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On a Parameter Dependence of Solvability of the Dirichlet Problem for Non-Parametric Surfaces of Prescribed Mean Curvature

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Introduction

The problem to find a surface having a given function as the mean curvature has been studied for a long time. A particular problem of this type, called the non-parametric problem, can be reduced to solve the Dirichlet problem for the following quasilinear elliptic equation

$$(*) \qquad -\operatorname{div}\{\nabla u/(1+|\nabla u|^2)^{1/2}\}=c \quad \text{in } \Omega\ , \quad u=\phi \text{ on } \partial\Omega\ .$$

Here Ω is a bounded domain in $\boldsymbol{R}^n$ ($n\geqq 2$), c is a given function on Ω and ϕ is a given boundary value. Since Eq. (*) is nonlinear and non-uniformly elliptic, we cannot expect in general that Eq. (*) has a classical solution for generic Ω, c and ϕ. In fact, some kinds of necessary condition on Ω and c are found by many people (see [3], [5], [7], [8], [9], [12]).

In this paper, as an approach to the problem, we introduce a parameter T (>0) into Eq. (*) as follows.

$$(*)_T \qquad -\operatorname{div}\{\nabla u/(1+|\nabla u|^2)^{1/2}\}=Tc \quad \text{in } \Omega\ , \quad u=\phi \text{ on } \partial\Omega\ ,$$

where c is supposed nonnegative and bounded in Ω. And we investigate how the solvability of Eq. $(*)_T$ depends on the parameter T. For this purpose we first consider the variational problem of finding a functional belonging to $BV(\Omega)$ which minimizes the functional

$$(0.1) \qquad J_T(u)=\int_\Omega (1+|\nabla u|^2)^{1/2}dx-T\int_\Omega cu\,dx+\int_{\partial\Omega}|u-\phi|\,dH_{n-1}$$

in $BV(\Omega)$. Here $BV(\Omega)$ is the space consisting of functions of bounded variation in Ω and H_{n-1} denotes $(n-1)$-dimensional Hausdorff measure. For this variational problem we first consider the condition when J_T is

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bounded from below on $BV(\Omega)$. Our result is that J_T is not bounded from below on $BV(\Omega)$ in case T is larger than a critical parameter T^* (>0) defined by

$$T^* = \inf_{E\subset\Omega} \{H_{n-1}(\partial E)/\text{meas}_c(E)\} \qquad \text{where} \quad \text{meas}_c(E) = \int_E c(x)dx\ .$$

The condition of this type was obtained by Mosolov [11] for the first time. However, his result [11] contains some inessential assumption because he formulated the problem in the Sobolev space $W^{1,1}(\Omega)$. And also since $W^{1,1}(\Omega)$ is not reflexive, he dose not give the solution of the variational problem. Though several people studied similar variational problems in the space $BV(\Omega)$ (for example, see [3], [5], [7]), their condition for the lowerboundedness of the functional less clarify the relation between the functional and the geometric property of the domain Ω than that of Mosolov. In this paper we return to Mosolov's formulation and show the existence of the solution of the variational problem for $T<T^*$ by using the space $BV(\Omega)$ instead of $W^{1,1}(\Omega)$. Using the result about the weak* topology on $BV(\Omega)$ ([1], [2]), we give a more direct proof than that in [3], [5], [7]. Next we consider the regularity property of the solution of the variational problem. Applying the regularity theorem due to Gerhardt [3] and Giaquinta [4], we give a partial result to this problem.

In section 1 we enumerate some properties of $BV(\Omega)$, which will be needed in the following sections. Section 2 is devoted to the proof of the lower boundedness theorem for the variational problem. In section 3 we discuss the existence and regularity property of the solution of the variational problem. In the final section 4 we show a dependence of solutions of Eq. $(*)_T$ on the parameter T and we state some results on Eq. $(*)_{T^*}$ for the critical parameter T^*. These theorems for Eq. $(*)_{T^*}$ are quite different from those of $T<T^*$.

§1. Definitions and properties of $BV(\Omega)$.

We present here the definition of the space $BV(\Omega)$ and some properties of its elements (for more detail, see [1], [2], [6]). Throughout this section Ω will denote a bounded domain in $\boldsymbol{R}^n$ ($n\geq 2$) with Lipschitz boundary.

The space of *functions of bounded variation* in Ω is defined as follows.

$$BV(\Omega) = \{u \in L^1(\Omega);\ \nabla u \in (C_0'(\Omega))^n\}\ .$$

Here $(C_0'(\Omega))^n$ denotes the dual space of $C_0(\Omega)^n$ and its norm is defined by

$$\|\omega\|_{(C_0^\prime(\Omega))^n}=\sup\{\omega(G);\ G\in C_0(\Omega)^n,\ |G|\leqq 1\}\ .$$

By virtue of Riesz's representation theorem, we observe that $BV(\Omega)$ is the function space consisting of L^1 functions whose gradient in distribution sense is a bounded vector-valued Radon measure. $BV(\Omega)$ is a Banach space under the norm

$$\|u\|_{BV(\Omega)}=\|u\|_{L^1(\Omega)}+\int_\Omega |\nabla u|$$

$$\text{where}\quad \int_\Omega |\nabla u|=\sup\left\{\int_\Omega u\,\mathrm{div}\,G\,dx;\ G\in C_0^1(\Omega)^n,\ |G|\leqq 1\right\}\ .$$

In the above definition $\int_\Omega |\nabla u|$ means the total variation of the vector-valued Radon measure ∇u in Ω. And it coincides with the norm $\|\nabla u\|_{(C_0^\prime(\Omega))^n}$.

EXAMPLE 1.1. (1) If u belongs to the Sobolev space $W^{1,1}(\Omega)$, we may easily show that

$$\int_\Omega |\nabla u|=\int_\Omega |\nabla u(x)|dx \quad \text{and} \quad \|u\|_{BV(\Omega)}=\|u\|_{W^{1,1}(\Omega)}\ .$$

Thus we also see that the Sobolev space $W^{1,1}(\Omega)$ is a closed subspace of $BV(\Omega)$.

(2) Suppose E be an open subset of Ω with C^2 boundary. We define the characteristic function χ_E of E

$$\chi_E(x)=1 \quad \text{if} \quad x\in E\ , \qquad =0 \quad \text{if} \quad x\in\Omega-E\ .$$

Then, the following results are known (see [6]).

$$\chi_E\in BV(\Omega)\ , \qquad \int_\Omega |\nabla\chi_E|=H_{n-1}(\Omega\cap\partial E) \quad \text{and} \quad \chi_E\notin W^{1,1}(\Omega)$$

where H_{n-1} denotes $(n-1)$-dimensional Hausdorff measure.

We define the *area functional*

$$\text{(1.1)}\qquad \int_\Omega (1+|\nabla u|^2)^{1/2}=\sup\left\{\int_\Omega (g_0+\mathrm{div}\,G)dx;\ G=(g_1,\cdots,g_n),\ g_i\in C_0^1(\Omega),\ i=0,\cdots,n,\ \sum_{i=0}^n g_i^2\leqq 1\right\}$$

on $BV(\Omega)$ according to [1], [5].

By this definition we may easily show that

$$\text{(1.2)}\qquad \int_\Omega |\nabla u|\leqq\int_\Omega (1+|\nabla u|^2)^{1/2}\leqq\int_\Omega |\nabla u|+\mathrm{meas}\,(\Omega)$$

where meas (Ω) denotes the n-dimensional Lebesgue measure of Ω. Furthermore we readily verify that

$$\int_{\Omega}(1+|\nabla u|^2)^{1/2}=\int_{\Omega}(1+|\nabla u(x)|^2)^{1/2}dx \quad \text{for} \quad u\in W^{1,1}(\Omega)\,. \tag{1.3}$$

We next state about some weak topology on $BV(\Omega)$. We define the following mapping.

$$\iota\colon BV(\Omega)\longrightarrow \boldsymbol{R}\oplus(C_0'(\Omega))^n=(\boldsymbol{R}\oplus C_0(\Omega)^n)'\,,$$
$$\iota(u)=\left(\int_{\Omega}udx,\ \nabla u\right).$$

It is easily seen that ι is an injective continuous linear mapping between Banach spaces. We identify $BV(\Omega)$ as a subspace of $\boldsymbol{R}\oplus(C_0'(\Omega))^n$ endowed with the weak* topology as the dual space of $\boldsymbol{R}\oplus C_0(\Omega)^n$. This weak* topology induces a topology of $BV(\Omega)$ as follows (see [1], [2]).

DEFINITION 1.2. A sequence $\{u_j\}$ of $BV(\Omega)$ converges to $u\in BV(\Omega)$ in the $\tilde{w}^*$ *topology* if and only if

$$\lim_{j\to\infty}\int_{\Omega}u_jdx=\int_{\Omega}udx \quad \text{and} \quad \lim_{j\to\infty}\int_{\Omega}G\cdot\nabla u_j=\int_{\Omega}G\cdot\nabla u \quad \text{for} \quad G\in C_0(\Omega)^n\,.$$

The $\tilde{w}^*$ topology has the following properties ([1], [2]).

PROPOSITION 1.3. (1) *$BV(\Omega)$ is a $\tilde{w}^*$-closed set in $\boldsymbol{R}\oplus(C_0(\Omega))^n$.*

(2) *If a sequence $\{u_j\}$ of $BV(\Omega)$ converges to $u\in BV(\Omega)$ in the $\tilde{w}^*$ topology, then $\{u_j\}$ converges to u in $L^1(\Omega)$.*

(3) *The closed balls of $BV(\Omega)$ are $\tilde{w}^*$-compact and their topology is metrizable.*

(4) *$W^{1,1}(\Omega)$ is $\tilde{w}^*$-dense in $BV(\Omega)$.*

Concerning the boundary value of a function belonging to $BV(\Omega)$, we state the following theorem (see [1], [6]).

THEOREM 1.4. *There exists a bounded operator γ (called the trace operator) from $BV(\Omega)$ to $L^1(\Omega)$ such that*

(1) *If $u\in W^{1,1}(\Omega)$, then $\gamma(u)$ coincides with the trace of u in the sense of the Sobolev space $W^{1,1}(\Omega)$.*

(2) *For every $G\in C_0^1(R^n)^n$ the following formula holds.*

$$\int_{\Omega}u\,\mathrm{div}\,Gdx+\int_{\Omega}G\cdot\nabla u=\int_{\partial\Omega}\gamma(u)G\cdot\nu dH_{n-1}$$

where ν denotes the outward unit normal vector of $\partial\Omega$.

§ 2. Lower boundedness of the variational problem.

In this paper we discuss the solvability of the Dirichlet problem for the quasilinear elliptic equation with a parameter T

$$(*)_T \qquad -\operatorname{div}\{\nabla u/(1+|\nabla u|^2)^{1/2}\}=Tc \quad \text{in } \Omega,\ u=\phi \quad \text{on } \partial\Omega .$$

Our purpose is to show the existence of the solution of Eq. $(*)_T$ belonging to $C^2(\Omega)\cap C^0(\bar{\Omega})$ assuming some kinds of conditions on Ω, c, ϕ and T if necessary. As our approach to this problem we use the variational method. We consider the following variational problem.

$$(2.1)_T \qquad \begin{aligned} &\text{Find} \quad u_T\in W_\phi^{1,1}(\Omega)=\{u\in W^{1,1}(\Omega);\ \gamma(u)=\phi\} \\ &\text{such that} \quad I_T(u_T)\leqq I_T(v) \quad \text{for all} \quad v\in W_\phi^{1,1}(\Omega) \\ &\text{where} \quad I_T(u)=\int_\Omega (1+|\nabla u|^2)^{1/2}dx-T\int_\Omega cudx . \end{aligned}$$

If a solution of Eq. $(*)_T$ may exist in $C^2(\Omega)\cap W_\phi^{1,1}(\Omega)$, it becomes a solution of $(2.1)_T$. Conversely, if there exists a solution of $(2.1)_T$ and it belongs to $C^2(\Omega)$, then it is also a solution of Eq. $(*)_T$. However, because the space $W^{1,1}$ is not reflexive, the general argument choosing a weakly convergent subsequence from a bounded sequence fails. We overcome this difficulty by considering the following problem instead of $(2.1)_T$.

$$(2.2)_T \qquad \begin{aligned} &\text{Find} \quad u_T\in BV(\Omega) \quad \text{such that} \quad J_T(u_T)\leqq J_T(v) \quad \text{for all} \quad v\in BV(\Omega) \\ &\text{where} \quad J_T(u)=\int_\Omega (1+|\nabla u|^2)^{1/2}-T\int_\Omega cudx+\int_{\partial\Omega}|\gamma(u)-\phi|dH_{n-1} . \end{aligned}$$

From Theorem 1.4 and (1.3) we readily see that $I_T(u)=J_T(u)$ whenever $u\in W_\phi^{1,1}(\Omega)$, that is, J_T is a extension of I_T to the space $BV(\Omega)$. The relation between $(2.1)_T$ and $(2.2)_T$ is stated in the following result due to Williams [14].

PROPOSITION 2.1. *Let Ω be a bounded open set of $\boldsymbol{R}^n$ with Lipschitz boundary and let $c\in L^n(\Omega)$ and $\phi\in L^1(\partial\Omega)$, Then, we have*

$$(2.3) \qquad \mu=\inf_{W_\phi^{1,1}(\Omega)} I_T=\inf_{BV_\phi(\Omega)} J_T=\inf_{BV(\Omega)} J_T$$

where $BV_\phi(\Omega)=\{u\in BV(\Omega);\ \gamma(u)=\phi\}$.

The remainder of this section is devoted to prove the following theorem about the existence of a finite infimum μ (cf. [11]).

THEOREM 2.2. *Let Ω be a bounded domain of $\boldsymbol{R}^n$ with Lipschitz* *boundary and suppose that* $c \in L^\infty(\Omega)$, $c \geqq 0$ *in* Ω *and* $\phi \in L^1(\partial\Omega)$. *Then, the functional* J_T *is bounded from below on* $BV_\phi(\Omega)$ *if and only if*

$$(2.4) \qquad 0 \leqq T \leqq T^* = \inf_{E \subset \Omega} \{H_{n-1}(\partial E)/\text{meas}_c(E)\} \quad \textit{where} \quad \text{meas}_c(E) = \int_E c\,dx\ .$$

In the right hand of (2.4) *the infimum is taken among open sets of* Ω *with* C^2 *boundary.*

PROOF. By (1.2) it is sufficient to show that the conclusion holds for the functional

$$(2.5) \qquad \Phi_T(u) = \int_\Omega |\nabla u| - T\int_\Omega cu\,dx + \int_{\partial\Omega} |\gamma(u) - \phi|\,dH_{n-1}$$

instead of J_T.

We first prove that the condition (2.4) is necessary. It is enough to show that if $T > T^*$ there exists a sequence $\{u_j\}$ of $BV_\phi(\Omega)$ such that $\lim_{j\to\infty} \Phi_T(u_j) = -\infty$. Since $T > T^*$, there exist $\lambda > 0$ and an open set G of Ω with C^2 boundary such that

$$(2.6) \qquad T > (H_{n-1}(\partial G) + \lambda)/\text{meas}_c(G)\ .$$

If ∂G intersects with $\partial\Omega$, we take an open set

$$G_\varepsilon = \{x \in \Omega;\ \text{dist}(x, \boldsymbol{R}^n - G) > \varepsilon\}\ , \quad (\varepsilon > 0)\ .$$

From the result of [9], Appendix we see that ∂G_ε is of class C^2 for sufficiently small ε. Furthermore, it is readily shown that (2.6) holds for such G_ε by replacing λ with smaller one if necessary, Hence, we may reduce the problem to the case $\partial G \cap \partial\Omega$ is empty.

By [14], Theorem 1 we take an extension $\tilde{\phi} \in W^{1,1}(\Omega)$ of the boundary value ϕ. We choose a cut off function $\eta \in C^\infty(\boldsymbol{R}^n)$ satisfying

$$\eta(x) = 1 \quad \text{if} \quad x \in \partial\Omega\ , \qquad = 0 \quad \text{if} \quad x \in U\ ,$$

where U is a fixed neighborhood of G such that $U \subset \Omega$.

We define

$$u_j(x) = j \cdot \chi_G(x) + \eta(x) \cdot \tilde{\phi}(x) \quad (j = 1, 2, \cdots)$$

where χ_G is the characteristic function of G. Example 1.1 (2) implies that $u_j \in BV_\phi(\Omega)$ for all j. Then,

$$\Phi_T(u_j) = \int_\Omega |\nabla u_j| - T\int_\Omega cu_j\,dx\ ,$$

$$\leqq j\left(\int_{\Omega} |\nabla\chi_G| - T\int_G c dx\right) + C\ ,$$
$$\leqq j(H_{n-1}(\partial G) - T\,\mathrm{meas}_c\,(G)) + C\ ,$$
$$< -j\lambda + C\ .$$

Here C denotes a constant independent of j. Hence, we have

$$\lim_{j\to\infty} \Phi_T(u_j) = -\infty\ .$$

Conversely, suppose that the condition (2.4) holds. We take an extension $\tilde{\phi} \in W^{1,1}(\Omega)$ of ϕ as the preceding case. For $u \in BV_\phi(\Omega)$ we set $v = u - \tilde{\phi}$ and then $\gamma(v) = 0$. We first consider the case $v \in C^\infty(\Omega)$. We set

$$A(t) = \{x \in \Omega;\ |v(x)| > t\}\ , \qquad a_t = \chi_{A(t)} \quad (t \geqq 0)\ .$$

Then the following formulas are known (see [5]).

$$|v(x)| = \int_0^\infty a_t(x) dt\ , \qquad \int_\Omega |\nabla|v|| = \int_0^\infty \left(\int_\Omega |\nabla a_t|\right) dt\ .$$

Using Sard's theorem we observe that the boundary $\partial A(t)$ of $A(t)$ is of class C^∞ for almost all $t > 0$. Furthermore $\overline{A(t)} \cap \partial\Omega$ is empty for all $t > 0$. From Example 1.1 (2) we obtain

$$\int_\Omega |\nabla|v|| = \int_0^\infty H_{n-1}(\partial A(t)) dt\ ,$$
$$\Phi_T(|v|) = \int_0^\infty H_{n-1}(\partial A(t)) dt - T\int_\Omega c(x)\left(\int_0^\infty a_t(x) dt\right) dx\ ,$$
$$= \int_0^\infty \left\{H_{n-1}(\partial A(t)) - T\int_\Omega c(x) a_t(x) dx\right\} dt\ ,$$
$$= \int_0^\infty \{H_{n-1}(\partial A(t)) - T\,\mathrm{meas}_c\,(A(t))\} dt \geqq 0\ .$$

Hence,

$$\Phi_T(u) \geqq \Phi_T(v) - \Phi_T(\tilde{\phi}) \geqq \Phi_T(|v|) - \Phi_T(\tilde{\phi}) \geqq -\Phi_T(\tilde{\phi})\ .$$

For general element u of $BV_\phi(\Omega)$ we approximate $v = u - \tilde{\phi}$ by smooth function. Using [6], 2.12 we can choose a sequence $\{v_j\}$ of $C^\infty(\Omega)$ such that $\{v_j\}$ converges to v in $L^1(\Omega)$, $\lim_{j\to\infty} \int_\Omega |\nabla v_j| = \int_\Omega |\nabla v|$ and $\gamma(v_j) = \gamma(v) = 0$. Therefore,

$$\Phi_T(u) \geqq \Phi_T(v) - \Phi_T(\tilde{\phi}) = \Phi_T(v_j) + \Phi_T(v) - \Phi_T(v_j) - \Phi_T(\tilde{\phi})\ ,$$
$$\geqq \Phi_T(|v_j|) + \Phi_T(v) - \Phi_T(v_j) - \Phi_T(\tilde{\phi})\ ,$$
$$\geqq \Phi_T(v) - \Phi_T(v_j) - \Phi_T(\tilde{\phi})\ .$$

Since $\lim_{j\to\infty}\Phi_T(v_j)=\Phi_T(v)$, we have the desired result

$$\Phi_T(v)\geqq-\Phi_T(\tilde{\phi}) \quad \text{for all} \quad u\in BV_\phi(\Omega)\,. \qquad \text{Q.E.D.}$$

REMARK 2.3. (1) Using the isoperimetric inequality, we have the lower estimate for the critical parameter T^*.

$$T^*\geqq n(\omega_n/\text{meas}\,(\Omega))^{1/n}\cdot\|c\|^{-1}_{L^\infty(\Omega)}>0 \tag{2.7}$$

where ω_n denotes n-dimensional Lebesgue measure of a unit ball in R^n.

(2) Since the functional J_T is convex, it cannot attain any critical value except for the minimum value. Hence, Eq. $(*)_T$ does not have any weak solution of $W^{1,1}(\Omega)$ for $T>T^*$.

§ 3. Existence and regularity of solutions of variational problem.

Here we consider the existence and the regularity of solutions of the variational problem $(2.2)_T$ for $T<T^*$. We first prove the following existence theorem.

THEOREM 3.1. *Let Ω be a bounded domain in R^n with Lipschitz boundary. Suppose that $c\in L^\infty(\Omega)$ with $c\geqq 0$, $\phi\in L^1(\partial\Omega)$ and $0\leqq T\leqq T^*$. Then, there exists $u_T\in BV(\Omega)$ such that u_T minimizes the functional J_T on $BV(\Omega)$.*

We state the following lemmas which will be needed in the proof of the above theorem.

LEMMA 3.2. *If a sequence $\{u_j\}$ of $BV(\Omega)$ converges to $u\in BV(\Omega)$ in the $\tilde{w}^*$ topology, then*

$$J_T(u)\leqq\liminf_{j\to\infty}J_T(u_j)$$

holds.

PROOF. From [1] the functional $\int_\Omega(1+|\nabla u|^2)^{1/2}+\int_{\partial\Omega}|\gamma(u)-\phi|dH_{n-1}$ is lower semicontinuous with respect to the $\tilde{w}^*$ topology. Hence, using Proposition 1.3 (2) we have

$$\begin{aligned}J_T(u)&=\int_\Omega(1+|\nabla u|^2)^{1/2}+\int_{\partial\Omega}|\gamma(u)-\phi|dH_{n-1}-T\int_\Omega cu\,dx\,,\\&\leqq\liminf_{j\to\infty}\left\{\int_\Omega(1+|\nabla u_j|^2)^{1/2}+\int_{\partial\Omega}|\gamma(u_j)-\phi|dH_{n-1}\right\}+\lim_{j\to\infty}T\int_\Omega cu_j dx\,,\\&\leqq\liminf_{j\to\infty}\left\{\int_\Omega(1+|\nabla u_j|^2)^{1/2}+\int_{\partial\Omega}|\gamma(u_j)-\phi|dH_{n-1}-T\int_\Omega cu_j dx\right\}\,,\\&=\liminf_{j\to\infty}J_T(u_j)\,.\end{aligned}$$

Q.E.D.

LEMMA 3.3 (Miranda [10]). *For any element u of $BV(\Omega)$, the following inequality holds.*

$$\int_\Omega |u|dx \leqq n(\text{meas}\,(\Omega)/\omega_n)^{1/n}\left(\int_\Omega |\nabla u| + \int_{\partial\Omega} |\gamma(u)|dH_{n-1}\right). \tag{3.1}$$

PROOF OF THEOREM 3.1. By virtue of Theorem 2.2 and $T<T^*$, we have

$$\mu = \inf_{BV(\Omega)} J_T = \inf_{BV_\phi(\Omega)} J_T > -\infty .$$

We choose a minimizing sequence $\{u_j\}$ of $BV_\phi(\Omega)$, that is, $J_T(u_j)$ converges to μ as j tends to infinity. We may assume

$\Phi_T(u_j) \leqq J_T(u_j) \leqq C_1$ where C_1 is a constant independent of j.

Then we have

$$(T^*/T)\Phi_T(u_j) = ((T^*/T)-1)\int_\Omega |\nabla u_j| + \Phi_{T^*}(u_j) ,$$

$$\leqq ((T^*/T)-1)\int_\Omega |\nabla u_j| - \Phi_{T^*}(\tilde{\phi}) .$$

$$\int_\Omega |\nabla u_j| \leqq (T^*\Phi_T(u_j) + T\Phi_{T^*}(\tilde{\phi}))/(T^*-T) \leqq C_2 ,$$

where C_2 is a constant independent of j. Using Lemma 3.3, we obtain

$$\|u_j\|_{L^1(\Omega)} \leqq n(\text{meas}\,(\Omega)/\omega_n)^{1/n}(C_2 + \|\phi\|_{L^1(\partial\Omega)}) .$$

Hence, $\{u_j\}$ is bounded in $BV(\Omega)$. By Proposition 1.3 (3). There exists a subsequence $\{u_k\}$ of $\{u_j\}$ which converges to some element u_T of $BV(\Omega)$ in the $\tilde{w}^*$ topology. Using Lemma 3.2, we obtain

$$\mu \leqq J_T(u_T) \leqq \liminf_{k\to\infty} J_T(u_k) = \mu . \qquad \text{Q.E.D.}$$

Concerning with the regularity property of the solution u_T of the variational problem $(2.2)_T$ obtained in the above theorem, we state the following theorem, which is derived from the result of Giaquinta [4].

THEOREM 3.4. *Let Ω be a bounded domain in $\boldsymbol{R}^n$ with C^2 boundary and $\phi \in C^0(\partial\Omega)$. Suppose that a nonnegative function $c \in C^1(\bar{\Omega})$ satisfies*

$$Tc(y) \leqq (n-1)H(y) \quad \textit{for any} \quad y \in \partial\Omega , \tag{3.2}$$

where H denotes the mean curvature of $\partial\Omega$ with respect to the inward unit normal vector of $\partial\Omega$. And suppose $0<T<T^*$. Then, the solution

u_T of $(2.2)_T$ belongs to $C^{2,\alpha}(\Omega)\cap C^0(\bar{\Omega})$ $(0\leq\alpha<1)$, $u_T=\phi$ on $\partial\Omega$ and u_T is a unique solution of Eq. $(*)_T$.

REMARK 3.5. (1) In the above theorem if we consider the interior regularity alone, we may assume that $c\in C^1(\Omega)$ (see [4]).

(2) The condition (3.2) is initially introduced by Serrin [12] and he shows that (3.2) is necessary to solve Eq. $(*)_T$ for any boundary value $\phi\in C^0(\partial\Omega)$ (see [9], [12]).

For the further regularity property we have the following theorem using the result due to Gerhardt [3].

THEOREM 3.6. *Let Ω be a bounded domain of $\boldsymbol{R}^n$ with $C^{2,\alpha}$ boundary for some $\alpha>0$ and ϕ can be extended to an element of $C^{2,\alpha}(\bar{\Omega})$. Suppose $c\in C^1(\bar{\Omega})$ is as in Theorem 3.4 and $0\leq T<T^*$. Then, $u_T\in C^{2,\alpha}(\bar{\Omega})$.*

PROOF. By virtue of Gerhardt's result ([3], Theorem 3) we first observe that $u_T\in W^{2,p}(\Omega)$ for any p with $n<p<\infty$. By Sobolev imbedding theorem u_T belongs to $C^{1,\lambda}(\bar{\Omega})$ for some $\lambda>0$. We may regard Eq. $(*)_T$ as a linear uniformly elliptic equation whose coefficients belong to $C^\lambda(\bar{\Omega})$ and we have the desired result using the regularity theory for linear elliptic equations. Q.E.D.

EXAMPLE 3.7. Let $\Omega=\{x\in\boldsymbol{R}^n;\ |x|<R\}$, $\phi=0$ and $c(x)=|x|^k$ $(k\geq0)$. Then, the solution u_T of Eq. $(*)_T$ is given by

$$\begin{aligned} u_T(x)&=\int_{|x|}^{R}[r^{k+1}/\{(k+n)^2/T^2-r^{2k+2}\}^{1/2}]dr\ , \\ 0&\leq T\leq T^*=(k+n)/R^{k+1}\ . \end{aligned} \tag{3.3}$$

In particular, when $c=1$ we have

$$\begin{aligned} u_T(x)&=((n^2/T^2)-|x|^2)^{1/2}-((n^2/T^2)-R^2)^{1/2}\ , \\ 0&\leq T\leq T^*=n/R\ . \end{aligned} \tag{3.4}$$

In this case Eq. $(*)_T$ is also solvable for $T=T^*$. The graph of u_T in (3.4) is a portion of a sphere in $\boldsymbol{R}^{n+1}$. We also see that the solution u_T for $T>T^*$ exists in geometric sense but it cannot be represented as a graph of some function over Ω.

§ 4. The case $T=T^*$.

In this section we discuss about the case $T=T^*$. First we provide a result on the global regularity property of Eq. $(*)_{T^*}$ which is in contrast with the case $T<T^*$.

THEOREM 4.1. *Let Ω be a bounded domain of $\boldsymbol{R}^n$ with Lipschitz boundary and let $c\in L^\infty(\Omega)$ with $c\geqq 0$ and c is not identically zero. Suppose that $u\in C^1(\Omega)$ is a weak solution of the equation*

$$-\operatorname{div}\{\nabla u/(1+|\nabla u|^2)^{1/2}\}=T^*c \quad in \quad \Omega\,,$$

where T^* is as in Theorem 2.2. Then, we have $\sup_\Omega|\nabla u|=\infty$, that is, $u\notin C^1(\bar{\Omega})$.

PROOF. By the definition of the weak solution we have

$$\int_\Omega \frac{\nabla u\cdot\nabla\eta}{(1+|\nabla u|^2)^{1/2}}dx=T^*\int_\Omega c\eta dx \quad \text{for any} \quad \eta\in C_0^1(\Omega)\,.$$

Hence,

$$T^*\int_\Omega c\eta dx\leqq M\int_\Omega|\nabla\eta|dx \quad \text{where} \quad M=\sup_\Omega\{|\nabla u|/(1+|\nabla u|^2)^{1/2}\}\,. \tag{4.1}$$

Using [6], 2.12 we observe that the above inequality can be extended for any element $\eta\in BV(\Omega)$ with $\gamma(\eta)=0$. Therefore, we choose η as follows:

$$\eta=\chi_E \quad \text{for any} \quad E\subset\Omega \quad \text{with} \quad \partial E\in C^2\,.$$

Then, we have

$$T^*\int_\Omega c\chi_E dx=T^*\operatorname{meas}_c(E)\leqq M\int_\Omega|\nabla\chi_E|=M\cdot H_{n-1}(\partial E)\,,$$
$$T^*\leqq M\cdot H_{n-1}(\partial E)/\operatorname{meas}_c(E)\,.$$

In this inequality we take infimum with respect to E. By the definition of T^* and $M\leqq 1$ we obtain

$$T^*\leqq M\cdot T^*\leqq T^*\,.$$

Therefore,

$$M=\sup_\Omega\{|\nabla u|/(1+|\nabla u|^2)^{1/2}\}=1$$

holds. This implies that $\sup_\Omega|\nabla u|=\infty$. Q.E.D.

We next treat the solvability of the Eq. $(*)_{T^*}$. However, we cannot apply the same method for Eq. $(*)_{T^*}$ as the case $T<T^*$. We so consider the problem whether the solution u_T of Eq. $(*)_T$ $(T<T^*)$ converges to a solution of Eq. $(*)_{T^*}$ as T tends to T^*. The behavior of solutions $\{u_T\}$ $(T<T^*)$ is proposed by the following proposition.

PROPOSITION 4.2. *Suppose that* $T_1 < T_2$ *and* $u_1, u_2 \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *are solutions of* $(*)_T$ *for* $T = T_1, T_2$ *respectively. Then,*

$$u_1(x) < u_2(x) \quad \textit{for} \quad x \in \Omega$$

holds in case c is not identically zero.

PROOF. By hypothesis $T_2 > T_1$

$$\operatorname{div}\{\nabla u_2/(1+|\nabla u_2|^2)^{1/2}\} - \operatorname{div}\{\nabla u_1/(1+|\nabla u_1|^2)^{1/2}\} \leqq 0 . \tag{4.2}$$

Using the mean value theorem, we can regard the left hand of (4.2) as a linear elliptic equation of divergence form for $u_2 - u_1$, that is, (4.2) can be written as follows:

$$\operatorname{div}\{A(x)\cdot\nabla(u_2 - u_1)\} \leqq 0 , \tag{4.3}$$

where $A(x) = (a^{ij}(x))$, $a^{ij} \in C^1(\Omega)$ $(i, j = 1, \cdots, n)$ is defined by

$$a^{ij}(x) = \int_0^1 \left\{ \frac{(1+|\nabla u_t(x)|^2)\cdot\delta_{ij} - \partial_i u_t(x)\cdot\partial_j u_t(x)}{(1+|\nabla u_t(x)|^2)^{3/2}} \right\} dt ,$$

$$u_t(x) = u_1(x) + t\cdot(u_2(x) - u_1(x)) .$$

From the maximum principle we first obtain

$$\inf_{\Omega}(u_2 - u_1) \geqq 0 , \qquad \text{that is,} \quad u_2 \geqq u_1 \quad \text{in} \quad \Omega .$$

We next consider a set $N = \{x \in \Omega;\ u_1(x) = u_2(x)\}$. We show that N is open and closed in Ω. By continuity of u_1, u_2, the closedness is evident. To prove, the openness we use the following weak Harnack inequality (see [9], Theorem 8.18).

For any $y \in \Omega$ and $R > 0$ with $B_{4R}(y) \subset \Omega$, there exists a constant $C > 0$ such that

$$R^{-n} \int_{B_{2R}(y)} (u_2 - u_1)dx \leqq C \inf_{B_R(y)} (u_2 - u_1) ,$$

where $B_r(y)$ is a open ball in R^n with center y and radius r. If $x \in N$ and we choose $R > 0$ with $B_{4R}(x) \subset \Omega$, then we obtain

$$R^{-n} \int_{B_{2R}(x)} (u_2 - u_1)dx \leqq C \inf_{B_R(x)} (u_2 - u_1) = 0 .$$

From $u_2 \geqq u_1$ we have $u_2 = u_1$ in $B_{2R}(x)$. This implies the openness of N. Since Ω is connected, the set N is either empty or Ω. Hence, in case c is not identically zero we obtain the desired result. Q.E.D.

According to the above proposition we observe that the next two cases may occur about the behavior of u_T as T tends to T^* in case c is not identically zero.

(1) $\sup_\Omega u_T \leqq K$ for some constant independent of T.

(2) $\sup_\Omega u_T \to \infty$ as $T \to T^*$.

The Example 3.7 is the case (1). Concerning with the case (2) we propose the following theorem.

THEOREM 4.3. *Let Ω be a bounded domain of $\boldsymbol{R}^n$ with C^3 boundary and $\phi \in C^{1,\alpha}(\partial\Omega)$ for some $\alpha > 0$. Suppose $c \in C^1(\bar{\Omega})$ satisfying $c \geqq 0$, c is not identically zero and*

$$T^* c(y) \leqq (n-1)H(y) \quad \text{for all} \quad y \in \partial\Omega . \tag{4.4}$$

Then,

$$\lim_{T \uparrow T^*} \sup u_T = \infty , \tag{4.5}$$

and the Eq. $(*)_{T^*}$ does not have any solution in $C^2(\Omega) \cap C^0(\bar{\Omega})$.

PROOF. Contrary to the theorem we assume that there exists a constant K independent of T such that

$$\sup_\Omega u_T \leqq K \quad \text{for all} \quad T < T^* .$$

Combining to Proposition 4.2 the sequence $\{u_T(x)\}$ is bounded and monotone increasing for any $x \in \bar{\Omega}$. Hence, the limiting value $u_{T^*}(x)$ exists for all $x \in \bar{\Omega}$ and $u_{T^*}(x) = \phi(x)$ for all $x \in \partial\Omega$. Furthermore we obtain

$$u_0(x) \leqq u_T(x) \leqq K \quad \text{for all} \quad T < T^* \quad \text{and} \quad x \in \bar{\Omega} ,$$
$$\sup_\Omega |u_T| \leqq C_1 = \max\{\sup_\Omega |u_0|, K\} \quad \text{for} \quad 0 \leqq T \leqq T^* ,$$

where u_0 is a unique solution of $(*)_T$ for $T=0$.

We first establish the interior regularity of u_{T^*}. We use the following a priori estimate due to Trudinger ([9], [13]).

For any $\Omega' \subset \Omega$ the following estimate holds.

$$|\nabla u_T(x)| \leqq C \exp\{C' \sup_{y \in \Omega} (u_T(y) - u_T(x))/d\} \quad \text{for} \quad x \in \Omega' \quad \text{and} \quad T < T^* ,$$

where $d = \operatorname{dist}(\Omega', \partial\Omega)$ and C, C' denote constant depending on n, $dT^* \sup_\Omega |c|$ and $d^2 T^* \sup_\Omega |\nabla c|$.

From this estimate we obtain the uniform gradient estimate

$$\sup_{\Omega'} |\nabla u_T| \leqq C_2(n, C_1, d, \|c\|_{C^1(\bar{\Omega})}) \quad \text{for all} \quad T < T^* .$$

Using the theorem of Ladyzhenskaya-Ural'tseva ([9], Theorem 12.1) we obtain the uniform Hölder estimate on each $\Omega'\subset\Omega$ of ∇u_T $(T<T^*)$. Combining with the Ascoli-Arzelà theorem we obtserve that $u_{T^*}\in C^{1,\beta}(\overline{\Omega'})$ for some β (>0) depending on d, C_1 and C_2. And we have

$$\int_{\Omega'}\left\{\frac{\nabla u_{T^*}\cdot\nabla\zeta}{(1+|\nabla u_{T^*}|^2)^{1/2}}-T^*c\zeta\right\}dx=0 \quad \text{for any} \quad \zeta\in C_0^1(\Omega')\,,$$

that is, u_{T^*} is a weak solution of $-\operatorname{div}\{\nabla u/(1+|\nabla u|^2)^{1/2}\}=T^*c$ in Ω'. By virtue of the regularity theory for linear elliptic equation we have $u_{T^*}\in C^{2,\alpha}(\Omega')$ $(0\leqq\alpha<1)$. Since $\Omega'\subset\Omega$ is arbitrarily chosen, we have $u_{T^*}\in C^{2,\alpha}(\Omega)$ $(0\leqq\alpha<1)$ and

$$-\operatorname{div}\{\nabla u_{T^*}/(1+|\nabla u_{T^*}|^2)^{1/2}\}=T^*c \quad \text{in} \quad \Omega\,.$$

We next show the continuity of u_{T^*} on the boundary $\partial\Omega$. We may claim the following by applying [9], Theorem 13.15 concerning with the boundary behavior of solutions $\{u_T\}$ $(T<T^*)$.

For any $x_0\in\partial\Omega$ and any $\varepsilon>0$, there exists a neighborhood V of x_0 and a function $w\in C^2(\Omega\cap V)\cap C^1(\bar{\Omega}\cap V)$ satisfying $w(x_0)=0$ and

$$|u_T(x)-\phi(x_0)|\leqq\varepsilon+w(x)+(2/\delta^2)(\sup_{\partial\Omega}|\phi|)|x-x_0|^2 \tag{4.6}$$

for all $x\in V\cap\Omega$ and all $T<T^*$ where V and w depend on n, δ, C_1, $\|c\|_{C^1(\bar{\Omega})}$ and Ω and $\delta>0$ is chosen so that any pair x, $y\in\partial\Omega$ with $|x-y|<\delta$ implies $|\phi(x)-\phi(y)|<\varepsilon$.

Making T tends to T^*, we get

$$|u_{T^*}(x)-\phi(x_0)|\leqq\varepsilon+w(x)+(2/\delta^2)(\sup_{\partial\Omega}|\phi|)|x-x_0|^2$$

for $x\in V\cap\Omega$. This implies $u_{T^*}\in C^0(\bar{\Omega})$.

Thus we construct the solution u_{T^*} of Eq. $(*)_{T^*}$ belonging to $C^2(\Omega)\cap C^0(\bar{\Omega})$. Furthermore, from the result of Giaquinta [5] we derive that u_{T^*} is Lipschitz continuous on $\bar{\Omega}$. However, this contradicts with Theorem 4.1. Therefore, (4.5) must hold. The rest of the theorem follows immediately. Q.E.D.

REMARK 4.4. In the above theorem the regularity hypothesis on $\partial\Omega$ and ϕ is needed only to apply the result of Giaquinta. His result is obtained by the maximum principle and nice choices of barrier functions.

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