TOKYO J. MATH.
VOL. 6, No. 2, 1983

# The Existence of an Invariant Stable Foliation and the Problem of Reducing to a One-Dimensional Map

Michiko YURI

*Tsuda College*

(Communicated by M. Mori)

## Introduction

In 1962, E. Lorenz found the first example of a strange attractor by investigating a hydrodynamical system. Recently, another equation has been proposed by Rössler, and by numerical solution, it was shown that these equations indicate the existence of a two-dimensional attractor which has a compact "ribbon-like" structure.

As the attractor can be treated as a "single-sheeted" quasi-two-dimensional object, we take a cut across the attractor and construct a Poincaré mapping by means of which we can reduce a three-dimensional continuous flow to a one-dimensional discrete mapping. Thus one-dimensional models serve as the simplest example of models for some dynamical systems and have become common. They appear in the original paper by Lorenz [1], and also in more recent works of Guckenheimer [2], Rössler [3], and others ([4], [5], [6]). But this procedure has not been justified rigorously so far. Our purpose here is to give some justification for reducing a three-dimensional flow which has a two-dimensional attractor to a one-dimensional mapping. To be more precise with the problem, let us consider a map $H_0$: $\mathrm{R}^2 \to \mathrm{R}^2$ defined by

$$H_0(x, y) = (f(x), \mu y)$$

where $f$ is a map of piecewise $C^2$-class such that $f(I) \subset I$ for an interval $I$ and $0 < \mu < 1$. The map $H_0$ has trivial stable foliation $\{x = Constant\}$, and hence the behavior of $H_0$ near the invariant set $I \times \{0\}$ is reduced to the one-dimensional map $f$ on $I$. Let $H$ be a perturbation of $H_0$ defined by

Received February 5, 1982
Revised June 23, 1983

$$H(x, y)=(f(x)+\varepsilon_1(x, y), \mu y+\varepsilon_2(x, y))$$

where each $\varepsilon_i$: $R^2 \to R$ is of $C^2$-class. Then this map could have an invariant set $\Gamma$ near $I\times\{0\}$ under some conditions on $\varepsilon_i(x, y)$ and $\mu$. So, if we could construct an invariant stable foliation on $\Gamma$, then we could say that the study of the behavior of $H$ near $\Gamma$ is reduced to the study of the one-dimensional map on $\Gamma$. (For the precise definition of the invariant stable foliation, see § 3.)

In this paper, we deal with a perturbation $H$ which leaves $I\times\{0\}$ invariant and obtain conditions about $H$ which imply the existence of an invariant stable foliation almost everywhere with respect to Lebesgue measure. Furthermore these conditions are expressed in terms of $\mu$ which measures the degree of contraction, perturbing terms $\varepsilon_i$ and the one-dimensional map $f$.

Recently, Ruelle [7] has proved that if $g$ is a diffeomorphism of a compact manifold, a stable foliation exists almost everywhere with respect to $g$ invariant measure, meanwhile Pessin [8] presented a stable manifold theorem under the existence of a smooth invariant measure. Anyway, since $g$ does not always have a smooth invariant measure, we could not apply their results to our problem directly. The proof of Ruelle's stable manifold theorem is based on the study of random matrix products and perturbations of such products occurring in the multiplicative ergodic theorem due to Oseledic ([9]). In contrast, since in our problem, the tangent mapping on $I\times\{0\}$ is an upper triangular matrix, it becomes possible for us to form a stable foliation without using the multiplicative ergodic theorem. We only need some assumptions on the ratio of eigenvalues of the tangent mappings.

The constitution of this paper is as follows:

(§ 1) Perturbations of upper triangular matrix product.
(§ 2) The existence of stable foliations for two-dimensional mappings.
(§ 3) Applications.

In conclusion the author would like to thank Professors T. Niwa, Y. Takahashi, S. Ito, and S. Ushiki for their encouragement and valuable advices.

## § 1. Perturbations of upper triangular matrix product.

Let $T=(T_n)_{n>0}$ be a sequence of upper triangular real $2\times 2$ matrices. We define $\tilde{\alpha}_n$, $\tilde{\beta}_n$, $\tilde{\gamma}_n$ by $T_n=\begin{pmatrix}\tilde{\alpha}_n & \tilde{\beta}_n \\ 0 & \tilde{\gamma}_n\end{pmatrix}$, and $\alpha_n$, $\beta_n$, $\gamma_n$ by $T^n=T_n\circ\cdots\circ T_1=\begin{pmatrix}\alpha_n & \beta_n \\ 0 & \gamma_n\end{pmatrix}$. Denote by $\mu_n<\lambda_n$ the eigenvalues of $\sqrt{{}^tT^nT^n}$ and $U_n$, $V_n$ the corresponding eigenspaces.

**PROPOSITION 1.** *Suppose* $T=(T_n)_{n>0}$ *satisfies the conditions* (A.1) *and* $(\mathrm{A}.2)_\xi$ *below*:

(A.1) $\limsup_n (1/n)\log \| T_n \| \leqq 0$,

$(\mathrm{A}.2)_\xi$ *there is* $\xi>0$ *such that*

$$\left|\frac{\beta_n}{\alpha_{n+1}}\right| < \exp(-n\xi) \quad (n>0) .$$

*Then* $\lim_n U_n = U$ *and* $\lim_n V_n = V$ *exist, and for any* $\varepsilon>0$ *there exists* $K(\varepsilon)>0$ *such that, for all* $u \in U$,

$$\| T^n u \| \leqq \| u \| \cdot K(\varepsilon) \cdot \lambda_n \exp(-n(\xi-\varepsilon)) \quad (n>0) . \tag{1.1}$$

PROOF. By (A.1) there exists $C_1(\varepsilon)>1$ such that for all $n>0$

$$\| T_n \| \leqq C_1(\varepsilon) \exp(n\varepsilon) .$$

For a unit vector $u \in U_n$, we write $u = a_{n+1}u_{n+1} + b_{n+1}v_{n+1}$. Here $u_{n+1} \in U_{n+1}$ and $v_{n+1} \in V_{n+1}$ are unit vectors. Then, since $U_n$ is the orthogonal complement of $V_n$, we have

$$\begin{aligned} \| T^{n+1}(b_{n+1}v_{n+1}) \| &\leqq \| T^{n+1}u \| \\ &\leqq \| T_{n+1} \| \, \| T^n u \| \\ &\leqq C_1(\varepsilon) \exp((n+1)\varepsilon) \cdot \mu_n . \end{aligned}$$

As $|\alpha_{n+1}| \leqq \lambda_{n+1}$ and $\mu_n \leqq |\beta_n|$, we obtain

$$\begin{aligned} |b_{n+1}| &\leqq C_1(\varepsilon) \exp(\varepsilon) \exp(n\varepsilon) \frac{\mu_n}{\lambda_{n+1}} \\ &\leqq C_1(\varepsilon) \exp(\varepsilon) \exp(n(-\xi+\varepsilon)) \end{aligned}$$

by $(\mathrm{A}.2)_\xi$. Therefore

$$\operatorname{dis}(U_n, U_{n+1}) \leqq C_1(\varepsilon) \exp(\varepsilon) \exp(n(-\xi+\varepsilon)) . \tag{1.2}$$

Here $\operatorname{dis}(W_1, W_2)$ is defined by $|\det(w_1, w_2)|$ where $w_1 \in W_1$, $w_2 \in W_2$ are unit vectors and $(w_1, w_2)$ denotes the matrix whose $j$th column is $w_j$. The inequality (1.2) implies that $(U_n)_{n>0}$ constitutes a Cauchy sequence and thus $\lim_n U_n = U$ exists. Hence also $\lim_n V_n = V$ exists. Moreover, for any $\varepsilon>0$, there is $K_1(\varepsilon)>0$ such that

$$\begin{aligned} \operatorname{dis}(U_n, U) &\leqq \sum_{k=n}^{\infty} \operatorname{dis}(U_k, U_{k+1}) \\ &\leqq K_1(\varepsilon) \exp(n(-\xi+\varepsilon)) \quad (n>0) . \end{aligned}$$

For $u \in U$, we write $u=a_n u_n+b_n v_n$, here $u_n \in U_n$ and $v_n \in V_n$ are unit vectors. Then we have

$$\begin{aligned}\|T^n u\| &\leqq |a_n|\mu_n+|b_n|\lambda_n \\ &\leqq \|u\|(\mu_n+\mathrm{dis}\,(U_n, U)\lambda_n) \\ &\leqq \|u\|(\mu_n+K_1(\varepsilon)\exp(-n(\xi-\varepsilon))\lambda_n)\,.\end{aligned}$$

Therefore, for any $\varepsilon>0$, there is $K(\varepsilon)>0$ such that

$$\|T^n u\| \leqq \|u\| K(\varepsilon)\exp(-n(\xi-\varepsilon))\lambda_n \quad (n>0)\,.$$

Here we used the inequality

$$\frac{\mu_n}{\lambda_n} \leqq \left|\frac{\beta_n}{\alpha_n}\right| \leqq \exp(-n\xi)\|T_{n+1}\|\,.$$

This gives the conclusion of Proposition 1.

A sequence $T'=(T'_n)_{n>0}$ of $2\times 2$ matrices is called a *$\nu$-perturbation of $T$*, for a positive integer $\nu$, if $T'_n=T_n$ for all $n>\nu$. Denote by $\mu'_n<\lambda'_n$ the eigenvalues of $\sqrt{{}^t(T'^n)(T'^n)}$, and by $U'_n$, $V'_n$ the corresponding eigenspaces. For $\eta>0$, define $\|T'-T\|_\eta$ by

$$\|T'-T\|_\eta \equiv \sup_n \|T'_n-T_n\|\exp(3n\eta)\,.$$

Let $T$ be a sequence of upper triangular matrices which satisfies the conditions (A.1), $(\mathrm{A.2})_\xi$ and also (A.3), (A.4), (A.5) below:

(A.3) $\det T_n \neq 0$ *for all* $n>0$,

(A.4) $\lim_n (1/n)\log|\det T_n|=0$,

$(\mathrm{A.5})_{\tilde\xi}$ $\limsup_n (1/n)\log|\alpha_n/\beta_n| \equiv \tilde\xi>\xi$.

Then, by Proposition 1 $\lim U_n=U$ and $\lim V_n=V$ exist and also we have the following

THEOREM 2. *For any $T$ above and constants $\eta>\tilde\xi-\xi$ and $\varepsilon>0$, there are constants $\delta>0$ and $F(\varepsilon,\eta)>1$ such that the following holds:*

*If $T'$ is a $\nu$-perturbation of $T$ such that $\|T'-T\|_\eta \leqq \delta$, then* $\lim U'_n=U'$ *and* $\lim V'_n=V'$ *exist and for all $u \in U'$*

$$\|T'^n u\|=\|u\|F(\varepsilon,\eta)\mu_n\exp(n(\tilde\xi-\xi+2\varepsilon)) \quad (n>0)\,.$$

For the proof of Theorem 2, we prepare some lemmas. Let $T$, $U$, $V$ and $\eta$ be as in Theorem 2.

LEMMA 1. *For any $\varepsilon>0$ there exists $E(\varepsilon,\xi)>1$ such that*

$$\frac{\lambda_n}{\mu_n} \leqq E(\varepsilon,\xi)\exp(n(\tilde\xi+\varepsilon)) \quad \textit{for all} \quad n>0\,.$$

PROOF. Indeed, as $\gamma_n=\sum_{k=0}^{n-1}\tilde{\alpha}_n\times\cdots\times\tilde{\alpha}_{k+2}\times\tilde{\gamma}_{k+1}\times\beta_k$ $(\beta_0=1)$, we have

$$\begin{aligned}\left|\frac{\gamma_n}{\alpha_n}\right|&\leqq\sum_{k=0}^{n-1}\left|\frac{\beta_k}{\alpha_{k+1}}\right||\tilde{\gamma}_{k+1}|\\&\leqq\sum_{k=1}^{n-1}\exp(-k\xi)\|T_{k+1}\|+\left|\frac{\tilde{\gamma}_1}{\alpha_1}\right|\\&\leqq\tilde{C}(\varepsilon)\sum_{k=0}^{\infty}\exp(-k(\xi-\varepsilon)),\end{aligned}$$

so that

$$|\gamma_n|\leqq D(\varepsilon,\xi)|\alpha_n|\quad\text{where}\quad D(\varepsilon,\xi)=\tilde{C}\sum_{k=0}^{\infty}\exp(-k(\xi-\varepsilon)).$$

Hence for all $n>0$, we obtain

$$\begin{aligned}\frac{\lambda_n^2}{\mu_n^2}&=\frac{\{(\alpha_n^2+\beta_n^2+\gamma_n^2)+\sqrt{(\alpha_n^2+\beta_n^2+\gamma_n^2)^2-4\alpha_n^2\beta_n^2}\}^2}{4\alpha_n^2\beta_n^2}\\&\leqq\frac{(\alpha_n^2+\beta_n^2+\gamma_n^2)^2}{\alpha_n^2\beta_n^2}\\&\leqq\frac{\alpha_n^2}{\beta_n^2}\left(1+\frac{\beta_n^2}{\alpha_n^2}+\frac{\gamma_n^2}{\alpha_n^2}\right)^2\\&\leqq\frac{\alpha_n^2}{\beta_n^2}(2+D^2(\varepsilon,\xi))^2\\&\leqq\exp(2(n(\tilde{\xi}+\varepsilon)))(2+D^2(\varepsilon,\xi))^2.\end{aligned}$$

Thus we can take $E(\varepsilon,\xi)=2+D^2(\varepsilon,\xi)>1$.

LEMMA 2. *Let $\delta$ be a constant with $0<\delta<1$ and suppose that a $\nu$-perturbation $T'$ of $T$ satisfies $\|T'-T\|_\eta<\delta$. Then for any $\varepsilon>0$ there is $C_2(\varepsilon)>1$ so that*

$$\operatorname{dis}(U_n',U_{n+1}')\leqq 2C_2(\varepsilon)\exp(n\varepsilon)\frac{\mu_n'}{\lambda_{n+1}'}\quad(n>0).$$

PROOF. As $\|T_n'-T_n\|<\delta\exp(-3n\eta)$ for all $n>0$, we have

$$\|T_n'\|\leqq\delta\exp(-3n\eta)+\|T_n\|.$$

Hence by (A.1) we obtain $C_2(\varepsilon)>1$ for any $\varepsilon>0$ so that

$$\begin{aligned}\|T_n'\|&\leqq\delta\exp(-3n\eta)+C_2(\varepsilon)\exp(n\varepsilon)\\&\leqq\delta+C_2(\varepsilon)\exp(n\varepsilon)\\&\leqq 2C_2(\varepsilon)\exp(n\varepsilon)\quad(n>0).\end{aligned}$$

Since $\operatorname{dis}(U'_n, U'_{n+1}) \leqq \| T'_{n+1} \| (\mu'_n/\lambda'_{n+1})$, the conclusion is obtained.

LEMMA 3. $\limsup_n (1/n) \log \| T_n^{-1} \| = 0$.

PROOF. Since $\| T_n^{-1} \| = \| T_n \| / |\det T_n|$, Lemma 3 is obtained from (A.1) and (A.4) immediately.

LEMMA 4. *Let $u \in U$ and $v \in V$ be unit vectors. Then for all $n > 0$ we have*

$$\mu_n \leqq \| T^n u \| \leqq K(\varepsilon) \lambda_n \exp(-n(\xi - \varepsilon)) \quad \textit{and}$$
$$d\lambda_n \leqq \| T^n v \| \leqq \lambda_n ,$$

*where the constant $d$ does not depend on $n$.*

PROOF. We write $v = c_n u_n + d_n v_n$. Here $c_n$ and $d_n$ are components along $u_n \in U_n$ and $v_n \in V_n$ which are unit vectors respectively. As $(V_n)_{n>0}$ is a Cauchy sequence, $^\sharp\{n; |d_n| \leqq |c_n|\}$ is finite. Put $N = \max\{n; |d_n| \leqq |c_n|\}$. If $|d_n| \leqq |c_n|$, then from Lemma 1

$$\begin{aligned} \| T^n v \| &= \sqrt{c_n^2 \mu_n^2 + d_n^2 \lambda_n^2} \\ &> |c_n| \mu_n \\ &= |c_n| \left( \frac{\mu_n}{\lambda_n} \right) \lambda_n \\ &\geqq \frac{1}{\sqrt{2}} \frac{\lambda_n}{E(\varepsilon, \xi) \exp(n(\tilde{\xi} + \varepsilon))} \\ &\geqq \frac{1}{\sqrt{2}} \frac{\lambda_n}{E(\varepsilon, \xi) \exp(N(\tilde{\xi} + \varepsilon))} . \end{aligned}$$

If $|d_n| > |c_n|$, then we have $\| T^n v \| \geqq |d_n| \lambda_n > (1/\sqrt{2}) \lambda_n$. Hence we can take $d = 1/(\sqrt{2} E(\varepsilon, \xi) \exp(N(\tilde{\xi} + \varepsilon))) > 0$. Since $\mu_n^2 = \inf_{\|x\|=1} ({}^t T^n T^n x, x)$, by Proposition 1 we have

$$\mu_n \leqq \| T^n u \| \leqq K(\varepsilon) \lambda_n \exp(-n(\xi - \varepsilon))$$

immediately.

We define positive numbers $t_n$, $s_n$ and $t_n^*$ by

$$t_n = \frac{\| T^n u \|}{\| T^{n-1} u \|} , \qquad s_n = \frac{\| T^n v \|}{\| T^{n-1} v \|} \quad \text{and} \quad t_n^* = \exp \tilde{\xi} \cdot t_n ,$$

where $u \in U$ and $v \in V$ are unit vectors.

LEMMA 5. *Let $\eta > \tilde{\xi} - \xi$. Then there is $C(\eta) > 1$ such that*

$$\frac{\prod_{k=1}^{n-1} s_k}{\prod_{k=1}^{n} t_k^*}\,,\ \frac{\prod_{k=1}^{n-1} t_k^*}{\prod_{k=1}^{n} s_k}\,,\ \frac{1}{t_n^*}\,,\ \frac{1}{t_n}\,,\ \frac{1}{s_n}<C(\eta)\exp(n\eta) \quad (n>0)\,.$$

PROOF. Note that

$$\frac{1}{t_n}=\frac{\|T^{n-1}u\|}{\|T^n u\|}=\frac{\|T_n^{-1}T^n u\|}{\|T^n u\|} \quad \text{and} \quad \frac{1}{s_n}=\frac{\|T^{n-1}v\|}{\|T^n v\|}=\frac{\|T_n^{-1}T^n v\|}{\|T^n v\|}\,.$$

From Proposition 1, Lemmas 1, 3 and 4, we have the conclusion easily.

PROOF OF THEOREM 2. Let $u\in U$ and $v\in V$ be unit vectors, and write

$${}^n u=\frac{T^n u}{\|T^n u\|} \quad \text{and} \quad {}^n v=\frac{T^n v}{\|T^n v\|}\,.$$

For $w\neq 0$ in $\mathrm{R}^2$ we write $T'^n w=a_n{}^n u+b_n{}^n v$ where $a_n$ and $b_n$ are components along ${}^n u$ and ${}^n v$ respectively. As $T'^n w=T_n'(T'^{n-1}w)$ and $T_n{}^{n-1}u=t_n{}^n u$, we have

$$T'^n w=(T_n'-T_n)(a_{n-1}{}^{n-1}u+b_{n-1}{}^{n-1}v)+a_{n-1}t_n{}^n u+b_{n-1}s_n{}^n v\,.$$

Hence we have

$$|a_{n-1}|t_n-\frac{\|T_n'-T_n\|}{|\det({}^n u,{}^n v)|}(|a_{n-1}|+|b_{n-1}|)$$
$$\leqq|a_n|\leqq|a_{n-1}|t_n+\frac{\|T_n'-T_n\|}{|\det({}^n u,{}^n v)|}(|a_{n-1}|+|b_{n-1}|)$$
$$|b_{n-1}|s_n-\frac{\|T_n'-T_n\|}{|\det({}^n u,{}^n v)|}(|a_{n-1}|+|b_{n-1}|)$$
$$\leqq|b_n|\leqq|b_{n-1}|s_n+\frac{\|T_n'-T_n\|}{|\det({}^n u,{}^n v)|}(|a_{n-1}|+|b_{n-1}|)\,.$$

As $|\det({}^n u,{}^n v)|=|\det T^n|/\|T^n u\|\,\|T^n v\|$, it follows from Lemma 4 that for $\eta>\tilde{\xi}-\xi$ we have:

$$D_\eta=\sup_n\frac{1}{|\det({}^n u,{}^n v)|}\exp(-n\eta)<+\infty\,.$$

We suppose $\|T'-T\|_\eta\leqq\delta$. Then we have

$$|a_n|\leqq|a_{n-1}|t_n+\delta D\exp(-2n\eta)(|a_{n-1}|+|b_{n-1}|)$$
$$\leqq|a_{n-1}|t_n^*+\delta D\exp(-2n\eta)(|a_{n-1}|+|b_{n-1}|)$$

and

$$|b_n|\leqq|b_{n-1}|s_n+\delta D\exp(-2n\eta)(|a_{n-1}|+|b_{n-1}|)$$

where $t_n^* = t_n \exp(\xi)$.

If there is $m \geqq 0$ snch that $|a_m| < |b_m|$, then by Lemma 5 we obtain, for $n > m$,

$$(2.1)\quad \begin{aligned} |a_n| &\leqq |b_m| \prod_{k=m+1}^{n} t_k^* \prod_{k=m+1}^{n} (1+2\delta C(\eta) D_\eta \exp(-k\eta)) \\ |b_n| &\leqq |b_m| \prod_{k=m+1}^{n} s_k \prod_{k=m+1}^{n} (1+2\delta C(\eta) D_\eta \exp(-k\eta)) . \end{aligned}$$

We choose $\delta = (1/(2C(\eta)D_\eta)) \prod_{k=1}^{\infty} (1-\exp(-k\eta))^2$. In this way $2CD\delta < 1$ and $C' = (\prod_{k=1}^{\infty} (1+2CD\delta \exp(-k\eta)))/(\prod_{k=1}^{\infty} (1-\exp(-k\eta))) < 1/2CD\delta$. Therefore (2.1) gives

$$(2.2)_a\quad \begin{aligned} |a_n| &\leqq C'|b_m| \prod_{k=m+1}^{n} t_k^* \prod_{k=m+1}^{n} (1-\exp(-k\eta)) \\ |b_n| &\leqq C'|b_m| \prod_{k=m+1}^{n} s_k \prod_{k=m+1}^{n} (1-\exp(-k\eta)) \quad (n>m) . \end{aligned}$$

Using Lemma 5 and $(2.2)_a$, we have for $n > m$

$$(2.2)_b\quad |b_n| \geqq |b_m| \prod_{k=m+1}^{n} s_k \prod_{k=m+1}^{n} (1-\exp(-k\eta)) .$$

Set $W = \{a_0 u + b_0 v : |a_0| < |b_0|\}$. Then if $w \in W$, we obtain

$$(2.2)_c\quad \begin{aligned} &|a_n| \leqq |b_0| \prod_{k=1}^{n} (1-\exp(-k\eta)) C' \prod_{k=1}^{n} t_k^* \\ &|b_0| \prod_{k=1}^{n} (1-\exp(-k\eta)) \prod_{k=1}^{n} s_k \leqq |b_n| \leqq |b_0| \prod_{k=1}^{n} (1-\exp(-k\eta)) C' \prod_{k=1}^{n} s_k . \end{aligned}$$

From $(2.2)_c$, we have

$$\begin{aligned} \| T'^n w \| &\geqq e(\eta) |b_n| \\ &\geqq e(\eta) |b_0| \prod_{k=1}^{n} (1-\exp(-k\eta)) \prod_{k=1}^{n} s_k \end{aligned}$$

where the constant $e(\eta)$ only depends on $\eta$. As $\prod_{k=1}^{n} s_k = \| T^n v \|$, by Lemma 4 we have

$$\| T'^n w \| > e(\eta) d |b_0| \prod_{k=1}^{\infty} (1-\exp(-k\eta)) \lambda_n .$$

Write $L_1(\eta) = \prod_{k=1}^{\infty} (1-\exp(-k\eta)) d e(\eta)$. Then we obtain

$$(2.3)\quad \lambda_n' > L_1(\eta) \lambda_n \quad (n>0) .$$

Let $T'$ be a *$\nu$-perturbation of* $T$ and write for $k \leqq \nu$

$$T'^k=\begin{pmatrix}\alpha'_k & \gamma'_k\\ \chi_k & \beta'_k\end{pmatrix}.$$

Then we have

$$T'^n=T_n\circ\cdots\circ T_{\nu+1}T'^\nu=\begin{pmatrix}\alpha_n/\alpha_\nu & *\\ 0 & \beta_n/\beta_\nu\end{pmatrix}T'^\nu$$
$$=\begin{pmatrix} * & *\\ (\beta_n/\beta_\nu)\chi_\nu & (\beta_n/\beta_\nu)\beta'_\nu\end{pmatrix}$$

and hence

$$(T'^n)({}^tT'^n)=\begin{pmatrix} * & *\\ * & ((\beta_n/\beta_\nu)\chi_\nu)^2+((\beta_n/\beta_\nu)\beta'_\nu)^2\end{pmatrix}.$$

As $\|T'^n\|=\|{}^tT'^n\|$, we obtain for $n>\nu$

$$\mu'_n<\sqrt{\left(\frac{\chi_\nu}{\beta_\nu}\right)^2+\left(\frac{\beta'_\nu}{\beta_\nu}\right)^2}|\beta_n|.$$

Therefore it follows from (2.3) that

$$\frac{\mu'_n}{\lambda'_{n+1}}<\frac{\sqrt{(\chi_\nu/\beta_\nu)^2+(\beta'_\nu/\beta_\nu)^2}}{L_1(\eta)}\left|\frac{\beta_n}{\alpha_{n+1}}\right|\quad(n>\nu).$$

As $\nu$ is fixed, by Lemma 2 we see that

$$\mathrm{dis}\,(U'_n,\,U'_{n+1})\leqq 2C_2(\varepsilon)\exp(n\varepsilon)\frac{\sqrt{(\chi_\nu/\beta_\nu)^2+(\beta'_\nu/\beta_\nu)^2}}{L_1(\eta)}\exp(-n\xi).$$

This implies that $\{U'_n\}_{n>0}$ is a Cauchy sequence and $\lim U'_n=U'$ and $\lim V'_n=V'$ exist.

Moreover if $|a_n|\geqq|b_n|$ for all $n\geqq 0$, then by using Lemma 5 we have

$$|a_n|\leqq t_n|a_{n-1}|+\frac{\|T'_n-T_n\|}{|\det({}^nu,\,{}^nv)|}(|a_{n-1}|+|b_{n-1}|)$$
$$\leqq t_n|a_{n-1}|(1+2CD\delta\exp(-n\eta)).$$

Hence by induction, we have the following:

$$(2.4)_a\qquad |b_n|\leqq|a_n|\leqq C'\prod_{k=1}^n t_k\prod_{k=1}^n(1-\exp(-k\eta))|a_0|\quad(n>0).$$

Similarly we have

$$(2.4)_b\qquad \prod_{k=1}^n t_k\prod_{k=1}^n(1-\exp(-k\eta))|a_0|\leqq|a_n|\quad(n>0).$$

Because of $(2.2)_b$, it follows that $u \in U'$ implies $|a_n| \geqq |b_n|$ for all $n \geqq 0$. Hence for $u \in U'$ we have

$$\begin{aligned} \| T'^n u \| &\leqq |a_n| + |b_n| \\ &\leqq 2|a_0|C'\| T^n u \| \\ &\leqq 2\| u \| C'(\eta)K(\varepsilon)\lambda_n \exp(-n(\xi-\varepsilon)) \quad (n>0)\,. \end{aligned}$$

Thus by Lemma 1 there is $F(\varepsilon, \eta)>1$ such that for all $u \in U'$,

$$\| T'^n u \| \leqq \| u \| \mu_n \exp(n(\tilde{\xi}-\xi)+2\varepsilon)F(\varepsilon, \eta) \quad (n>0)\,. \tag{2.5}$$

The proof of Theorem 2 shows that $U'$ can not intersect with $W$. Moreover we have the following

THEOREM 3. *Let $T$, $T'$ and $\delta$ be as in Theorem 2 and suppose that $\| T'-T \|_\eta \leqq \delta \cdot \alpha$ where $0<\alpha\leqq 1$. Then $W_\alpha \cap U' = \emptyset$, where $W_\alpha = \{a_0(u/\alpha) + b_0 v : |a_0| < |b_0|\}$. Furthermore there is $A(\delta) \equiv A > 0$ so that*

$$\| P^\lambda(T') - P^\lambda(T) \| \leqq \| T'-T \|_\eta A \quad \textit{and}$$
$$\| P^\mu(T') - P^\mu(T) \| \leqq \| T'-T \|_\eta A$$

*where $P^\mu(T')$ and $P^\lambda(T')$ denote the orthogonal projections to $U'$ and $V'$ respectively.*

COROLLARY 4. *If $\| T''-T \|_\eta \leqq \delta$, then there is $A(\delta) \equiv A > 0$ so that*

$$\| P^\lambda(T') - P^\lambda(T'') \| \leqq \| T'-T'' \|_\eta A \quad \textit{and}$$
$$\| P^\mu(T') - P^\mu(T'') \| \leqq \| T'-T'' \|_\eta A\,.$$

PROOF OF THEOREM 3. By estimating the components $|a_n|$, $|b_n|$ of $T'^n w$ along ${}^n u/\alpha$, ${}^n v$ for $w \neq 0$ in $\mathbf{R}^2$, we have $W_\alpha \cap U' = \emptyset$ similarly to the case $\alpha=1$. Therefore, for $u' \in U'$, $\| P^\lambda(T)u' \| \leqq \alpha \| u' \|$ holds. Indeed if $w = a_0(u/\alpha) + b_0 v \notin W_\alpha$, then we have

$$\begin{aligned} \| P^\lambda(T)w \| &= |b_0| \lneqq |a_0| \\ &\leqq \left\| a_0 \frac{u}{\alpha} + b_0 v \right\| \alpha \\ &= \alpha \| w \|\,. \end{aligned}$$

Thus for $v \in V$ we have

$$\| P^\mu(T')v \| \leqq \alpha \| v \|\,,$$

and then it is easy to see that

$$\|(1-P^\lambda(T'))P^\lambda(T)\|=\|(1-P^\lambda(T'))v\|\leqq\alpha ,$$

where $v$ is a unit vector in the range of $P^\lambda(T)$. Hence it follows that

$$\begin{aligned}\|P^\lambda(T)-P^\lambda(T')\|&=\|(1-P^\lambda(T'))P^\lambda(T)-P^\lambda(T')(1-P^\lambda(T))\|\\&\leqq\|(1-P^\lambda(T'))P^\lambda(T)\|+\|(1-P^\mu(T'))P^\mu(T)\|\\&\leqq 2\alpha .\end{aligned}$$

If $\|T'-T\|_\eta\leqq\delta$, then we can take $\alpha=\|T'-T\|_\eta/\delta\leqq 1$. Put

$$A=\frac{2}{\delta} . \tag{3.1}$$

Thus we have

$$\|P^\lambda(T)-P^\lambda(T')\|\leqq A\|T'-T\|_\eta .$$

PROOF OF COROLLARY 4. We obtain Corollary 4 from Theorem 3 by replacing $T$ by $T''$ if the constant $A$ does not depend on $T''$. In view of (3.1) this is achieved if we can replace $T$ by $T''$ in Theorem 2 and get bounds on $1/\delta''$ uniform in $T''$. In view of the choice of $\delta$ in the proof of Theorem 2, $\delta''$ is given by

$$\delta''=\frac{1}{2C''(\eta)D''_\eta}\prod_{k=1}^{\infty}(1-\exp(-k\eta))^2 .$$

Therefore it suffices to find upper bounds of $C''(\eta)$ and $D''_\eta$, which do not depend on $T''$. In fact, in view of (2.2) and (2.4), by using Lemma 5, the conclusion is obtained easily.

## § 2. The existence of stable foliations for two-dimensional mappings.

Denote by $B(z,\alpha)$ the open ball of radius $\alpha$ centered at $z$ in $\mathrm{R}^2$ and by $\overline{B(z,\alpha)}$ its closure.

Let $H$: $\mathrm{R}^2\to\mathrm{R}^2$ be a differentiable map of $\mathrm{C}^2$-class.

We write, for $z\in\mathrm{R}^2$,

$$\begin{aligned}&F_n(u)=H(H^{n-1}(z)+u)-H^n(z) , &&F^n=F_n\circ\cdots\circ F_1\\&T_n=DF_n(0)\ (=DH(H^{n-1}(z))) , &&T^n=T_n\circ\cdots\circ T_1 .\end{aligned}$$

Now, we can state our main result.

MAIN THEOREM. *Let $H$: $\mathrm{R}^2\to\mathrm{R}^2$ be a differentiable map of $\mathrm{C}^2$-class. Assume that there is a compact set $\Gamma\subset\mathrm{R}^2$ such that $H(\Gamma)\subset\Gamma$ and that the tangent map $DH(z)$ is expressed by an upper triangular matrix for* *all* $z\in\Gamma$. *We write for* $z\in\Gamma$,

$$T_n=\begin{pmatrix}\tilde{\alpha}_n & \tilde{\gamma}_n\\ 0 & \tilde{\beta}_n\end{pmatrix} \quad \text{and} \quad T^n=\begin{pmatrix}\alpha_n & \gamma_n\\ 0 & \beta_n\end{pmatrix}.$$

*Suppose the following*:

(B)$_1$ *There exists* $\xi>0$ *such that* $|\beta_n/\alpha_{n+1}|<\exp(-n\xi)$ *for all* $n>0$,

(B)$_2$ $\tilde{\xi}\equiv\limsup_n (1/n)\log|\alpha_n/\beta_n|>\xi$,

(B)$_3$ $\det T_n\neq 0$ *for all* $n>0$,

(B)$_4$ $\lim_n (1/n)\log|\det T_n|=0$,

(B)$_5$ $\hat{\beta}\equiv\limsup_n (1/n)\log|\beta_n|<0$ *and* $5(\tilde{\xi}-\xi)<-\hat{\beta}$.

*Let* $\theta$ *be a constant such that* $-4(\tilde{\xi}-\xi)>\theta>\hat{\beta}+(\tilde{\xi}-\xi)$. *Under these conditions, there are constants* $\alpha_z>0$, $\gamma_z>0$ *and* $\pi_z>0$ *with the following properties*:

(I) $S(\pi_z)=\{u\in\overline{B(z,\alpha_z)}: \|H^n(z+u)-H^n(z)\|\leqq\pi_z\exp(n\theta)$ *for all* $n\geqq 0\}$ *is a one-dimensional* $C^1$*-submanifold of* $\overline{B(z,\alpha_z)}$, *tangent at* $z$ *to* $U_z$.

(II) *If* $u, v\in S(\pi_z)$, *then* $\|H^n(z+u)-H^n(z+v)\|\leqq\gamma_z\|u-v\|\exp(n\theta)$ *for all* $n\geqq 0$.

PROOF OF MAIN THEOREM. Let $\eta>0$ such that $\tilde{\xi}-\xi<\eta<-\theta/4$. We may then write

$$G=\sup_k M_k\exp(-k\eta)\exp(4\eta)\,, \tag{1}$$

where $M_k$ denotes the Lipschitz constant of $DF_k$ in $\overline{B(H^{k-1}(z),1)}$. Given $\pi$ with $0<\pi\leqq 1$, we put

$$\begin{aligned} S^\nu(\pi)&=\{u\in\mathbf{R}^2: \|F^n u\|\leqq\pi\exp(n\theta) \text{ for } 0\leqq n\leqq\nu\}\\ S(\pi)&=\{u\in\mathbf{R}^2: \|F^n u\|\leqq\pi\exp(n\theta) \text{ for } n\geqq 0\}\,. \end{aligned} \tag{2}$$

There are $\delta$, $A>0$ such that Theorems 2 and 3 hold for the above $\eta$ and for $T_n$. We can make $\delta$ smaller so that

$$A\delta<\frac{1}{\sqrt{2}} \tag{3}$$

and then choose $\pi$ satisfying

$$0<\pi<1\,,\qquad G\pi<\delta\,.$$

Take $\kappa>1$ such that $\kappa\pi\leqq 1$, $G(\kappa\pi)\leqq\delta$.

ASSERTION. *There is* $\alpha>0$ *such that, for all* $\nu>0$,

$$B(\alpha)\cap S^\nu(\pi)\cap(F^\nu)^{-1}(T^\nu U)\supset B(\alpha)\cap S^\nu(\kappa\pi)\cap(F^\nu)^{-1}(T^\nu U)\,, \tag{4}$$

*where* $U=\lim U_n$.

**Proof.** For $u\in S^{\nu}(\kappa\pi)\cap(F^{\nu})^{-1}(T^{\nu}U)$, the $2\times 2$ matrices:

$$ {}^{(u)}_{(\nu)}T'_n=\begin{cases}\int_0^1 DF_n(tF^{n-1}u)dt & \text{if} \quad n\leqq\nu\\ T_n & \text{if} \quad n>\nu\,,\end{cases} $$

satisfy ${}^{(u)}_{(\nu)}T'^n u=F^n u$ if $n\leqq\nu$, and thus by using (1), (2) we obtain

$$ \begin{aligned} \|T'_n-T_n\| &\leqq\int_0^1 \|DF_n(tF^{n-1}u)-DF_n(0)\|dt\\ &\leqq\int_0^1 M_n\|tF^{n-1}u\|dt\\ &\leqq M_n\exp((n-1)\theta)\frac{\kappa\pi}{2}\\ &\leqq\delta\exp(-3n\eta)\quad(n>0)\,. \end{aligned} $$

Therefore Theorems 2, 3 can be applied. Moreover, since

$$ {}^{(u)}_{(\nu)}T'^{k+\nu}u=T_{k+\nu}\circ\cdots\circ T_{\nu+1}(F^{\nu}u)=T^{k+\nu}((T^{\nu})^{-1}F^{\nu}u)\,, $$

it follows from (1.1) and (2.3) that $u\in(F^{\nu})^{-1}T^{\nu}U$ implies $u\in{}^{(u)}_{(\nu)}U'$ for all $\nu>0$.

Let $\varepsilon_0=(\theta-(\hat{\beta}+\tilde{\xi}-\xi))/2>0$. Then (2.5) gives

$$ \begin{aligned} \|{}^{(u)}_{(\nu)}T'^n u\| &\leqq\|u\|F(\varepsilon_0,\eta)\mu_n\exp(n((\tilde{\xi}-\xi)+2\varepsilon_0))\\ &\leqq\|u\|F(\varepsilon_0,\eta)\exp(n(\tilde{\xi}-\xi+\hat{\beta}+2\varepsilon_0))\\ &=\|u\|F(\varepsilon_0,\eta)\exp(n\theta)\quad(n>0)\,. \end{aligned} $$

It holds uniformly in $\nu$ and in $u\in S^{\nu}(\kappa\pi)\cap(F^{\nu})^{-1}(T^{\nu}U_{\varepsilon})$. If $\alpha=\pi/F(\varepsilon_0,\eta)$ $(<\pi)$, (4) is obtained.

**Lemma.** $\overline{B(\alpha)}\cap S^{\nu}(\pi)\cap(F^{\nu})^{-1}(T^{\nu}U)\equiv D^{\nu}(\alpha)$ *is open and closed in* $\overline{B(\alpha)}\cap(F^{\nu})^{-1}(T^{\nu}U)$. *Indeed, the boundary of* $S^{\nu}(\kappa\pi)$ *is disjoint with* $S^{\nu}(\pi)$, *and hence with* $D^{\nu}(\alpha)$.

Let now $u, v\in D^{\nu}(\alpha)$ (or $u, v\in S(\pi)\cap\overline{B(\alpha)}$). The $2\times 2$ matrices

$$ {}^{(u,v)}_{(\nu)}T'_n=\begin{cases}\int_0^1 DF_n(tF^{n-1}u+(1-t)F^{n-1}v)dt & \text{if} \quad n\leqq\nu\\ T_n & \text{if} \quad n>\nu\end{cases} $$

satisfy

$$ {}^{(u,v)}_{(\nu)}T'^n(u-v)=F^n u-F^n v \quad \text{if} \quad n\leqq\nu $$

and using (1) and (2), we obtain

$$\begin{aligned}\| T'_n - T_n \| &\leqq 2G\pi \exp(-3n\eta) \\ &\leqq \delta \exp(-3n\eta) \quad (n>0) .\end{aligned}$$

(We choose $\pi$ satisfying $2G\pi<\delta$). Hence Theorems 2, 3 can be applied. Since

$$^{(u,v)}_{(\nu)}T'^{k+\nu}(u-v) = T^{k+\nu}(T^\nu)^{-1}(F^\nu u - F^\nu v)$$

and $F^\nu u$, $F^\nu v \in T^\nu U$, it follows that

$$u-v \in {}^{(u,v)}_{(\nu)}U' \quad \text{for all} \quad \nu>0 .$$

Therefore by (2.5) we have for all $n \leqq \nu$

$$\begin{aligned}\| {}^{(u,v)}_{(\nu)}T'^n(u-v) \| &= \| F^n u - F^n v \| \\ &\leqq \| u-v \| F(\varepsilon_0, \eta) \exp(n\theta) .\end{aligned}$$

Writting $\gamma_z = F(\varepsilon_0, \eta) > 1$, we obtain, for all $n \leqq \nu$,

(5) $$\| F^n u - F^n v \| \leqq \gamma_z \| u-v \| \exp(n\theta) .$$

This proves part (II) of the Main Theorem.

By Theorem 3 we obtain

$$\begin{aligned}\| (1-P^\mu(T))(u-v) \| &= \| (P^\mu(T') - P^\mu(T))(u-v) \| \\ &\leqq A \| T'-T \|_\eta \| u-v \| \\ &\leqq A\delta \| u-v \| .\end{aligned}$$

Since $\| u-v \|^2 = \| P^\mu(T)(u-v) \|^2 + \| P^\lambda(T)(u-v) \|^2$, we have

$$\| P^\mu(T)(u-v) \|^2 \geqq \| u-v \|^2 (1-A^2\delta^2)$$

and hence

(6) $$\| (1-P^\mu(T))(u-v) \| \leqq \frac{A\delta}{\sqrt{1-(A\delta)^2}} \| P^\mu(T)(u-v) \| \quad \text{holds} .$$

We write $u=u_1+u_2$ and $v=v_1+v_2$ where $u_1, v_1 \in U$ and $u_2, v_2 \in V$. Since $u \in {}^{(u)}_{(\nu)}U'$ and $v \in {}^{(v)}_{(\nu)}U'$, by Theorem 3 we have

(7) $$\begin{aligned}\| u_2 \| &= \| u-u_1 \| \\ &= \| P^\mu({}^{(u)}_{(\nu)}T')u - P^\mu(T)u \| \\ &\leqq A \| T'-T \|_\eta \| u \| \\ &\leqq A\delta\alpha .\end{aligned}$$

Similarly we have

$$(8) \qquad \|v_2\| \leqq A\delta\alpha\ .$$

Moreover it follows from (6) that

$$(9) \qquad \|u_2-v_2\|=\|(1-P^{\mu}(T))(u-v)\| \leqq \frac{A\delta}{\sqrt{1-(A\delta)^2}}\|P^{\mu}(T)(u-v)\|$$

$$=\frac{A\delta}{\sqrt{1-(A\delta)^2}}\|u_1-v_1\|\ .$$

This situation will be considered in $(U\cap\overline{B(\alpha)})\times(V\cap\overline{B(\alpha)})$.

Define $\Phi\colon (U\cap\overline{B(\alpha)})\times(V\cap\overline{B(\alpha)})\to\overline{B(\alpha)}$ by putting

$$\Phi(u_1, u_2)=\left(\frac{u_1}{\alpha}\sqrt{\alpha^2-\|u_2\|^2},\ u_2\right).$$

Let $\Phi(u_1, u_2)$, $\Phi(v_1, v_2)\in D^{\nu}(\alpha)$ or $\in\overline{B(\alpha)}\cap S(\pi)$. Then it follows from (7), (8) and (9) that

$$\|u_2-v_2\|\frac{\sqrt{1-(A\delta)^2}}{A\delta}\leqq\left\|\frac{u_1}{\alpha}\sqrt{\alpha^2-\|u_2\|^2}-\frac{v_1}{\alpha}\sqrt{\alpha^2-\|v_2\|^2}\right\|$$

$$\leqq\frac{\|u_1\|}{\alpha}\left|\sqrt{\alpha^2-\|u_2\|^2}-\sqrt{\alpha^2-\|v_2\|^2}\right|$$

$$+\|u_1-v_1\|\frac{\sqrt{\alpha^2-\|v_2\|^2}}{\alpha}$$

$$=\frac{\|u_1\|}{\alpha}\left|\frac{\|v_2\|^2-\|u_2\|^2}{\sqrt{\alpha^2-\|u_2\|^2}+\sqrt{\alpha^2-\|v_2\|^2}}\right|+\|u_1-v_1\|$$

$$\leqq\frac{\|u_1\|}{\alpha}\frac{|(\|v_2\|+\|u_2\|)(\|v_2\|-\|u_2\|)|}{2\alpha\sqrt{1-(A\delta)^2}}+\|u_1-v_1\|$$

$$\leqq\frac{A\delta}{\sqrt{1-(A\delta)^2}}\|v_2-u_2\|+\|u_1-v_1\|\ .$$

Thus we have

$$(10) \qquad \frac{\|u_2-v_2\|}{\|u_1-v_1\|}\leqq\frac{A\delta\sqrt{1-(A\delta)^2}}{1-2(A\delta)^2}\ .$$

In view of (3)

$$\frac{A\delta\sqrt{1-(A\delta)^2}}{1-2(A\delta)^2}>0\ .$$

As $D^{\nu}(\alpha)$ is open and closed in $\overline{B(\alpha)}\cap(F^{\nu})^{-1}T^{\nu}U$ by Lemma, we conclude from (10) that $D^{\nu}(\alpha)$ is the connected component of 0 in $\overline{B(\alpha)}\cap(F^{\nu})^{-1}T^{\nu}U$. Furthermore $\Phi^{-1}(D^{\nu}(\alpha))$ is the graph of Lipschitz-continuous function $\varphi_{\nu}$: $U\cap\overline{B(\alpha)}\rightarrow V\cap\overline{B(\alpha)}$ with the Lipschitz constant bounded uniformly with respect to $\nu$.

Let $\varphi$ be a limit of a uniformly convergent subsequence of $\{\varphi_{\nu}\}$. As $\Phi(\text{graph }\varphi_{\nu})=D^{\nu}(\alpha)\subset\overline{B(\alpha)}\cap S^{\nu}(\pi)$, we obtain

$$\Phi(\text{graph }\varphi)\subset\overline{B(\alpha)}\cap S(\pi)\ .$$

Since $D^{\infty}(\alpha)\supset S(\pi)\cap\overline{B(\alpha)}$, it follows that $\Phi(\text{graph }\varphi)=\overline{B(\alpha)}\cap S(\pi)$ and by the uniqueness of $\varphi$, $\lim_{\nu}\varphi_{\nu}=\varphi$ uniformly. Therefore it follows from (10) that $\overline{B(\alpha)}\cap S(\pi)$ is Lipschitz-continuous (since (10) holds for $\nu=\infty$, similarly).

Finally, we show that $\Phi^{-1}(D^{\nu}(\alpha))$ is the graph of a $C^1$-function $\varphi_{\nu}$ and thus $\lim\varphi_{\nu}=\varphi$ is of $C^1$-class.

Let $u, v\in D^{\nu}(\alpha)$ and define $2\times 2$ matrices;

$$T'_n=DF_n(F^{n-1}u)\ ,\qquad T''_n=DF_n(F^{n-1}v)\quad\text{if}\quad n\leqq\nu\ ,$$
$$T'_n=T''_n=T_n\quad\text{if}\quad n>\nu\ .$$

Then it is easy to see that $\|T'-T\|_{\eta}$, $\|T''-T\|_{\eta}<\delta$. Using (5), we also have

$$\|T'_n-T''_n\|\leqq M_n\gamma_s\|u-v\|\exp((n-1)\theta)$$
$$\leqq G\gamma_s\|u-v\|\exp(-3n\eta)\quad(n\leqq\nu)\ .$$

Therefore $\|T'-T''\|_{\eta}\leqq G\gamma_s\|u-v\|$. By Corollary 4, we have

$$\|P^{\mu}(T')-P^{\mu}(T'')\|\leqq A\|T'-T''\|_{\eta}$$
$$\leqq AG\gamma_s\|u-v\|\ ,$$

where the ranges of $P^{\mu}(T')$ and $P^{\mu}(T'')$ are the tangent spaces to $D^{\nu}(\alpha)$ at $u$ and $v$. Thus $\{\varphi'_{\nu}\}$ is a family of Lipschitz-continuous functions with the Lipschitz constant bounded uniformly with respect to $\nu$ and thus $\lim\varphi_{\nu}=\varphi$ is of $C^1$-class.

## § 3. Applications.

In this section, we apply Main Theorem to our problem. We recall the two-dimensional mapping $H$: $\mathbf{R}^2\rightarrow\mathbf{R}^2$ defined by

$$H(x, y)=(f(x)+\varepsilon_1(x, y),\ \mu y+\varepsilon_2(x, y))\ ,$$

where $\varepsilon_i(x, y)$ $(i=1, 2)$ is of $C^2$-class, and $f$: $\mathbf{R}\rightarrow\mathbf{R}$ is a one-dimensional mapping of piecewise $C^2$-class such that $f(I)\subset I$. Hence $I=[0,1]$. We assume further the following conditions on $\varepsilon_i(x, y)$ $(i=1, 2)$ and on $\mu$:

$$\varepsilon_i(x, 0)=0\,, \qquad \left|\frac{\partial\varepsilon_i}{\partial y}(x, 0)\right|<\hat{\varepsilon}_i \quad (i=1, 2) \quad \text{for} \quad (x, 0)\in I\times\{0\}$$

and

$$\mu>\hat{\varepsilon}_2\,, \qquad \mu+\hat{\varepsilon}_1+\hat{\varepsilon}_2<1\,.$$

Namely, $H$ is supposed to leave the set $I\times\{0\}\subset\mathrm{R}^2$ invariant, and the tangent mapping $DH(x, 0)$ at $(x, 0)\in I\times\{0\}$ is expressed by an upper triangular matrix

$$\begin{pmatrix} f'(x) & \dfrac{\partial\varepsilon_1}{\partial y}(x, 0) \\ 0 & \mu+\dfrac{\partial\varepsilon_2}{\partial y}(x, 0) \end{pmatrix}.$$

Hereafter, we use the notation $f'(x)$ also to denote the right or left differential coefficient at a discontinuity point $x$, instead of $f'_{\pm}(x)$.

DEFINITION. We say that $H$ has an invariant stable foliation on $J\times\{0\}$ for $f$-invariant set $J\subset I$, if there exists a family of curves $\{C_x\colon x\in J\}$, called leaves, such that

(1) $C_x$ are Lipschitz-continuous curves,

(2) $(x, 0)\in C_x$,

(3) $H(C_x)\subset C_{f(x)}$ and

(4) there are $\lambda<0$ and $\gamma_x>0$ such that for $z_1, z_2\in C_x$

$$\|H^n(z_1)-H^n(z_2)\|\leqq\gamma_x \exp(n\lambda)\|z_1-z_2\| \quad (n\geqq 0)\,.$$

In this situation, we have the following results.

THEOREM 4. *Let $f$ be of piecewise $C^2$-class and let $\inf_{x\in I}|f'(x)|>0$. Let $\mu$, $\hat{\varepsilon}_i$ $(i=1, 2)$ be sufficiently small. (More precisely, we choose $\mu$, $\hat{\varepsilon}_i$ as*

$$\inf_{x\in I}|f'(x)|>\mu+\hat{\varepsilon}_2 \quad \textit{and} \quad \left\{\frac{\sup_{x\in I}|f'(x)|}{\inf_{x\in I}|f'(x)|}\right\}^5<\frac{(\mu-\hat{\varepsilon}_2)^5}{(\mu+\hat{\varepsilon}_2)^6}\Big)\,.$$

*Then the map $H$ has an invariant stable foliation on $I\times\{0\}$.*

THEOREM 5. *Let $f(x)=Ax(1-x)$, where $0<A\leqq 4$. Suppose that $f$ has a stable periodic point $x_0\in I\backslash\Lambda$, where $\Lambda$ denotes the set $\{x\in I\colon f^k(x)=1/2$ for some $k\geqq 0\}$, and $\mu$, $\hat{\varepsilon}_i$ are sufficiently small. Then there exists $f$-invariant set $\tilde{I}$ such that the Lebesgue measure of $\tilde{I}$ is equal to 1 and*

*the map $H$ has an invariant stable foliation on $\tilde{I}\times\{0\}$ whose leaves are of $C^1$-class.*

We remark that the conditions on $f$ can be considerably weakened regardless of the simplicity of their expression. For the proof of Theorems 4 and 5, it is useful to prepare the following theorem which gives sufficient conditions for the existence of an invariant stable foliation.

THEOREM 6. *Suppose the following properties hold for $x\in I$:*
$(C)_1$ *There exists*

$$\Theta_1(x)\equiv\liminf_n \frac{1}{n}\log|(f^n)'(x)| \quad \textit{and} \quad \Theta_1(x)>\log(\mu+\hat{\varepsilon}_2)\,,$$

$(C)_2$ $f'(f^{n-1}(x))\neq 0$ *for all* $n\geqq 1$,
$(C)_3$ $\lim(1/n)\log|f'(f^{n-1}(x))|=0$,
$(C)_4$ *there exists*

$$\Theta_2(x)\equiv\limsup_n \frac{1}{n}\log|(f^n)'(x)| \quad \textit{and}$$

$$5(\Theta_2(x)-\Theta_1(x))<\log\frac{(\mu-\hat{\varepsilon}_2)^5}{(\mu+\hat{\varepsilon}_2)^6}\,.$$

*Then, an invariant stable foliation exists on the orbit of* $(x, 0)$.

PROOF OF THEOREM 6. Let

$$T^n=T_n\circ\cdots\circ T_1=\begin{pmatrix}\alpha_n & \gamma_n\\ 0 & \beta_n\end{pmatrix},$$

where

$$T_n=DH(H^{n-1}(x, 0))=\begin{pmatrix} f'(f^{n-1}(x)) & \dfrac{\partial\varepsilon_1}{\partial y}(f^{n-1}(x), 0)\\ 0 & \mu+\dfrac{\partial\varepsilon_2}{\partial y}(f^{n-1}(x), 0)\end{pmatrix}.$$

It is easy to see that

$$\alpha_n=(f^{n\prime})(x) \qquad \text{and} \qquad \beta_n=\prod_{k=0}^{n-1}\left(\mu+\frac{\partial\varepsilon_2}{\partial y}(f^k(x), 0)\right).$$

We can verify easily that $(C)_2$ and $(C)_3$ imply $(B)_3$ and $(B)_4$ respectively. Note that

$$\left|\frac{\beta_n}{\alpha_{n+1}}\right|\leqq\frac{(\mu+\hat{\varepsilon}_2)^n}{|(f^{n+1})'(x)|} \quad \text{and} \quad \left|\frac{\alpha_n}{\beta_n}\right|\leqq\frac{|(f^n)'(x)|}{(\mu-\hat{\varepsilon}_2)^n}\,.$$

We replace in Main Theorem the constants $\xi$ and $\tilde{\xi}$ by $(\Theta_1(x)-\log(\mu+\hat{\varepsilon}_2))$ and $(\Theta_2(x)-\log(\mu-\hat{\varepsilon}_2))$ respectively. Then Theorem 6 is an immediate consequence of Main Theorem.

LEMMA. *As $f$ is of piecewise $C^2$-class, we can take* $\log\{\inf_{x\in I}|f'(x)|\}$, $\log\{\sup_{x\in I}|f'(x)|\}$ *as* $\Theta_1(x)$, $\Theta_2(x)$.

PROOF OF THEOREM 4. Immediate by Theorem 6 and the above lemma.

PROOF OF THEOREM 5. At first, we assume that $f$ has a stable fixed point $x_0$ (i.e., $1<A\leqq 3$). In the case that $x_0$ is a periodic point with period $k>1$, a similar argument would be valid. Since the Lebesgue measure of those points not converging to the stable "periodic orbit" is zero, we have

$$\lim_n \frac{1}{n}\log|f'(f^{n-1}(x))|=0\ , \quad \text{and}$$

$$\lim_n \frac{1}{n}\log|(f^n)'(x)|=\log|f'(x_0)| \quad \text{almost everywhere}\ .$$

Choosing $\mu$, $\hat{\varepsilon}_2$ satisfying

$$|f'(x_0)|=\left|f'\left(\frac{A-1}{A}\right)\right|=|A-2|>\mu+\hat{\varepsilon}_2 \quad \text{and} \quad \frac{(\mu-\hat{\varepsilon}_2)^5}{(\mu+\varepsilon_2)^6}>1\ ,$$

we complete the proof of Theorem 5.

EXAMPLE. Let us consider the one-dimensional mapping $f:[0,1]\to[0,1]$, defined by

$$f(x)=\begin{cases}4\left(\dfrac{\sqrt{5}-1}{2}x+\dfrac{1}{4}\right)^2-\dfrac{1}{4} & \text{on } \left[0,\dfrac{1}{2}\right]\\[2ex] 4\left(\dfrac{1-\sqrt{5}}{2}x+\dfrac{2\sqrt{5}-1}{4}\right)^2-\dfrac{1}{4} & \text{on } \left[\dfrac{1}{2},1\right],\end{cases}$$

and take $\mu$, $\hat{\varepsilon}_i$ with

$$(\sqrt{5})^5<\frac{(\mu-\hat{\varepsilon}_2)^5}{(\mu+\hat{\varepsilon}_2)^6}\ .$$

In this case $H$ satisfies the assumption of Theorem 4.

## References

[1] E. N. LORENZ, Deterministic nonperiodic flow, J. Atmospheric Sci., **20** (1963), 130-141.

[2] J. GUCKENHEIMER, G. OSTER and A. IPAKTCHI, The dynamics of density dependent population models, J. Math. Biol., **4** (1977), 101-147.

[3] O. E. RÖSSLER, An equation for continuous chaos, Phys. Lett., **57** A, (1976), 397-398.

[4] I. SHIMADA and T. NAGASHIMA, On the *C*-system-like property of the Lorenz system, Progr. Theoret. Phys., **58** (1978), 1318-1319.

[5] R. M. MAY, Simple mathematical models with very complicated dynamics, Nature, **261** (1976), 459-467.

[6] R. SHAW, Strange attractors, chaotic behavior and information flow, Z. Naturforsch., **36** (a) (1981), 80-112.

[7] D. RUELLE, Ergodic Theory of Differentiable Dynamical System, Publ. Math. I.H.E.S., **50** (1981), 27-58.

[8] YA. B. PESIN, Lyapunov characteristic exponents and smooth ergodic theory, Uspehi Mat. Nauk., **32** n. 4 (196) (1977), 55-112.

[9] V. I. OSELEDIC, A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical system, Trudy Mosk. Mat. Obšč., **19** (1968), 179-231.

[10] M. YURI, A construction of an invariant stable foliation by the shadowing lemma, Tokyo J. Math., preprint.

*Present Address*:
DEPARTMENT OF MATHEMATICS
TSUDA COLLEGE
KODAIRA, TOKYO 187