TOKYO J. MATH.
VOL. 6, No. 2, 1983

A Construction of an Invariant Stable Foliation by the Shadowing Lemma

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(Communicated by M. Mori)

Introduction

There are many studies on the dynamical properties of one-dimensional maps. For instance, asymptotic behavior and the existence of invariant measures were studied in [1], [2] and [3]. In contrast, in the case of two-dimensional maps the results obtained are not so many. So, it would be usuful to investigate whether there exist two-dimensional maps which can be reduced to one-dimensional maps.

In this paper, to clarify how the behavior of not necessarily differentiable two-dimensional maps is related to that of one-dimensional maps, we investigate the existence of an invariant stable foliation of two-dimensional maps by using the shadowing lemma.

Let $I=[0, 1]$ and f be a map of piecewise C-^{2}class from I into itself; i.e., there is a finite sequence $0=c_0<c_1<\cdots<c_N=1$ of points in I such that if $I_i=[c_i, c_{i+1})$ then the restriction of f to I_i is C^2 and there exist $\lim_{x\to c_{i+1}-0}(d^n/dx^n)f(x)$ $(n=0, 1, 2)$. A sequence of points $\{x_n\}_{n\geq 0}$ is called an *ε-pseudo-orbit* of f iff $|f(x_n)-x_{n+1}|<\varepsilon$ for $n\geqq 0$. Denote sometimes by I_x the interval I_i that contains a point x. A sequence $\{x_n\}_{n\geq 0}$ is called *β-traced* by $\xi\in I$ iff $|f^n(\xi)-x_n|<\beta$ and $f^n(\xi)\in I_{x_n}$ for $n\geqq 0$. We say that (I, f) has the *pseudo-orbit-tracing property* (abbrev. P.O.T.P) iff for every $\beta>0$ there exists $\varepsilon=\varepsilon(\beta)>0$ such that every $\varepsilon(\beta)$-pseudo-orbit is β-traced by some point $\xi\in I$. We claim that our definition of P. O. T. P is not the same as in R. Bowen (p. 74, (4)).

Throughout this paper, we denote by $f'(x)$ the right or left differential coefficient ($f'_+(x)$ or $f'_-(x)$, respectively) at a discontinuity point x if there is no confusion.

For convenience we write

Received June 30, 1982
Revised May 10, 1983

$$\inf_{x\in I}|f'(x)|=\inf_{x\in I}\{|f'_+(x)|,\ |f'_-(x)|\}\quad\text{and}$$

$$|(f^{n\prime})(x)|=\prod_{k=0}^{n-1}\min\{|f'_+(f^k(x))|,\ |f'_-(f^k(x))|\}\ .$$

Let $\mu>0$ and let $H: I\times \boldsymbol{R}\to I\times\boldsymbol{R}$ be a map defined by

$$H(x,\ y)=(f(x),\ \mu y)\ .$$

Then for $\varepsilon>0$ a perturbation of H can be defined by

$$H_\varepsilon(x,\ y)=(f(x)+\varepsilon_1(x,\ y),\ \mu y+\varepsilon_2(x,\ y))$$

where each ε_i: $I\times\boldsymbol{R}\to\boldsymbol{R}$ is C^1 and $\|\varepsilon_i\|\leqq\varepsilon$ holds. Here the notation $\|\ \ \|$ denotes the C^1-norm.

Our purpose is to show the following

THEOREM. *Under the above notations, assume that H satisfies the conditions.*

(1) *(I, f) has P.O.T.P. and*

$$\sup_{x\in I}\left\{\limsup_{n}\frac{1}{n}\log\frac{1}{|(f^n)'(x)|}\right\}<0\ ,$$

(2) $\mu<\inf_{x\in I}|f'(x)|$.

Then there exists $\beta>0$ such that for every $L>0$ there are $\varepsilon(\beta, L)\equiv\varepsilon>0$ and a map $\varphi_{\beta,\varepsilon}$: $I\times\boldsymbol{R}\to I$ so that if $\|\varepsilon_i\|\leqq\varepsilon$ $(i=1, 2)$ and $\xi\in\varphi_{\beta,\varepsilon}(I\times\boldsymbol{R})$, then for (x, y), $(x', y')\in\varphi_{\beta,\varepsilon}^{-1}(\xi)$,

(A) $|x-x'|\leqq L|y-y'|$,

(B) *if $\mu<1$ then $|H_\varepsilon^n(x, y)-H_\varepsilon^n(x', y')|\to 0$ exponentially as $n\to\infty$.*

EXAMPLE. Let f be a piecewise C^2-map such that $\inf_{x\in I}|f'(x)|>1$ and $f(I_i)=I$. In this case f satisfies the assumption of the theorem.

§1. Proof of Theorem.

For the proof of our result, we need the following four Lemmas.

LEMMA 1. *As before let I_n be the finite sequence of subintervals of I. Then there exists $\beta>0$ such that if for $x, y\in I$ and $n\geqq 0$, $f^n(x)$ and $f^n(y)$ are contained in the same interval I_{k_n} and if $|f^n(x)-f^n(y)|<2\beta$ holds, then $x=y$.*

PROOF. Put $G=\sup_{x\in I}|f''(x)|$. Then by (1) and (2) we can find $\beta>0$ such that

(3) $\mu>3G\cdot\beta$ and

$$(4) \quad \sup_{x\in I}\left\{\limsup_{n}\frac{1}{n}\log\frac{1}{|(f^n)'(x)|}\right\}<\log\left\{1-\frac{3G\cdot\beta}{\inf_{x\in I}|f'(x)|}\right\}.$$

Since $f^n(x)$, $f^n(y)\in I_{k_n}(n\geqq 0)$ by assumption, from the mean value theorem it follows that

$$f^n(y)-f^n(x)=\int_0^1 f'(f^{n-1}(x)+t(f^{n-1}(y)-f^{n-1}(x)))(f^{n-1}(y)-f^{n-1}(x))dt\,,$$

so that

$$|f^n(y)-f^n(x)|=|f^{n-1}(y)-f^{n-1}(x)|\cdot\left|f'(f^{n-1}(x))+\{f^{n-1}(y)-f^{n-1}(x)\}\int_0^1 f''(\cdot)tdt\right|$$

$$=|f^{n-1}(y)-f^{n-1}(x)|\cdot|f'(f^{n-1}(x))|\cdot\left|1+\frac{\{f^{n-1}(y)-f^{n-1}(x)\}\int_0^1 f''(\cdot)tdt}{f'(f^{n-1}(x))}\right|.$$

By (3) and assumption of the Lemma we have

$$|f^n(y)-f^n(x)|\geqq|f^{n-1}(y)-f^{n-1}(x)|\cdot|f'(f^{n-1}(x))|\cdot\left\{1-\frac{2G\cdot\beta}{\inf_{x\in I}|f'(x)|}\right\},$$

and by induction on n

$$|f^n(y)-f^n(x)|\geqq|y-x|\prod_{k=0}^{n-1}|f'(f^k(x))|\left\{1-\frac{2G\cdot\beta}{\inf_{x\in I}|f'(x)|}\right\}^n$$

$$\geqq|y-x|\,|(f^n)'(x)|A^n\,,$$

where

$$A=\left\{1-\frac{3G\cdot\beta}{\inf_{x\in I}|f'(x)|}\right\}.$$

From this inequality together with (4), it follows that $|(f^n)'(x)|\times A^n\to\infty$ (as $n\to\infty$). Therefore we obtain $x=y$.

LEMMA 2. *Let $\varepsilon>0$ and put $(x_{\varepsilon,n}, y_{\varepsilon,n})=H_\varepsilon^n(x, y)$ for $(x, y)\in I\times \boldsymbol{R}$ and $n\geqq 0$. Then the sequence $\{x_{\varepsilon,n}\}_{n\geqq 0}$ is an ε-pseudo-orbit of f.*

PROOF. Since $x_{\varepsilon,n}=f(x_{\varepsilon,n-1})+\varepsilon_1(x_{\varepsilon,n-1}, y_{\varepsilon,n-1})$ for $n\geqq 1$, we have $x_{\varepsilon,n}-f(x_{\varepsilon,n-1})=\varepsilon_1(x_{\varepsilon,n-1}, y_{\varepsilon,n-1})$, and so $|x_{\varepsilon,n}-f(x_{\varepsilon,n-1})|\leqq\varepsilon$ (since $\|\varepsilon_1\|\leqq\varepsilon$).

LEMMA 3. *Let β be as in Lemma 1. Then there exists $\varepsilon(\beta)\equiv\varepsilon>0$ such that every ε-pseudo-orbit of f is β-traced by a unique point $\xi\in I$.*

Proof. If an ε-pseudo-orbit $\{x_{\varepsilon,n}\}_{n\geq 0}$ is β-traced by two points ξ and ξ', then $|f^n(\xi)-f^n(\xi')|<2\cdot\beta$ and $f^n(\xi), f^n(\xi')\in I_{x_n}$ for $n\geqq 0$. Therefore the conclusion is obtained by Lemma 1.

By Lemmas 2 and 3 there is a unique point $\xi\in I$ for $(x, y)\in I\times R$. Hence a map $\varphi_{\beta,\varepsilon}: I\times R\to I$ is defined by putting

$$\xi=\varphi_{\beta,\varepsilon}(x, y) \quad ((x, y)\in I\times R)\ .$$

Note that $\varphi_{\beta,\varepsilon}$ is not necessarily continuous. It is easy to see that there is $(x', y')\in I\times R$ such that $f(\xi)=\varphi_{\beta,\varepsilon}(x', y')$. Hence we have

$$\varphi_{\beta,\varepsilon}^{-1}(\xi)=\{(x, y)\in I\times R: |x_{\varepsilon,n}-f^n(\xi)|<\beta \quad\text{and}\quad x_{\varepsilon,n}\in I_{f^n(\xi)} \quad\text{for}\quad n\geqq 0\}\ ,$$
$$\varphi_{\beta,\varepsilon}^{-1}(f(\xi))=\{(x', y')\in I\times R: |x'_{\varepsilon,n}-f^{n+1}(\xi)|<\beta \quad\text{and}\quad x'_{\varepsilon,n}\in I_{f^{n+1}}(\xi) \quad\text{for}\quad n\geqq 0\}\ .$$

Lemma 4. *Let β be as in Lemma* 1 *and ε be as in Lemma* 3. *Then*

$$H_\varepsilon(\varphi_{\beta,\varepsilon}^{-1}(\xi))\subset\varphi_{\beta,\varepsilon}^{-1}(f(\xi)) \quad \textit{for} \quad {}^\forall\xi\in\varphi_{\beta,\varepsilon}(I\times R)\ .$$

Proof. Put $x_{\varepsilon,n}^{(1)}=x_{\varepsilon,n+1}$ for $n\geqq 0$. Then $\{x_{\varepsilon,n}^{(1)}\}_{n\geq 0}$ is β-traced by $f(\xi)$. Therefore $(x_{\varepsilon,0}^{(1)}, y_{\varepsilon,0}^{(1)})=H_\varepsilon(x, y)\in\varphi_{\beta,\varepsilon}^{-1}(f(\xi))$.

Proof of Theorem. By using (1) and (2) we can find $\beta>0$ such that (3) and (4) hold; i.e., $\mu>3G\cdot\beta$ and

$$\sup_{x\in I}\left\{\limsup_{n}\frac{1}{n}\log\frac{1}{|(f^n)'(x)|}\right\}<\log A\ .$$

Remark that for every $L>0$ there is $\varepsilon(\beta, L)=\varepsilon>0$ so that

$$(5)\quad \inf_{x\in I}|f'(x)|>\mu+\varepsilon\frac{(1+L)^2}{L} \quad\text{and}$$

$$(6)\quad \sup_{x\in I}\left\{\limsup_{n}\frac{1}{n}\log\frac{1}{|(f^n)'(x)|}\right\}<\log B\ ,$$

where

$$B=\left\{1-\frac{3G\beta+\varepsilon(1+L^{-1})}{\inf_{x\in I}|f'(x)|}\right\}\ .$$

For simplicity we write

$$x_{\varepsilon,k}=x_k \quad\text{and}\quad y_{\varepsilon,k}=y_k \quad (k\geqq 0)\ .$$

To get the statement (A), assume there exists $k\geqq 0$ such that for (x, y), $(x', y')\in\varphi_{\beta,\varepsilon}^{-1}(\xi)$

$$|x_k - x'_k| > L|y_k - y'_k| \ .$$

Since $x_{n+1} = f(x_n) + \varepsilon_1(x_n, y_n)$ and $y_{n+1} = \mu y_n + \varepsilon_2(x_n, y_n)$, we have

$$\begin{aligned} |x_{k+1} - x'_{k+1}| &\geqq |f(x_k) - f(x'_k)| - |\varepsilon_1(x_k, y_k) - \varepsilon_1(x'_k, y'_k)| \\ &\geqq |f(x_k) - f(x'_k)| - (\|\varepsilon_1\| \, |x_k - x'_k| + \|\varepsilon_1\| \, |y_k - y'_k|) \\ &> |f(x_k) - f(x'_k)| - \varepsilon(1 + L^{-1})|x_k - x'_k| \\ &> |x_k - x'_k| \left\{ \inf_{x \in I} |f'(x)| - \varepsilon(1 + L^{-1}) \right\} \end{aligned}$$

and moreover

$$\begin{aligned} L \cdot |y_{k+1} - y'_{k+1}| &= L|\mu(y_k - y'_k) + \varepsilon_2(x_k, y_k) - \varepsilon_2(x'_k, y'_k)| \\ &< \mu|x_k - x'_k| + L\{\varepsilon|x_k - x'_k| + \varepsilon|y_k - y'_k|\} \\ &< |x_k - x'_k|\{\mu + \varepsilon(1 + L)\} \ . \end{aligned}$$

Hence the above inequalities and (5) imply $|x_{k+1} - x'_{k+1}| > L|y_{k+1} - y'_{k+1}|$. And by induction we have $|x_n - x'_n| > L \cdot |y_n - y'_n|$ for $n \geqq k$. On the other hand, from the above inequality we have for every $n > k$

$$\begin{aligned} |x_n - x'_n| &= |f(x_{n-1}) - f(x'_{n-1}) + \varepsilon_1(x_{n-1}, y_{n-1}) - \varepsilon_1(x'_{n-1}, y'_{n-1})| \\ &> |f(x_{n-1}) - f(x'_{n-1})| - \varepsilon(1 + L^{-1}) \cdot |x_{n-1} - x'_{n-1}| \\ &= |x_{n-1} - x'_{n-1}| \cdot \{|f'(x'_{n-1} + t(x_{n-1} - x'_{n-1}))| - \varepsilon(1 + L^{-1})\} \\ &\qquad\qquad (\text{by some} \quad t \in (0, 1)) \\ &= |x_{n-1} - x'_{n-1}| \cdot \{|f'(f^{n-1}(\xi) + (x'_{n-1} - f^{n-1}(\xi)) + t(x_{n-1} - x'_{n-1}))| - \varepsilon(1 + L^{-1})\} \ . \end{aligned}$$

Therefore we have

$$\begin{aligned} &|x_n - x'_n| \\ &\quad > |x_{n-1} - x'_{n-1}|\{|f'(f^{n-1}(\xi))| - G(|x'_{n-1} - f^{n-1}(\xi)| + |x_{n-1} - x'_{n-1}|) - \varepsilon(1 + L^{-1})\} \\ &\quad > |x_{n-1} - x'_{n-1}| \cdot |f'(f^{n-1}(\xi))| \cdot \left\{1 - \frac{3G\beta + \varepsilon(1 + L^{-1})}{\inf_{x \in I} |f(x)|}\right\} \ . \end{aligned}$$

This inequality follows from the facts that $|x'_n - f^n(\xi)| < \beta$ and $|x_n - x'_n| < 2 \cdot \beta$ for $n \geqq 0$. By induction we have

$$|x_n - x'_n| > |x_k - x'_k| \cdot \prod_{j=k}^{n-1} |f'(f^j(\xi))| \cdot B^{n-k} \quad (n > k) \ .$$

Since $(x, y), (x', y') \in \varphi^{-1}_{\beta,\varepsilon}(\xi)$, we have $|x_n - x'_n| < 2\beta$ for $n \geqq 0$. Hence the last inequality contradicts to (6). Therefore for (x, y), $(x', y') \in \varphi^{-1}_{\beta,\varepsilon}(\xi)$ we have

(7) $|x_n - x'_n| \leqq L \cdot |y_n - y'_n| \quad (n \geqq 0)$.

Since $x = x_0$ and $y = y_0$, (7) implies the statement (A). Moreover by (7) we have

$$\begin{aligned} |y_n - y'_n| &= |\mu(y_{n-1} - y'_{n-1}) + \varepsilon_2(x_{n-1}, y_{n-1}) - \varepsilon_2(x'_{n-1}, y'_{n-1})| \\ &\leqq \mu \cdot |y_{n-1} - y'_{n-1}| + \|\varepsilon_2\| \cdot |x_{n-1} - x'_{n-1}| + \|\varepsilon_2\| \cdot |y_{n-1} - y'_{n-1}| \\ &\leqq |y_{n-1} - y'_{n-1}| \cdot \{\mu + \varepsilon(1+L)\} \\ &\leqq |y - y'| \{\mu + \varepsilon(1+L)\}^n \quad (n \geqq 0) . \end{aligned}$$

For the case when $\mu < 1$, taking $\varepsilon > 0$ with $1 > \mu + \varepsilon(1+L)$, we obtain the statement (B). The proof of Theorem is completed.

ACKNOWLEDGEMENT. The author wishes to thank Professor Y. Takahashi for advising me to make use of the shadowing lemma, and the referee for many suggestions.

References

[1] J. GUCKENHEIMER, Sensitive dependence on initial conditions for one-dimensional maps, Commun. Math. Phys., **70** (1979), 113–160.

[2] S. ITO, S. TANAKA and H. NAKADA, On unimodal linear transformations and Chaos, (I) and (II), TOKYO J. Math., **2** (1979), 221–259.

[3] M. JAKOBSON, Construction of Invariant Measures Absolutely Continuous with Respect to dx for Some Maps of the Interval, Lecture Notes in Math., **819** (1980), Springer, 246–257.

[4] R. BOWEN, Equilibrium States and the Ergodic Theory of Anosov Diffeomorphism, Lecture Note in Math., **470** (1975), Springer.

[5] YA. B. PESIN, Lyapunov characteristic exponents and smooth ergodic theory, Russian Math. Surveys, **32** (1977), 55–114.

[6] R. SHAW, Strange attractors, Chaotic behavior and information flow, Z. Naturforsch., **36** (a) (1981), 80–112.

[7] M. YURI, The existence of an invariant stable foliation and the problem of reducing to a one-dimensional map, to appear in Proc. Japan. Acad..

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