{a,b}\overset{\alpha'}{\sim} y'_{a,b}$ for $\alpha'>0$. Hence $y'_{a,b}\in R=\Omega$ (since α' is arbitrary). If a subsequence of $\{y'_{a,b}\}$ converges to y as $a\to-\infty$ and $b\to\infty$, then $y\in\Omega$ and $d(f^i(y), x_i)\leqq\varepsilon$ for i: i.e. (Ω, f) has the P.O.T.P..

COROLLARY 1. *If (Ω, f) has the P.O.T.P., then there exists a f-invariant probability Borel measure which is positive on all nonempty open sets of Ω.*

PROOF. For $\varepsilon>0$, let $\delta>0$ be the number with property of the P.O.T.P.. Put $U_\delta(x)=\{y\in\Omega: d(y, x)<\delta\}$ for $x\in\Omega$. Then there are $n_0>0$ and $z\in\Omega$ with $z, f^{n_0}(z)\in U_\delta(x)$. Define a sequence $\{z_i\}_{i\in Z}$ by $z_{k+i}=f^i(z)$ $(k\in \boldsymbol{Z},\ 0\leqq i\leqq n_0-1)$. Obviously $\{z_i\}$ is a δ-p.o.. Since (Ω, f) has the P.O.T.P., $d(f^j(y), z_j)<\varepsilon$ $(j\in \boldsymbol{Z})$ for some $y\in\Omega$. Put $\mu_m=\sum_{j=0}^{m-1}\delta(f^jy)/m$ for $m\geqq 1$ ($\delta(x)$ denotes the measure with support $\{x\}$). Then $\mu_m(U_\varepsilon(x))\geqq 1/n_0$. Since the set of all probability Borel measures is a compact metric space with respect to the weak topology (cf. see p. 10, [3]), a subsequence of $\{\mu_m\}$ converges to some f-invariant measure μ, and clearly $\mu(U_\varepsilon(x))\geqq 1/n_0$.

Assume that (Ω, f) is expansive. If $\varepsilon>0$ and $x\in\Omega$, let $W^s_\varepsilon(x)$, $W^u_\varepsilon(x)$ be the local stable and unstable sets defined by

$$W^s_\varepsilon(x)=\{y\in\Omega: d(f^n(x), f^n(y))\leqq\varepsilon,\ n\geqq 0\}\,,$$
$$W^u_\varepsilon(x)=\{y\in\Omega: d(f^{-n}(x), f^{-n}(y))\leqq\varepsilon,\ n\geqq 0\}\,.$$

Let $e>0$ be an expansive constant for f and fix $0<\varepsilon<e/2$. Now define the stable and unstable sets $W^s(x)$, $W^u(x)$ as

$$W^s(x)=\bigcup_{n\geqq 0} f^n W^s_\varepsilon(f^n x)\,,\qquad W^u(x)=\bigcup_{n\geqq 0} f^{-n} W^u_\varepsilon(f^{-n}x)\,.$$

It is known (Lemma 1, [4]) that for all $r>0$ there is $N>0$ such that $f^n W^s_\varepsilon(x)\subset W^s_r(f^n x)$ and $f^{-n}W^u_\varepsilon(x)\subset W^u_r(f^{-n}x)$ for all $x\in X$ and $n\geqq N$. From this we have

$$W^s(x)=\{y\in\Omega: \lim_{n\to\infty} d(f^n(x), f^n(y))=0\}\,,$$
$$W^u(x)=\{y\in\Omega: \lim_{n\to\infty} d(f^{-n}(x), f^{-n}(y))=0\}\,.$$

If (Ω, f) has the P.O.T.P., then Ω splits into the equivalence classes B_λ under the relation $\sim$. Each B_λ is closed and f-invariant (by (L. 1) and (L. 2)), and $\Omega=\bigcup_\lambda B_\lambda$. We now have:

THEOREM 2 (Smale's Spectral Decomposition). *If (Ω, f) is expansive and has the P.O.T.P., then Ω contains a finite sequence B_i $(1\leqq i\leqq k)$ of f-invariant closed subsets such that (B_i, f) is topologically transitive and $\Omega=\bigcup_1^k B_i$.*

Theorem 2 is obtained by the following (L. 4) and (L. 5). In the rest we assume that (Ω, f) is expansive and has the P.O.T.P.. It is clear that $\overline{\text{per}(f)}=\Omega$.

(L. 4) *Each B_λ is open in Ω.*

PROOF. As before define $U_\delta(B_\lambda)=\{y\in\Omega: d(y, B_\lambda)<\delta\}$ where δ is the number with property of the P.O.T.P.. Then for $p\in U_\delta(B_\lambda)\cap \text{per}(f)$ there is $y\in B_\lambda$ such that $d(y, p)<\delta$. Since (Ω, f) has the P.O.T.P., we get $W^u(p)\cap W^s(y)\neq\emptyset$ and $W^s(p)\cap W^u(y)\neq\emptyset$. Therefore there is $y_0\in B_\lambda$ with $y_0\sim p$; i.e. $p\in B_\lambda$. And so $B_\lambda\supset\overline{U_\delta(B_\lambda)\cap\text{per}(f)}\supset U_\delta(B_\lambda)\cap\overline{\text{per}(f)}=U_\delta(B_\lambda)$.

By compactness and (L. 4), Ω is expressed as a union of a finite set of $\{B_\lambda\}$; i.e. $\Omega=\bigcup_1^k B_i$.

(L. 5) *Each (B_i, f) is topologically transitive.*

PROOF. Since B_i is open in Ω by (L. 4), (B_i, f) has the P.O.T.P.. Let U and V be nonempty open sets in B_i. Since $x\sim y$ for $x\in U$ and $y\in V$, we can always find in B_i a tracing point for a p.o. from x to y; i.e. $U\cap f^l(V)\neq\emptyset$ for some $l>0$.

THEOREM 3 (Bowen's Decomposition). *Under the assumptions and the notations of Theorem 2, for $1\leqq i\leqq k$ there exists a subset X_p of B_i and $a>0$ such that $f^a(X_p)=X_p$, $X_p\cap f^j(X_p)=\emptyset$ $(0<j<a)$, (X_p, f^a) is topologically mixing and $B_i=\bigcup_0^{a-1} f^j(X_p)$.*

This theorem is conducted by the following known lemmas.

(L. 6) *Define $X_p=\overline{W^u(p)\cap B_i}$ for $p\in B_i\cap\text{per}(f)$, then X_p is open in B_i.*

PROOF. Let m be a period of p $(f^m(p)=p)$ and put $U_\delta(X_p)=\{y\in B_i: d(y, X_p)<\delta\}$ $(\delta>0$ is the number chosen as above). For $q\in U_\delta(X_p)\cap\text{per}(f)$ $(f^n(q)=q$ for some $n>0)$, we can find $x\in W^u(p)\cap B_i$ with $d(q, x)<\delta$. Since (B_i, f) has the P.O.T.P., there is x' such that $x'\in W^s(q)\cap W^u(x)\cap B_i$.

Since $W^u(x)=W^u(p)$, $f^{kmn}(x')\in f^{kmn}W^u(p)=W^u(p)$ for every $k>0$. On the other hand, $d(f^{kmn}(x'), f^{kmn}(q))=d(f^{kmn}(x'), q)\to 0$ $(k\to\infty)$. Therefore $q\in\overline{W^u(p)\cap B_i}=X_p$; i.e. $U_\delta(X_p)=X_p$.

Since $f(X_p)=X_{f(p)}$, obviously $f^m(X_p)=X_p$ so that there is $a>0$ with $a\leqq m$ such that $f^a(X_p)=X_p$. Since (B_i, f) is topologically transitive, we have $B_i=X_p\cup f(X_p)\cup\cdots\cup f^{a-1}(X_p)$.

(L. 7) $X_p=X_q$ *for* $q\in X_p\cap \mathrm{per}(f)$.

Proof. As we saw in the proof of (L. 6), $U_\delta(X_p)=X_p$ and so $W^u_\delta(q)\subset X_p$. As before let m and n be periods of p and q, respectively. Then we claim that $W^u(q)=\bigcup_{j\geqq 0} f^{nmj}W^u_\delta(q)$. So $X_q\subset X_p$. To get the conclusion, assume $p\notin X_q$. Then we get $0<d(K, X_q)$ where $K=X_p\backslash X_q$. Since $q\in X_p=\overline{W^u(p)\cap B_i}$, there is $z\in W^u(p)\cap B_i$ with $d(z, q)<d(K, X_q)$. Clearly $z\in X_q$. On the other hand, $d(f^{-nmj}(z), p)=d(f^{-nmj}(z), f^{-nmj}(p))\to 0$ $(j\to\infty)$, so that $f^{-nmj}(z)\notin X_q$ for sufficiently large j. We get that $z\notin f^{nmj}(X_q)=X_q$, which is a contradiction.

(L. 8) (X_p, f^a) *is topologically mixing.*

Proof. Let U, V be nonempty open subsets of X_p. Since $X_p=X_{p'}$ for all $p'\in X_p\cap\mathrm{per}(f)$, for $p'\in V\cap\mathrm{per}(f)$ we have $U\cap W^u(f^{aj}p')\neq\emptyset$ for $j\in \boldsymbol{Z}$. Now let $l>0$ be a period of p'. Then for $0\leqq j\leqq l-1$ there is $z_j\in U\cap W^u(f^{aj}p')$, so that $f^{-alt}(z_j)\to f^{aj}(p')$ as $t\to\infty$. Obviously $f^{aj}(p')\in f^{aj}(V)$. Fix j with $0\leqq j\leqq l-1$. Then there is $N_j>0$ such that for every $t\geqq N_j$, $f^{-alt}(z_j)\in f^{aj}(V)$. Put $N=\max\{N_j\colon 0\leqq j\leqq l-1\}$. For every $t\geqq N$, we get $t=ls+j$. If $s\geqq N$ then $f^{-at}(z_j)=f^{-als-aj}(z_j)\in V$. Since $z_j\in U$, $f^{at}(V)\cap U\neq\emptyset$ for $t\geqq lN$.

As an easy conclusion we have

Corollary 2. *If* (Ω, f) *is topologically transitive and if there is* $p\in\mathrm{per}(f)$ *with* $f(p)=p$, *then* (Ω, f) *is topologically mixing.*

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