Helical Geodesic Immersions of Compact Rank One Symmetric Spaces into Spheres

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Introduction

Let $\varphi: M \to \tilde{M}$ be an isometric immersion of a connected complete Riemannian manifold M into a Riemannian manifold $\tilde{M}$. If, for each geodesic γ of M, the curve $\varphi \circ \gamma$ in $\tilde{M}$ has constant curvatures of osculating order d which are independent of γ, then φ is called a *helical geodesic immersion of order* d (cf. Sakamoto [12]). In this paper we shall study a helical geodesic immersion into a unit sphere.

We give an example of a helical geodesic immersion according to Besse [2]. Suppose that the Riemannian manifold (M, g) is compact. Then it has a unique kernel for the heat equation $K: M \times M \times R_+^* \to R$. If there exists a map $\Xi: R_+ \times R_+^* \to R$ such that $K(p, q, t) = \Xi(\delta(p, q), t)$ for every $p, q \in M$ and $t \in R_+^*$, then M is called a *strongly harmonic manifold*, where δ denotes the distance function. Compact symmetric spaces of rank one are its examples. Following Besse we construct a helical geodesic and minimal immersion of a strongly harmonic manifold (M, g) into a unit sphere. Let V_k be the eigenspace of the Laplacian on (M, g) corresponding to the k-th eigenvalue $\lambda_k(>0)$ and $\dim V_k = m(k) + 1$. We define an inner product $\langle , \rangle$ on V_k by $\langle f, h \rangle = \int_M f \cdot h \, d\mu$, $f, h \in V_k$. For simplicity, we normalize the canonical measure $d\mu$ of (M, g) in such a way that $\int_M d\mu = \dim V_k$. Denote an orthonormal basis of V_k by $\{f_i\}(i = 0, 1, \cdots, m(k))$ and define a map $\psi: M \to R^{m(k)+1}$ by $\psi(p) = (f_0(p), \cdots, f_{m(k)}(p))$, $p \in M$. The definition of a strongly harmonic manifold implies that there exists a function $F: R_+ \to R$ such that $(\psi(p), \psi(q)) = \sum_{i=0}^{m(k)} f_i(p) f_i(q) = F(\delta(p, q))$. Owing to this equation, the map ψ enjoys the following properties.

(i) ψ is an isometric immersion of $(M, c_k \cdot g)$ into $R^{m(k)+1}$, where $c_k = \lambda_k / \dim M$.

(ii) $\psi(M)$ is contained in the unit sphere $S_1^{m(k)}$ of $R^{m(k)+1}$ and ψ is minimal in $S_1^{m(k)}$.

(iii) ψ is helical geodesic in $S_1^{m(k)}$.

We call this map the *k-th standard minimal immersion* of a strongly harmonic manifold and denote it by ψ_k.

A helical geodesic immersion of order 1 means a totally geodesic immersion. As for order 2 and order 3, the following result is already gotten: If $\varphi: M \to S_1^m$ is a helical geodesic minimal immersion of order 2 or order 3, then M is a compact symmetric space of rank one and φ is equivalent to a standard minimal immersion (Sakamoto [11], Nakagawa [8]). On the other hand, harmonic manifolds have been studied by many authors, but the conjecture that harmonic manifolds are rank one symmetric spaces or Euclidean spaces is still open. The study of helical geodesic immersions will be useful for the study of this conjecture. From this point of view Sakamoto ([12]) proposed the following conjecture.

CONJECTURE. If φ is a helical geodesic minimal immersion of a compact Riemannian manifold into a unit sphere, then M is isometric to a compact symmetric space of rank one and φ is equivalent to a standard minimal immersion.

In this paper we show that a helical geodesic minimal immersion of a real analytic strongly harmonic manifold into a unit sphere is equivalent to a standard minimal immersion (§2). And we determine helical geodesic immersions of compact rank one symmetric spaces (§3).

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§1. Rigidity Lemma.

First of all we shall define the higher fundamental forms and the degree of an isometric immersion (Wallach [17]). Let $\tilde{M}$ be a Riemannian manifold of constant curvature. Let $\varphi: M \to \tilde{M}$ be an isometric immersion of a Riemannian manifold M into $\tilde{M}$. Let B_2 be the second fundamental form of φ at $p \in M$ and O_p^2 be the linear span of the image of B_2 in the normal space $N_p(M)$ of the immersion φ at $p \in M$. We call $\varphi_* T_p M + O_p^2$ the *second osculating space* at $p \in M$. We say that $p \in M$ is degree 2 regular if O_p^2 is of maximal dimension. Let $R_2 \subset M$ be the set of all degree 2 regular points of M. Then R_2 is open in M. Let $p \in R_2$. Let N_2 be the normal projection in $N_p(M)$ relative to $N_p(M) = O_p^2 + (O_p^2)^\perp$ (we write $v \to v^{N_2} \in (O_p^2)^\perp$). We define $B_3(u_1, u_2, u_3) = (\tilde{\nabla}_{u_1}(B_2(u_2, u_3)))^{N_2}$ for $u_1, u_2, u_3 \in T_p M$ arbitrarily extended to vector fields on M, where $\tilde{\nabla}$

denotes the Riemannian connection on $\tilde{M}$. B_3 is well-defined and defines a symmetric tensor field on R_2. Let O_p^3 be the linear span of the image of B_3. We call B_3 the *third fundamental form* of φ at p and $\varphi_* T_p M + O_p^2 + O_p^3$ the *third osculating space*. We call a point $p \in R_2$ degree 3 regular if $\dim O_p^3$ is maximal. We define B_j, O^j for $j = 2, 3, \cdots$ by recursion as above on the space R_{j-1} of all degree j-1 regular points of M. We call B_j the *j-th fundamental form* of φ and $\varphi_* T_p M + O_p^2 + \cdots + O_p^j$ the *j-th osculating space*. Clearly the above process must eventually stop since $\dim(\varphi_* T_p M + O_p^2 + \cdots + O_p^j) \leq \dim T_p \tilde{M}$. Let d be the first integer ≥ 2 such that $B_d \not\equiv 0$ but $B_{d+1} \equiv 0$. Then we call d the *degree* of φ and the set of all d-regular points will be called the set of all completely regular points of M, denoted by $M' = R_d$. In particular, when φ is totally geodesic, i.e., $B_2 \equiv 0$, we say that φ has degree 1.

In this paper, from now on we assume that the ambient space $\tilde{M}$ is one of the standard models of Riemannian manifolds of constant curvature, namely, the Euclidean space R^m, the sphere S^m, the real projective space $P_m(R)$, and the hyperbolic space H^m. It will be covenient to say that an isometric immersion $\varphi: M \to \tilde{M}$ is *full* if $\varphi(M)$ is not contained in any proper totally geodesic submanifold of $\tilde{M}$ and that two isometric immersions φ and $\hat{\varphi}$ of M into $\tilde{M}$ are *equivalent* if there exists an isometry ρ of $\tilde{M}$ such that $\rho \circ \varphi = \hat{\varphi}$.

In this section we show the following rigidity lemma, which is the generalization of a Nomizu's Theorem in [9]. This result can be derived from the general treatment of connections in the bundles of normal spaces given by Kowalski [6]. But since we need the process of its proof for later applications, we shall prove it here.

LEMMA 1.1 (Rigidity Lemma). *Let φ and $\hat{\varphi}$ be two full isometric immersions of a connected Riemannian manifold M into $\tilde{M}$. We assume that every point of M is completely regular for both φ and $\hat{\varphi}$ and that the degree of φ is equal to that of $\hat{\varphi}$. If there exists a bundle isomorphism f of the normal bundle N_φ for φ onto the normal bundle $N_{\hat{\varphi}}$ for $\hat{\varphi}$ which preserves the bundle metrics and the higher fundamental forms, i.e., for each point $p \in M$, $f(B_j(u_1, \cdots, u_j)) = \hat{B}_j(u_1, \cdots, u_j)$, $u_1, \cdots, u_j \in T_p M$ $2 \leq j \leq$ the degree of $\varphi =$ the degree of $\hat{\varphi}$, where B_j and $\hat{B}_j$ denote the j-th fundamental forms of φ and $\hat{\varphi}$ respectively, then φ is equivalent to $\hat{\varphi}$.*

PROOF. By the assumptions of Lemma 1.1 and a Theorem of Erbacher [5], N_φ and $N_{\hat{\varphi}}$ are decomposed respectively as follows:

$$N_\varphi = O^2 + \cdots + O^d$$

$$N_{\hat{\varphi}} = \hat{O}^2 + \cdots + \hat{O}^d ,$$

where d = the degree of φ = the degree of $\hat{\varphi}$. Since $f(B_j(u_1, \cdots, u_j)) = \hat{B}_j(u_1, \cdots, u_j)$, f induces a bundle isomorphism of O^j onto $\hat{O}^j$. Let $\nabla^\perp$ and $\hat{\nabla}^\perp$ be the normal connections in N_φ and $N_{\hat{\varphi}}$ respectively. By Rigidity Theorem (cf. Chen [3] p. 49) it is sufficient to prove that f preserves the connections. Define a linear connection $\overset{\circ}{\nabla}{}^\perp$ in the normal bundle N_φ as follows. For any section ξ of N_φ and for any vector field X on M, we set $\overset{\circ}{\nabla}{}^\perp_X \xi = f^{-1}(\hat{\nabla}^\perp_X(f\xi))$.

Since f preserves the bundle metrics, we easily see

Assertion 1. $\overset{\circ}{\nabla}{}^\perp$ is a metric connection of N_φ.

Next we show

Assertion 2. For any vector fields X, Y and Z on M, $\overset{\circ}{\nabla}{}^\perp_X(B_2(Y, Z)) = \nabla^\perp_X(B_2(Y, Z))$ holds.

(Proof) Using Codazzi equation, we have

$$\langle \overset{\circ}{\nabla}{}^\perp_X B_2(Y, Z), B_2(U, V) \rangle - \langle \nabla^\perp_X B_2(Y, Z), B_2(U, V) \rangle$$
$$= \langle \hat{\nabla}^\perp_X \hat{B}_2(Y, Z), \hat{B}_2(U, V) \rangle - \langle \nabla^\perp_X B_2(Y, Z), B_2(U, V) \rangle$$
$$= \langle \hat{\nabla}^\perp_Y \hat{B}_2(X, Z), \hat{B}_2(U, V) \rangle - \langle \nabla^\perp_Y B_2(X, Z), B_2(U, V) \rangle$$
$$= \langle \overset{\circ}{\nabla}{}^\perp_Y B_2(X, Z), B_2(U, V) \rangle - \langle \nabla^\perp_Y B_2(X, Z), B_2(U, V) \rangle ,$$

that is,

(1.1) $\langle (\overset{\circ}{\nabla}{}^\perp_X - \nabla^\perp_X) B_2(Y, Z), B_2(U, V) \rangle = \langle (\overset{\circ}{\nabla}{}^\perp_Y - \nabla^\perp_Y) B_2(X, Z), B_2(U, V) \rangle .$

Using Assertion 1 and (1.1), we get

$$\langle (\overset{\circ}{\nabla}{}^\perp_X - \nabla^\perp_X) B_2(Y, Z), B_2(U, V) \rangle = - \langle B_2(Y, Z), (\overset{\circ}{\nabla}{}^\perp_X - \nabla^\perp_X) B_2(U, V) \rangle$$
$$= - \langle B_2(Y, Z), (\overset{\circ}{\nabla}{}^\perp_U - \nabla^\perp_U) B_2(X, V) \rangle$$
$$= \langle (\overset{\circ}{\nabla}{}^\perp_U - \nabla^\perp_U) B_2(Y, Z), B_2(X, V) \rangle$$
$$= - \langle B_2(U, Z), (\overset{\circ}{\nabla}{}^\perp_Y - \nabla^\perp_Y) B_2(X, V) \rangle$$
$$= \langle (\overset{\circ}{\nabla}{}^\perp_Y - \nabla^\perp_Y) B_2(U, Z), B_2(X, Y) \rangle$$
$$= - \langle B_2(U, V), (\overset{\circ}{\nabla}{}^\perp_Z - \nabla^\perp_Z) B_2(X, Y) \rangle$$
$$= - \langle B_2(U, V), (\overset{\circ}{\nabla}{}^\perp_X - \nabla^\perp_X) B_2(Y, Z) \rangle .$$

Therefore we have

(1.2) $\langle \overset{\circ}{\nabla}{}^\perp_X B_2(Y, Z), B_2(U, V) \rangle = \langle \nabla^\perp_X B_2(Y, Z), B_2(U, V) \rangle .$

We remark that (1.1) and (1.2) are proved by using only the fact

that there exists a bundle isomorphism f of O^2 onto $\hat{O}^2$ which preserves bundle metrics and the second fundamental forms. Under this situation (1.1) and (1.2) show that, for a section ξ of O^2, f maps the O^2-component of $\nabla^{\perp}_X \xi$ to the $\hat{O}^2$-component of $\hat{\nabla}^{\perp}_X(f\xi)$. Following definitions, we get

$$[\overset{\circ}{\nabla}{}^{\perp}_X B_2(Y, Z)]^{N_2} = f^{-1}[f\overset{\circ}{\nabla}{}^{\perp}_X B_2(Y, Z)]^{N_2}$$
$$= f^{-1}[\hat{\nabla}^{\perp}_X \hat{B}_2(Y, Z)]^{N_2}$$
$$= f^{-1}\hat{B}_3(X, Y, Z)$$
$$= B_3(X, Y, Z) ,$$

that is,

(1.3) $$[\overset{\circ}{\nabla}{}^{\perp}_X B_2(Y, Z)]^{N_2} = B_3(X, Y, Z) .$$

Since $B_3(X, Y, Z) = [\nabla^{\perp}_X B_2(Y, Z)]^{N_2}$, Assertion 2 follows from (1.2) and (1.3).

Assertion 3. Generally we have $\overset{\circ}{\nabla}{}^{\perp}_X B_j(U_1, \cdots, U_j) = \nabla^{\perp}_X B_j(U_1, \cdots, U_j)$ for the j-th fundamental form, $3 \leq j \leq d$.

(Proof) We will apply the mathematical induction. We assume that Assertion 3 holds for any k, $2 \leq k \leq j-1$. Since $\overset{\circ}{\nabla}{}^{\perp}$ is a metric connection, we get for any k, $2 \leq k \leq j-1$

$$\langle \overset{\circ}{\nabla}{}^{\perp}_X B_j(U_1, \cdots, U_j), B_k(V_1, \cdots, V_k) \rangle$$
$$= X\langle B_j(U_1, \cdots, U_j), B_k(V_1, \cdots, V_k) \rangle - \langle B_j(U_1, \cdots, U_j), \overset{\circ}{\nabla}{}^{\perp}_X B_k(V_1, \cdots, V_k) \rangle$$
$$= X\langle B_j(U_1, \cdots, U_j), B_k(V_1, \cdots, V_k) \rangle - \langle B_j(U_1, \cdots, U_j), \nabla^{\perp}_X B_k(V_1, \cdots, V_k) \rangle$$
$$= \langle \nabla^{\perp}_X B_j(U_1, \cdots, U_j), B_k(V_1, \cdots, V_k) \rangle ,$$

that is,

(1.4) $$\langle \overset{\circ}{\nabla}{}^{\perp}_X B_j(U_1, \cdots, U_j), B_k(V_1, \cdots, V_k) \rangle$$
$$= \langle \nabla^{\perp}_X B_j(U_1, \cdots, U_j), B_k(V_1, \cdots V_k) \rangle$$

for $2 \leq k \leq j-1$. Since f preserves metrics and the second fundamental forms, we easily see that $f^{-1}\hat{R}^{\perp}(X, Y)f\xi = R^{\perp}(X, Y)\xi$ for any section ξ of N_{φ}, where $R^{\perp}$ and $\hat{R}^{\perp}$ denote the curvature tensors of N_{φ} and $N_{\hat{\varphi}}$ with respect to $\nabla^{\perp}$ and $\hat{\nabla}^{\perp}$ respectively. We set the decomposition of $\nabla^{\perp}_X B_{j-1}(U_2, \cdots, U_j)$ as follows:

$$\nabla^{\perp}_X B_{j-1}(U_2, \cdots, U_j) = B_j(X, U_2, \cdots, U_j) + D_j(X, U_2, \cdots, U_j) ,$$

where $B_j(X, U_2, \cdots, U_j) \in O^j$ and $D_j(X, U_2, \cdots, U_j) \in O^2 + \cdots + O^{j-1}$. By the assumption of the induction, we have

$$\mathring{\nabla}^{\perp}_{X}\mathring{\nabla}^{\perp}_{Y}B_{j-1}(U_2, \cdots, U_j) = \mathring{\nabla}^{\perp}_{X}(\nabla^{\perp}_{Y}B_{j-1}(U_2, \cdots, U_j))$$

$$= \mathring{\nabla}^{\perp}_{X}(B_j(Y, U_2, \cdots, U_j) + D_j(Y, U_2, \cdots, U_j)) \ .$$

By the definition, we get

$$\mathring{\nabla}^{\perp}_{X}\mathring{\nabla}^{\perp}_{Y}B_{j-1}(U_2, \cdots, U_j) - \mathring{\nabla}^{\perp}_{Y}\mathring{\nabla}^{\perp}_{X}B_{j-1}(U_2, \cdots, U_j) - \mathring{\nabla}^{\perp}_{[X,Y]}B_{j-1}(U_2, \cdots, U_j)$$

$$= f^{-1}\hat{R}^{\perp}(X, Y)fB_{j-1}(U_2, \cdots, U_j)$$

and

$$\nabla^{\perp}_{X}\nabla^{\perp}_{Y}B_{j-1}(U_2, \cdots, U_j) - \nabla^{\perp}_{Y}\nabla^{\perp}_{X}B_{j-1}(U_2, \cdots, U_j) - \nabla^{\perp}_{[X,Y]}B_{j-1}(U_2, \cdots, U_j)$$

$$= R^{\perp}(X, Y)B_{j-1}(U_2, \cdots, U_j) \ .$$

Noticing that $\mathring{\nabla}^{\perp}_{X}D_j(Y, U_2, \cdots, U_j) = \nabla^{\perp}_{X}D_j(Y, U_2, \cdots, U_j)$, we have

$$\langle \mathring{\nabla}^{\perp}_{X}B_j(Y, U_2, \cdots, U_j) - \mathring{\nabla}^{\perp}_{Y}B_j(X, U_2, \cdots, U_j), B_j(V_1, \cdots, V_j)\rangle$$

$$= \langle \nabla^{\perp}_{X}B_j(Y, U_2, \cdots, U_j) - \nabla^{\perp}_{Y}B_j(X, U_2, \cdots, U_j), B_j(V_1, \cdots, V_j)\rangle \ ,$$

that is,

$$(1.5) \qquad \langle (\mathring{\nabla}^{\perp}_{X} - \nabla^{\perp}_{X})B_j(Y, U_2, \cdots, U_j), B_j(V_1, \cdots, V_j)\rangle$$

$$= \langle (\mathring{\nabla}^{\perp}_{Y} - \nabla^{\perp}_{Y})B_j(X, U_2, \cdots, U_j), B_j(V_1, \cdots, V_j)\rangle \ .$$

By the same calculations as (1.2), we have

$$(1.6) \qquad \langle \mathring{\nabla}^{\perp}_{X}B_j(U_1, \cdots, U_j), B_j(V_1, \cdots, V_j)\rangle$$

$$= \langle \nabla^{\perp}_{X}B_j(U_1, \cdots, U_j), B_j(V_1, \cdots, V_j)\rangle \ .$$

Again we remark that (1.4), (1.5) and (1.6) are obtained by using only the fact that there exists a bundle isomorphism f of $O^2 + \cdots + O^j$ onto $\hat{O}^2 + \cdots + \hat{O}^j$ which preserves bundle metrics and fundamental forms. So for a section ξ of $O^2 + \cdots + O^j$, f maps the $O^2 + \cdots + O^j$-component of $\nabla^{\perp}_{X}\xi$ to the $\hat{O}^2 + \cdots + \hat{O}^j$-component of $\mathring{\nabla}^{\perp}_{X}f\xi$.

By the same reason as (1.3), we have

$$(1.7) \qquad [\mathring{\nabla}^{\perp}_{X}B_j(U_1, \cdots, U_j)]^{N_j} = B_{j+1}(X, U_1, \cdots U_j) \ .$$

This, together with (1.4) and (1.6), implies that $\mathring{\nabla}^{\perp}_{X}B_j(U_1, \cdots, U_j) = \nabla^{\perp}_{X}B_j(U_1, \cdots, U_j)$ and hence Assertion 3 is proved.

By Assertion 2 and Assertion 3, $\mathring{\nabla}^{\perp}_{X}\xi = \nabla^{\perp}_{X}\xi$ holds for any section ξ of N_φ and any vector field X on M. Therefore Lemma 1.1 is proved.

§2. Rigidity of helical geodesic minimal immersions of strongly harmonic manifolds.

As an application of Lemma 1.1, we show the following.

THEOREM 2.1. Let φ and $\hat{\varphi}$ be two isometric immersions of a connected Riemannian manifold M into a space form $\tilde{M}$. Assume that φ is a full immersion into $\tilde{M}$ and every point of M is completely regular with respect to φ. If, for any point $p \in M$ and any unit vector $x \in T_pM$, there exists $\varepsilon > 0$ such that for the geodesic $\gamma(t)$ of M which is defined on $-\varepsilon < t < \varepsilon$ and satisfies $\gamma(0) = p$, $\dot{\gamma}(0) = x$, the curve $\varphi \circ \gamma$ in $\tilde{M}$ is equivalent to the curve $\hat{\varphi} \circ \gamma$ in $\tilde{M}$, i.e., there exists an isometry ρ of $\tilde{M}$ such that $\hat{\varphi} \circ \gamma = \rho(\varphi \circ \gamma)$, then φ is equivalent to $\hat{\varphi}$.

REMARK. We can not remove the condition that every point of M is completely regular with respect to φ. As for a counter example, we refer to Fig 10.1 (p. 366) in [14].

For the geodesic γ of M in the assumptions of the above Theorem, we set $\sigma(t) = \varphi \circ \gamma(t)$ and $\tau(t) = \hat{\varphi} \circ \gamma(t)$. Then evidently the following holds.

LEMMA 2.2. $\langle \tilde{\nabla}_{\dot{\sigma}}^{(k)} \dot{\sigma}, \tilde{\nabla}_{\dot{\sigma}}^{(l)} \dot{\sigma} \rangle = \langle \tilde{\nabla}_{\dot{\tau}}^{(k)} \dot{\tau}, \tilde{\nabla}_{\dot{\tau}}^{(l)} \dot{\tau} \rangle$, where $\tilde{\nabla}$ denotes the Riemannian connection of $\tilde{M}$.

We denote the normal bundles for φ and $\hat{\varphi}$ by N_φ and $N_{\hat{\varphi}}$ respectively and the normal connections of N_φ and $N_{\hat{\varphi}}$ by $\nabla^\perp$ and $\hat{\nabla}^\perp$ respectively. We write the covariant differentiations on $TM + N_\varphi$ and $TM + N_{\hat{\varphi}}$ by $\bar{\nabla}$ and $\hat{\nabla}$ respectively. The assumption of the Theorem, together with Erbacher's Theorem, implies that the normal bundle N_φ is decomposed as $N_\varphi = O^2 + \cdots + O^d$, where d is the degree of φ. Moreover, for $2 \leq j \leq d$, the j-th fundamental form B_j of φ is defined on M. We shall prove the Theorem by calculating the higher fundamental forms $\hat{B}_j$ of $\hat{\varphi}$ and simultaneously constructing the bundle isomorphism f of N_φ onto $N_{\hat{\varphi}}$.

For an arbitrary unit tangent vector x at p, we take the geodesic γ of M which satisfies the assumption of the Theorem. Noticing that $(\tilde{\nabla}_{\dot{\sigma}} \dot{\sigma})(0) = B_2(x, x)$ and $(\tilde{\nabla}_{\dot{\tau}} \dot{\tau})(0) = \hat{B}_2(x, x)$, we have, by Lemma 2.2, $\langle B_2(x, x), B_2(x, x) \rangle = \langle \hat{B}_2(x, x), \hat{B}_2(x, x) \rangle$ for any unit tangent vector x at p. This equation is equivalent to

$$\langle B_2(u_1, u_2), B_2(u_3, u_4) \rangle + \langle B_2(u_1, u_3), B_2(u_4, u_2) \rangle + \langle B_2(u_1, u_4), B_2(u_2, u_3) \rangle$$
$$= \langle \hat{B}_2(u_1, u_2), \hat{B}_2(u_3, u_4) \rangle + \langle \hat{B}_2(u_1, u_3), \hat{B}_2(u_4, u_2) \rangle + \langle \hat{B}_2(u_1, u_4), \hat{B}_2(u_2, u_3) \rangle$$

for $u_1, u_2, u_3, u_4 \in T_p M$.

By Gauss equation, we have

$$3\langle B_2(u_1, u_2), B_2(u_3, u_4)\rangle + \langle R(u_1, u_4)u_2, u_3\rangle$$
$$+c\{\langle u_1, u_2\rangle\langle u_4, u_3\rangle - \langle u_1, u_3\rangle\langle u_4, u_2\rangle\} + \langle R(u_1, u_3)u_2, u_4\rangle$$
$$+c\{\langle u_1, u_2\rangle\langle u_3, u_4\rangle - \langle u_1, u_4\rangle\langle u_3, u_2\rangle\}$$
$$=3\langle \hat{B}_2(u_1, u_2), \hat{B}_2(u_3, u_4)\rangle + \langle R(u_1, u_4)u_2, u_3\rangle$$
$$+c\{\langle u_1, u_2\rangle\langle u_4, u_3\rangle - \langle u_1, u_3\rangle\langle u_4, u_2\rangle\} + \langle R(u_1, u_3)u_2, u_4\rangle$$
$$+c\{\langle u_1, u_2\rangle\langle u_3, u_4\rangle - \langle u_1, u_4\rangle\langle u_3, u_2\rangle\} ,$$

where c is the sectional curvature of $\tilde{M}$ and R denotes the curvature tensor of M. Thus we have

(2.1) $\langle \hat{B}_2(u_1, u_2), \hat{B}_2(v_1, v_2)\rangle = \langle B_2(u_1, u_2), B_2(v_1, v_2)\rangle$

$$\text{for } u_1, u_2, v_1, v_2 \in T_p M .$$

We define a linear map f of O_p^2 onto $\hat{O}_p^2$ by $f(B_2(x, y)) = \hat{B}_2(x, y)$, $x, y \in T_p M$. (2.1) shows that f is well-defined and is an isometry of O_p^2 onto $\hat{O}_p^2$. Especially we have $\dim O_p^2 = \dim \hat{O}_p^2$. By the assumption, every point of M is degree 2 regular for $\hat{\varphi}$. Consequently f is a bundle isomorphism of O^2 onto $\hat{O}^2$ which preserves the bundle metrics and the second fundamental forms. For convenience we denote by $D_3(x, y, z)$(resp. $\hat{D}_3(x, y, z)$) the O_p^2-component (resp. $\hat{O}_p^2$-component) of $\bar{\nabla} B_2(x, y, z)$(resp. $\hat{\nabla}\hat{B}_2(x, y, z)$). Then, by the remark in the proof of Lemma 1.1, $f(D_3(x, y, z)) = \hat{D}_3(x, y, z)$ holds for any x, y, z.

For the geodesic γ of M such that $\dot{\gamma}(0) = x$, we have

$$\tilde{\nabla}_\sigma^{(2)}\dot{\sigma}(0) = \tilde{\nabla}_x(B_2(\dot{\sigma}, \dot{\sigma})) = \bar{\nabla} B_2(x, x, x) - A_{B_2(x,x)}x$$
$$= B_3(x, x, x) + D_3(x, x, x) - A_{B_2(x,x)}x ,$$

where A_ξ denotes the second fundamental tensor corresponding to the normal vector ξ. Then we have

$$\langle \tilde{\nabla}_\sigma^{(2)}\dot{\sigma}, \tilde{\nabla}_\sigma^{(2)}\dot{\sigma}\rangle = \|B_3(x, x, x)\|^2 + \|D_3(x, x, x)\|^2 + \|A_{B_2(x,x)}x\|^2 .$$

Similarly we have

$$\langle \tilde{\nabla}_\tau^{(2)}\dot{\tau}, \tilde{\nabla}_\tau^{(2)}\dot{\tau}\rangle = \|\hat{B}_3(x, x, x)\|^2 + \|\hat{D}_3(x, x, x)\|^2 + \|\hat{A}_{\hat{B}_2(x,x)}x\|^2 .$$

By (2.1), we easily see that $A_{B_2(x,y)}z = \hat{A}_{\hat{B}_2(x,y)}z$. Since $f(D_3(x, x, x)) = \hat{D}_3(x, x, x)$, Lemma 2.2 implies that

(2.2) $\quad \langle \hat{B}_3(x, x, x), \hat{B}_3(x, x, x) \rangle = \langle B_3(x, x, x), B_3(x, x, x) \rangle$

$$\text{for any unit tangent vector } x \, .$$

Let u, v, x_1, x_2, y_1 and y_2 be tangent vectors at p of M. We write by γ the curve which is tangent to u. Let V, X_1, X_2, Y_1 and Y_2 be parallel vector fields along γ which coincide with v, x_1, x_2, y_1 and y_2 at p respectively. Then

$$\langle \bar{\nabla} B_2(V, X_1, X_2), B_2(Y_1, Y_2) \rangle = \langle \hat{\nabla} \hat{B}_2(V, X_1, X_2), \hat{B}_2(Y_1, Y_2) \rangle$$

holds along γ. Differentiating this equation with respect to u, we have

$$u \langle \bar{\nabla} B_2(V, X_1, X_2), B_2(Y_1, Y_2) \rangle$$
$$= \langle \nabla^\perp_u (\bar{\nabla} B_2(V, X_1, X_2)), B_2(Y_1, Y_2) \rangle + \langle \bar{\nabla} B_2(V, X_1, X_2), \nabla^\perp_u(B_2(Y_1, Y_2)) \rangle$$
$$= \langle \bar{\nabla}^2 B_2(u, v, x_1, x_2), B_2(y_1, y_2) \rangle + \langle \bar{\nabla} B_2(v, x_1, x_2), \bar{\nabla} B_2(u, y_1, y_2) \rangle$$
$$= \langle \bar{\nabla}^2 B_2(u, v, x_1, x_2), B_2(y_1, y_2) \rangle + \langle B_3(v, x_1, x_2), B_3(u, y_1, y_2) \rangle$$
$$+ \langle D_3(v, x_1, x_2), D_3(u, y_1, y_2) \rangle \, .$$

Similarly we get

$$u \langle \hat{\nabla} \hat{B}_2(V, X_1, X_2), \hat{B}_2(Y_1, Y_2) \rangle$$
$$= \langle \hat{\nabla}^2 \hat{B}_2(u, v, x_1, x_2), \hat{B}_2(y_1, y_2) \rangle + \langle \hat{B}_3(v, x_1, x_2), \hat{B}_3(u, y_1, y_2) \rangle$$
$$+ \langle \hat{D}_3(v, x_1, x_2), \hat{D}_3(u, y_1, y_2) \rangle \, .$$

Therefore

$$\langle \bar{\nabla}^2 B_2(u, v, x_1, x_2), B_2(y_1, y_2) \rangle + \langle B_3(v, x_1, x_2), B_3(u, y_1, y_2) \rangle$$
$$= \langle \hat{\nabla}^2 \hat{B}_2(u, v, x_1, x_2), \hat{B}_2(y_1, y_2) \rangle + \langle \hat{B}_3(v, x_1, x_2), \hat{B}_3(u, y_1, y_2) \rangle$$

holds. On the other hand, Ricci formula implies that

$$\langle \bar{\nabla}^2 B_2(u, v, x_1, x_2) - \bar{\nabla}^2 B_2(v, u, x_1, x_2), B_2(y_1, y_2) \rangle$$
$$= \langle R^\perp(u, v) B_2(x_1, x_2) - B_2(R(u, v)x_1, x_2) - B_2(x_1, R(u, v)x_2), B_2(y_1, y_2) \rangle$$

and

$$\langle \hat{\nabla}^2 \hat{B}_2(u, v, x_1, x_2) - \hat{\nabla}^2 \hat{B}_2(v, u, x_1, x_2), \hat{B}_2(y_1, y_2) \rangle$$
$$= \langle \hat{R}^\perp(u, v) \hat{B}_2(x_1, x_2) - \hat{B}_2(R(u, v)x_1, x_2) - \hat{B}_2(x_1, R(u, v)x_2), \hat{B}_2(y_1, y_2) \rangle \, ,$$

where $R^\perp$ and $\hat{R}^\perp$ denote the curvature tensors of N_φ and $N_{\hat{\varphi}}$ respectively.

By (2.1) and Ricci equation, we get

$$\langle \bar{\nabla}^2 B_2(u, v, x_1, x_2) - \bar{\nabla}^2 B_2(v, u, x_1, x_2), B_2(y_1, y_2) \rangle$$
$$= \langle \hat{\bar{\nabla}}^2 \hat{B}_2(u, v, x_1, x_2) - \hat{\bar{\nabla}}^2 \hat{B}_2(v, u, x_1, x_2), \hat{B}_2(y_1, y_2) \rangle .$$

Thus we have

(2.3) $\langle \hat{B}_3(u, x_1, x_2), \hat{B}_3(v, y_1, y_2) \rangle - \langle \hat{B}_3(v, x_1, x_2), \hat{B}_3(u, y_1, y_2) \rangle$

$= \langle B_3(u, x_1, x_2), B_3(v, y_1, y_2) \rangle - \langle B_3(v, x_1, x_2), B_3(u, y_1, y_2) \rangle$

for $u, v, x_1, x_2, y_1, y_2 \in T_pM$.

Symmetrizing (2.2), we have

$$\mathscr{S}_6 \langle \hat{B}_3(u_1, u_2, u_3), \hat{B}_3(u_4, u_5, u_6) \rangle = \mathscr{S}_6 \langle B_3(u_1, u_2, u_3), B_3(u_4, u_5, u_6) \rangle ,$$

where $\mathscr{S}_6$ denotes the symmetrizer of order 6. This, together with (2.3), gives

(2.4) $\langle \hat{B}_3(u_1, u_2, u_3), \hat{B}_3(v_1, v_2, v_3) \rangle = \langle B_3(u_1, u_2, u_3), B_3(v_1, v_2, v_3) \rangle$

for $u_1, u_2, u_3, v_1, v_2, v_3 \in T_pM$.

By (2.4), we can define an isometric linear map f of O_p^3 onto $\hat{O}_p^3$ which satisfies $f(B_3(x, y, z)) = \hat{B}_3(x, y, z)$. Especially we have $\dim O_p^3 = \dim \hat{O}_p^3$, which implies that every point of M is degree 3 regular for $\hat{\varphi}$. Consequently f is a bundle isomorphism of $O^2 + O^3$ onto $\hat{O}^2 + \hat{O}^3$ which preserves the bundle metrics and the second and third fundamental forms.

Calculating inductively, we see that every point of M is completely regular for $\hat{\varphi}$ and the degree of $\hat{\varphi}$ is equal to that of φ. Moreover we can construct the bundle isomorphism f of N_φ onto $N_{\hat{\varphi}}$ which preserves the bundle metrics and the higher fundamental forms. Therefore Lemma 1.1 gives Theorem 2.1.

COROLLARY 2.3.   Let (M, g) be an n-dimensional real analytic strongly harmonic manifold. If φ is a full helical geodesic minimal immersion of $(M, c_k \cdot g)$ into a unit sphere S_1^l, then φ is equivalent to the standard minimal immersion ψ_k. In particular l is equal to $m(k)$. (For notations we refer to Introduction.)

We prepare some lemmas before the proof of Corollary 2.3. Let $X \in T_{x_0}M - \{0\}$ and $\gamma: s \to \exp_{x_0}(s/\|X\|)X$ be the geodesic. Let $\{Y_i\}$ $i = 2, \cdots, n$ be Jacobi fields along γ such that $Y_i(0) = 0$ for every i and $\{Y_i'(0)\}_{i=2,\cdots,n}$ forms an orthonormal basis of the orthogonal complement of X in $T_{x_0}M$. Then we define $\theta: TM \to R$ by

$$\begin{cases} \theta(0)=1 \\ \theta(X)=\|X\|^{-n+1}\det(Y_2(\|X\|), \cdots, Y_n(\|X\|)) , \end{cases}$$

where the determinant should be understood with respect to the parallel frame field of $\{Y_i'(0)\}$. It is known that a strongly harmonic manifold is a globally harmonic manifold, i.e., there exists a C^∞-function $\Theta\colon R_+ \to R$ such that $\theta(X)=\Theta(\|X\|)$ for every $x \in M$ and every $X \in T_x M$.

LEMMA 2.4 (Berger-Gauduchon-Mazet [1] p. 134). *Let f be a C^∞-function on M of the form $f(x)=F(\delta(x, x_0))$ (i.e., which depends only on the distance to x_0), where δ denotes the distance function. Then we have*

$$\Delta f = -\frac{d^2 F}{ds^2} - \Big(\frac{\theta'_{x_0}}{\theta_{x_0}}+\frac{n-1}{s}\Big)\frac{dF}{ds} ,$$

where Δ denotes the Laplacian and θ'_{x_0} is the radial derivative of θ_{x_0} in $T_{x_0}M$.

LEMMA 2.5 (Sakamoto [12]). *Let $\varphi\colon M \to S_1^l$ be a helical geodesic immersion. Then there exists a C^∞-function $F\colon R_+ \to R$ such that the Euclidean inner product of position vectors $\varphi(x)$ and $\varphi(y)$ is given by $\langle\varphi(x), \varphi(y)\rangle = F(\delta(x, y))$.*

LEMMA 2.6 (Besse [2] p. 177, p. 178). *Let $\varphi\colon M \to S_1^l$ be a helical geodesic immersion and let $\gamma\colon I \to M$ be a geodesic parametrized by the arc-length. Then the curvatures of the curve $\sigma=\varphi\circ\gamma$ in S_1^l are completely determined by $F^{(k)}(0)$, $k=1, 2, \cdots$, where F is the function introduced in Lemma 2.5.*

LEMMA 2.7 (Do Carmo and Wallach [4]). *Let $\varphi\colon (M, c_k\cdot g) \to S_1^l$ be a minimal immersion. Then there exists a symmetric positive semi-definite linear map A of $R^{m(k)+1}$ such that φ is equivalent to $A\circ\psi_k$. Futhermore φ is equivalent to ψ_k if and only if the associated symmetric linear mapping A of φ is equal to the identity map.*

PROOF OF COROLLARY 2.3. Since l is not greater than $m(k)$, we may regard φ as a helical geodesic minimal immersion into $S_1^{m(k)}$. We denote by F and $\hat{F}$ the functions introduced in Lemma 2.5 associated with helical geodesic immersions ψ_k and φ respectively. For a fixed point $x_0 \in M$, we define the functions f and $\hat{f}$ on M by

$$f(x) = \langle\psi_k(x), \psi_k(x_0)\rangle = F(\delta(x, x_0))$$
$$\hat{f}(x) = \langle\varphi(x), \varphi(x_0)\rangle = \hat{F}(\delta(x, x_0)) ,$$

where $\langle \, , \, \rangle$ denotes the Euclidean inner product of $R^{m(k)+1}$. By a Theorem of Takahashi [15], $f(x)$ and $\hat{f}(x)$ are eigenfuctions of the Laplacian on $(M, c_k \cdot g)$ with eigenvalue n. By Lemma 2.4, we have

$$-\frac{d^2 F}{ds^2} - \frac{dF}{ds}\left(\frac{\theta'_{x_0}}{\theta_{x_0}} + \frac{n-1}{s}\right) = nF$$

and

$$-\frac{d^2 \hat{F}}{ds^2} - \frac{d\hat{F}}{ds}\left(\frac{\theta'_{x_0}}{\theta_{x_0}} + \frac{n-1}{s}\right) = n\hat{F} \, .$$

Since $F(0) = \hat{F}(0) = 1$ and $F'(0) = \hat{F}'(0) = 0$, we obtain $F \equiv \hat{F}$. Then Lemma 2.6 implies that, for any geodesic γ of M, $\varphi \circ \gamma$ is equivalent to $\psi_k \circ \gamma$ in $S_1^{m(k)}$. Let A be a symmetric positive semi-definite linear map of $R^{m(k)+1}$ such that $A \circ \psi_k$ is equivalent to φ. Let M' be a connected open sub-manifold of M whose points are all completely regular for ψ_k. Since ψ_k is analytic and is a full immersion into $S_1^{m(k)}$, ψ_k is a full immersion into $S_1^{m(k)}$ on M', too. We apply Theorem 2.1 to the immersions ψ_k and φ of M' into $S_1^{m(k)}$. By Lemma 2.7, A is the identity map. Therefore Corollary 2.3 holds.

Since a compact symmetric space of rank one is a real analytic strongly harmonic manifold, we immediately get

COROLLARY 2.8. *A helical geodesic minimal immersion of a compact symmetric space of rank one into a unit sphere is equivalent to a standard minimal immersion.*

This result in the case of a sphere has already been shown in [16].

§3. Helical geodesic immersions of a compact rank one symmetric space.

First we prepare two notions.

DEFINITION 3.1 (O'Neill [10]). Let B be an R^k-valued symmetric multilinear form on R^n. We say that B is *isotropic* if $\|B(u, \cdots, u)\| =$ constant for any unit vector u in R^n.

DEFINITION 3.2. Let $\varphi: M \to \tilde{M}$ be an isometric immersion of a Riemannian homogeneous space $M = G/K$ into a space form $\tilde{M}$. We say that φ is *equivariant* if there exists a continous homomorphism ρ of G into the isometry group $I(\tilde{M})$ of $\tilde{M}$ such that $\varphi(g \cdot p) = \rho(g)\varphi(p)$ $p \in M$, $g \in G$.

It is easily seen that the standard minimal immersion of a compact rank one symmetric space into a sphere is naturally equivariant. For the higher fundamental forms of an equivariant immersion, the following properties hold.

LEMMA 3.3 (cf. Mashimo [7]). Let $\varphi\colon M \to \tilde{M}$ be an equivariant isometric immersion of a Riemannian homogeneous space $M = G/K$ into a space form $\tilde{M}$. Then every point of M is completely regular. Moreover the j-th fundamental form B_j is G-invariant, i.e.,

$$B_{j|gp}(g \cdot u_1, \cdots, g \cdot u_j) = \rho(g)B_{j|p}(u_1, \cdots, u_j)$$
$$\rho(g)O_p^j = O_{gp}^j$$
$$N_j \circ \rho(g) = \rho(g) \cdot N_j, \quad g \in G.$$

THEOREM 3.4. Let M be a compact symmetric space of rank one and (G, K) be its symmetric pair. Let $\varphi\colon M \to S_1^l$ be an isometric immersion. Then the following three statements are mutually equivalent.

(1) φ is a G-equivariant immersion.

(2) For $2 \le j \le$ the degree of φ, the j-th fundamental form B_j is constant isotropic at every degree j-1 regular point $p \in R_{j-1}$, i.e., there exists a positive constant $\lambda_j > 0$ such that $\|B_j(u, \cdots, u)\| = \lambda_j$ for any unit tangent vector u at $p \in R_{j-1}$.

(3) φ is a helical geodesic immersion.

PROOF. "(1)$\Rightarrow$(2)" By Lemma 3.3, we have at eK, $\|B_j(k \cdot u, \cdots, k \cdot u)\| = \|\rho(k)B_j(u, \cdots, u)\| = \|B_j(u, \cdots, u)\|$ for $k \in K$. Since K acts transitively on the unit sphere of $T_{eK}M$, B_j is isotropic at eK. Again by Lemma 3.3, we see that B_j is constant isotropic.

"(1)$\Rightarrow$(3)" For an arbitrary unit tangent vector x at the origin $o = eK$ of M, we denote by γ the geodesic of M such that $\gamma(0) = o$ $\dot\gamma(0) = x$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of the symmetric pair (G, K). Then the geodesic γ is described as $\gamma(t) = (\exp tX) \cdot o$, where $\exp tX$ denotes a one-parameter subgroup of G and X is a vector in $\mathfrak{m}$ corresponding to x. Since $(\varphi \circ \gamma)(t) = \rho(\exp tX) \cdot \varphi(o)$, the curve $\varphi \circ \gamma$ in S_1^l has constant curvatures. Any geodesic γ parametrized by arc-length is described as $\gamma(t) = (g \exp tX) \cdot o$ for some $g \in G$. Therefore we have $(\varphi \circ \gamma)(t) = \rho(g)\rho(\exp tX) \cdot \varphi(o)$, which gives the statement of (3).

"(2)$\Rightarrow$(1)" We assume without loss of generality that φ is full. We shall prove the following.

Assertion. Under the condition of (2), we see that every point of M is completely regular. That is, for $2 \le j \le$ the degree of φ, the j-th

fundamental form B_j is defined on M. Moreover, for any $g \in G$, we have

$$\langle B_{j|gp}(g \cdot u_1, \cdots, g \cdot u_j), B_{j|gp}(g \cdot v_1, \cdots, g \cdot v_j) \rangle$$
$$= \langle B_{j|p}(u_1, \cdots, u_j), B_{j|p}(v_1, \cdots, v_j) \rangle, \quad u_1, \cdots, u_j, v_1, \cdots, v_j \in T_pM.$$

This Assertion is proved essentially by the same method as Theorem 2.1. For an arbitrary $g \in G$, we define another isometric immersion $\hat{\varphi}$ by $\hat{\varphi} = \varphi \circ g$. Then we have

$$\hat{B}_{j|p}(u_1, \cdots, u_j) = B_{j|gp}(g \cdot u_1, \cdots, g \cdot u_j)$$

at a degree $j-1$ regular point gp for φ, where $\hat{B}_j$ denotes the j-th fundamental form of $\hat{\varphi}$. By the assumption,

$$\langle B_2(u, u), B_2(u, u) \rangle = \langle B_2(g \cdot u, g \cdot u), B_2(g \cdot u, g \cdot u) \rangle = \langle \hat{B}_2(u, u), \hat{B}_2(u, u) \rangle$$

holds for any unit tangent vector u. By the same argument as in the proof of Theorem 2.1, we obtain

$$\langle \hat{B}_2(u_1, u_2), \hat{B}_2(v_1, v_2) \rangle = \langle B_2(u_1, u_2), B_2(v_1, v_2) \rangle.$$

Therefore we have

$$\langle B_{2|p}(u_1, u_2), B_{2|p}(v_1, v_2) \rangle = \langle B_{2|gp}(g \cdot u_1, g \cdot u_2), B_{2|gp}(g \cdot v_1, g \cdot v_2) \rangle,$$

which implies $\dim O_p^2 = \dim O_{gp}^2$. Since G acts transitively on M, the set of all degree 2 regular points for φ coincides with M. Following the same way as the proof of Theorem 2.1, we obtain Assertion.

In the process of the proof we can construct the bundle isomorphism $f(g)$ of the normal bundle N_φ for any $g \in G$ which preserves the bundle metrics and satisfies $B_{j|gp}(g \cdot u_1, \cdots, g \cdot u_j) = f(g)B_{j|p}(u_1, \cdots, u_j)$. By Lemma 1.1, there exists an isometry $\rho(g)$ of S_1^l such that $\rho(g) \circ \varphi = \varphi \circ g$. We immediately see that the map $\rho: G \to I(S_1^l)$ is a continuous homomorphism. Therefore the statement of (1) is proved.

"(3) $\Rightarrow$ (2)" By the assumption, for any geodesic γ of M and any $g \in G$, the curve $\varphi(g \circ \gamma)$ is equivalent to the curve $\varphi(\gamma)$ in S_1^l. We apply the methods of Theorem 2.1 to isometric immersions φ and $\varphi \circ g$. Moreover noticing that for two unit tangent vectors $u \in T_pM$ and $v \in T_qM$ there exists an element $g \in G$ such that $g \cdot u = v$, we obtain our result.

We construct a G-equivariant immersion of a compact rank one symmetric space M into a unit sphere. Let ψ_k be the k-th standard

minimal immersion of M into a unit sphere. We define

$$\psi: M \to R^{m(k_1)+\cdots+m(k_r)+r} \quad \text{by}$$

$$\psi(p) = (c_1 \psi_{k_1}(p), \cdots, c_r \psi_{k_r}(p)) \quad p \in M,$$

where each c_i is a positive number and $c_1^2 + \cdots + c_r^2 = 1$. By a suitable homothetic change of the metric of M, ψ becomes an isometric immersion into a unit sphere $S_1^{m(k_1)+\cdots+m(k_r)+r-1}$. ψ is evidently a G-equivariant immersion. For a compact rank one symmetric space G acts irreducibly on each eigenspace of the Laplacian. Therefore any G-equivariant isometric immersion of M into a unit sphere is described as above. For convenience, we denote by $\psi_{k_1,\cdots,k_r}$ the map which is obtained as above. Here we remark that $\psi_{k_1,\cdots,k_r}$ is full in a sphere only if $k_1, \cdots, k_r$ are distinct.

By Theorem 3.4, we have

COROLLARY 3.5.   *Let $\varphi: M \to S_1^l$ be a helical geodesic isometric immersion of a compact rank one symmetric space into a unit sphere. Assume that φ is full. Then there exist non-negative integers $k_1, \cdots, k_r$ such that φ is equivalent to $\psi_{k_1,\cdots,k_r}$, where $k_1, \cdots, k_r$ are distinct and may contain zero.*

§4.   Osculating order of a helical geodesic immersion $\psi_{k_1,\cdots,k_r}$.

In this section we calculate the osculating order of a helical geodesic immersion $\psi_{k_1,\cdots,k_r}$ of $M = $ a sphere S^n, a complex projective space $P_n(C)$, or a quaternion projective space $P_n(H)$ into a unit sphere. We assume that $n \geq 2$ in all case.

THEOREM 4.1.   *When $M = S^n$, the order of $\psi_{k_1,\cdots,k_r}$*

is
$\begin{cases} \text{the maximum of } k_1, \cdots, k_r \text{ if } k_1, \cdots, k_r \text{ are all odd or all even,} \\ 1 + \text{the maximum of odd numbers in } k_1, \cdots, k_r + \text{the maximum of} \\ \text{even numbers in } k_1, \cdots, k_r \text{ if } k_1, \cdots, k_r \text{ contain both odd numbers} \\ \text{and even numbers.} \end{cases}$

When $M = P_n(C)$ or $P_n(H)$, the order of $\psi_{k_1,\cdots,k_r}$ is $2 \times$ (the maximum of $k_1, \cdots, k_r$).

EXAMPLES.   We describe the degrees and the orders of some examples of $\psi_{k_1,\cdots,k_r}$ in the case of S^n.

	degree	order
$\psi_{0,1}$	2	2
$\psi_{0,2}$	2	2
$\psi_{0,3}$	3	4
$\psi_{1,2}$	3	4
$\psi_{1,3}$	3	3
$\psi_{2,3}$	4	6
$\psi_{1,2,3}$	5	6

REMARK. It is known that the degree is not greater than the order for a helical geodesic immersion (Sakamoto [12]). In the above examples there exists a helical geodesic immersion whose degree is less than the order. On the other hand, when ψ is a helical geodesic minimal immersion of $M=S^n$, $P_n(C)$ or $P_n(H)$, the order is equal to the degree.

Before the proof of Theorem 4.1, we describe the eigenfunctions of the Laplacian on S^n, $P_n(C)$ and $P_n(H)$. Let R^{n+1} be an $(n+1)$-dimensional Euclidean space $(n \geq 1)$ with coordinates $(x_0, x_1, \cdots, x_n)$. Let $S^k(R^{n+1})$ be the space of homogeneous polynomials of degree k on R^{n+1} and $H^k(R^{n+1})$ be the space of harmonic homogeneous polynomials of degree k on R^{n+1}.

LEMMA 4.2 (Berger-Gauduchon-Mazet [1] p. 160). *For any* $k \geq 0$, $S^k(R^{n+1})$ *has the following direct decomposition*:

$$S^{2k}(R^{n+1}) = H^{2k}(R^{n+1}) + r^2 H^{2(k-1)}(R^{n+1}) + \cdots + r^{2k} H^0(R^{n+1}) ,$$

and

$$S^{2k+1}(R^{n+1}) = H^{2k+1}(R^{n+1}) + r^2 H^{2k-1}(R^{n+1}) + \cdots + r^{2k} H^1(R^{n+1}) ,$$

where $r^2 = x_0^2 + x_1^2 + \cdots + x_n^2$.

LEMMA 4.3 (Berger-Gauduchon-Mazet [1] p. 160). *The k-th eigenspace of the Laplacian on S_1^n is given by the restriction of $H^k(R^{n+1})$ on S_1^n.*

Let C^{n+1} be an $(n+1)$-dimensional complex Euclidean space with coordinates $(z_0, z_1, \cdots, z_n)$ and $H^{k,k}$ be the space of harmonic homogeneous polynomials of type (k, k). Let $\pi: S^{2n+1} \to P_n(C)$ be the Hopf fibration. It is known that the space given by the restriction of $H^{k,k}$ on S^{2n+1} is π-invariant ([1] p. 172). Then it is regarded as the space of functions on $P_n(C)$, which is denoted by $\hat{H}^{k,k}$. Let H^{n+1} be an $(n+1)$-dimensional quaternion Euclidean space and H^{n+1} be naturally regarded as $2(n+1)$-

dimensional complex Euclidean space $C^{2(n+1)}$. This identification gives the Riemannian submersion $\pi: P_{2n+1}(C) \to P_n(H)$ with totally geodesic fibres $P_1(C) = S^2$.

LEMMA 4.4 ([1] p. 173, Smith [13]). *The k-th eigenspace of the Laplacian on $P_n(C)$ is given by $\hat{H}^{k,k}$. There exists the subspace of $\hat{H}^{k,k}$ which is invariant by the fibration $\pi: P_{2n+1}(C) \to P_n(H)$ and it gives the k-th eigenspace of the Laplacian on $P_n(H)$.*

REMARK. Let $H_R^{k,k}$ be the subspace of $H^{k,k}$ which consists of real-valued functions. Then we have $H_R^{k,k} \otimes_R C = H^{k,k}$, and $\hat{H}_R^{k,k}$ coincides with the space of real-valued eigenfunctions with the k-th eigenvalue of the Laplacian on $P_n(C)$.

First we shall prove Theorem 4.1 in the case of S^n. Let ψ_k be the k-th standard minimal immersion of S^n. We fix the geodesic γ of S^n. It is easily seen that $1+$the order of ψ_k is equal to the dimension of the minimum subspace of $R^{m(k)+1}$ which contains the curve $\psi_k(\gamma)$. Let $\{f_0 \circ \gamma, \cdots, f_{m(k)} \circ \gamma\}_R$ be the subspace of functions on γ which is generated by the functions given by the restriction of the coordinate functions $f_0, \cdots, f_{m(k)}$ of S^n on γ. Evidently the dimension of $\{f_0 \circ \gamma, \cdots, f_{m(k)} \circ \gamma\}_R$ is equal to the dimension of the minimum subspace of $R^{m(k)+1}$ which contains $\psi_k \circ \gamma$. For simplicity, we take the geodesic $\gamma = S^1$ as follows: $S^1 = \{x \in S^n \subset R^{n+1}; x_2 = x_3 = \cdots = x_n = 0\}$. Here we recall that the k-th eigenspace of the Laplacian on S^n is given by the restriction of $H^k(R^{n+1})$ on S^n (Lemma 4.3). Putting $x_2 = x_3 = \cdots = x_n = 0$, we get the linear homomorphism $\pi: H^k(R^{n+1}) \to S^k(R^2)$. It is easily checked that π is surjective. It is well-known that $\dim S^k(R^2) = k+1$ and hence $\dim\{f_0 \circ \gamma, \cdots, f_{m(k)} \circ \gamma\}_R = k+1$. Therefore the order of the k-th standard minimal immersion of S^n into a unit sphere $S_1^{m(k)}$ is k.

When $k_1, \cdots, k_r$ are all even or all odd, we assume that k_r is the maximum of $k_1, \cdots, k_r$. We write the coordinate functions of $c_j \psi_{k_j}$ of $\psi_{k_1,\cdots,k_r}$ by $f_0^j, \cdots, f_{m(k_j)}^j$. By Lemmas 4.2 and 4.3, $\{f_0^j \circ \gamma, \cdots, f_{m(k_j)}^j \circ \gamma\}_R$ is contained in $\{f_0^r \circ \gamma, \cdots, f_{m(k_r)}^r \circ \gamma\}_R$ for any $j = 1, \cdots, r$. Thus we get

$$\dim\{f_0^1 \circ \gamma, \cdots, f_{m(k_1)}^1 \circ \gamma, \cdots, f_0^r \circ \gamma, \cdots, f_{m(k_r)}^r \circ \gamma\}_R \leq k_r + 1 .$$

If its dimension is less than $k_r + 1$, $c_r \psi_{k_r}$, which is given by composing $\psi_{k_1,\cdots,k_r}$ and the projection $R^{m(k_1)+\cdots+m(k_r)+r} \to R^{m(k_r)+1}$ has the order less than k_r. It is a contradiction. Therefore the order of $\psi_{k_1,\cdots,k_r}$ is k_r. When $\{k_1, \cdots, k_r\}$ contains both even numbers and odd numbers, similarly we get $\dim\{f_0^1 \circ \gamma, \cdots, f_{m(k_r)}^r \circ \gamma\}_R =$ the maximum of even numbers + the

maximum of odd numbers$+2$. Thus we obtain Theorem 4.1 in the case of S^n.

It is well-known that $P_n(C)$ and $P_n(H)$ contain an n-dimensional real projective space $P_n(R)$ as a totally geodesic submanifold. We shall briefly recall it. Let R^{n+1} be included in C^{n+1} and H^{n+1} as a real part respectively. Let $\pi\colon S^{2n+1}\subset C^{n+1}\to P_n(C)$ and $\pi\colon S^{4n+3}\subset H^{n+1}\to P_n(H)$ be the Hopf fibrations. $S^n=R^{n+1}\cap S^{2n+1}$ is a totally geodesic submanifold of S^{2n+1}. Similarly $S^n=R^{n+1}\cap S^{4n+3}$ is a totally geodesic submanifold of S^{4n+3}. Restricting Hopf fibrations π on S^n, we get totally geodesic immersions $\pi\colon S^n\to P_n(C)$ and $\pi\colon S^n\to P_n(H)$, and its image $\pi(S^n)$ is $P_n(R)$. Let ψ_k be the k-th standard minimal immersion of $P_n(C)$ or $P_n(H)$. ψ_k is a helical geodesic minimal immersion into a unit sphere. Trivially $\psi_k\circ\pi$ is also a helical geodesic immersion of S^n. Moreover the order of ψ_k is equal to the order of $\psi_k\circ\pi$. So we shall calculate the order of the helical geodesic immersion $\psi_k\circ\pi$ of S^n. Restricting $H_R^{k,k}(C^{n+1})$ on a real part R^{n+1}, we get a linear homomorphism $P\colon H_R^{k,k}(C^{n+1})\to S^{2k}(R^{n+1})$. We remark that the homomorphism is not always surjective. But, for any $f\in H^{2k}(R^{n+1})$, there exists $\tilde{f}$ in $H_R^{k,k}(C^{n+1})$ such that $P\tilde{f}=f$. When ψ_k is the k-th standard minimal immersion of $P_n(C)$, by the same arguments as S^n, we see that the order of $\psi_k\circ\pi$ is $2k$, and hence the order of ψ_k is $2k$. In the case of $P_n(H)$, by the above argument we see that the order of the k-th standard minimal immersion ψ_k is not greater than $2k$. But it is known that the degree of ψ_k is $2k$ (Mashimo [7]). Since the order of ψ_k is not less than the degree of ψ_k, we see that the order of ψ_k is $2k$. We can obtain the order of $\psi_{k_1,\cdots,k_r}$ by the same argument as the case of S^n.

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