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# Fundamental Properties of Modified Fourier Hyperfunctions

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## Introduction

In the present paper we discuss the fundamental properties of modified Fourier hyperfunctions.

It is about a quarter of a century ago that Professor Sato introduced and developed the theory of hyperfunctions in [42], [43] and [44]. In [42], among many important discussions, he introduced Fourier hyperfunctions to define the Fourier transformation of hyperfunctions in the case of one variable. Roughly speaking, a Fourier hyperfunction is presented as a difference of boundary values of holomorphic functions with infra-exponential growth from a complex domain to a real domain. (These holomorphic functions which present a Fourier hyperfunction are called the defining functions of the Fourier hyperfunction.) Indicated by Sato [42], Kawai [16], [17] treated the theory of Fourier hyperfunctions of several variables. Kawai [16], [17] also discussed its applications to linear partial differential equations with constant coefficients. Modified Fourier hyperfunctions were proposed by Professors Sato and Kawai for the purpose of their applications to the so called division problem in the theory of linear partial differential equations. Kawai [17] referred the matter, and announced the publishment of the paper on the theory of modified Fourier hyperfunctions. (See pp-468, 469 in Kawai [17].) But it has not been published.

On the other hand, the theory of vector valued Fourier hyperfunctions was developed by Ito-Nagamachi [10], Nagamachi-Mugibayashi [31] and Junker [11], [12]. Nagamachi-Mugibayashi [31] also introduced (axiomatic) hyperfunction fields. Further, Nagamachi-Mugibayashi [32], [33] introduced (vector valued) Fourier hyperfunctions of the second type and mixed type to show the equivalence of the relativistic and Euclidean field theory. Afterward, Nagamachi [29], [30] treated the theory of (vector

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valued) Fourier hyperfunctions of mixed type. Saburi [38] also treated the theory of modified Fourier hyperfunctions.

Here we have to give an explanation on the terminology. What Nagamachi-Mugibayashi called the Fourier hyperfunctions of the second type are the modified Fourier hyperfunctions in our terminology. A Fourier hyperfunction of mixed type is a hyperfunction which is a (original) Fourier hyperfunction in some variables and a modified Fourier hyperfunction in other variables. Therefore there are three types of Fourier hyperfunctions at present: the (original) Fourier hyperfunctions, modified Fourier hyperfunctions and the Fourier hyperfunctions of mixed type.

The sheaves of these three types of Fourier hyperfunctions are defined on the radial compactification $\boldsymbol{D}^n$ of the Euclidean space $\boldsymbol{R}^n$. (See Definition 1.1.1 in Kawai [17] (p-227) for the definition of the radial compactification $\boldsymbol{D}^n$ of $\boldsymbol{R}^n$.) Each sheaf of these three types of Fourier hyperfunctions is flabby. Each space of global sections of sheaves of these three types of Fourier hyperfunctions are stable under the Fourier transformation. The distinction of these three types of Fourier hyperfunctions is essentially in the distinction of types of the domain of the defining functions. (Compare the definitions of the sheaves $\tilde{\mathcal{O}}$ of defining holomorphic functions of Fourier hyperfunctions, $\mathcal{O}_{inc}$ of defining holomorphic functions of modified Fourier hyperfunctions and $\tilde{\mathcal{O}}'_{k,l}$ of defining holomorphic functions of Fourier hyperfunctions of mixed type: Definition 1.1.2 in Kawai [17], Definition 1.1.2 in the present paper and Definition 2.3 in Nagamachi [29].)

From the functional analytic view point, each Fourier hyperfunction of these three types can be regarded as an analytic functional with non compact carrier contained in the real domain. Therefore our theory can be regarded as a part of the theory of analytic functionals. The theory of analytic functionals and their Fourier transformation was developed by Martineau [24] in details. Afterward Morimoto [25], [27] introduced the theory of analytic functionals with noncompact carrier and their Fourier transformation. There are several works on the theory of analytic functionals with noncompact carrier besides Morimoto [25], [27]. Those are Zharinov [46], [47], de Roever [37], Morimoto-Yoshino [28], Sargos-Morimoto [41], Yoshino [45] and others.

The aim of the present paper is to give complete proofs of the theorems on fundamental properties of the sheaf $\mathscr{R}$ of modified Fourier hyperfunctions on $\boldsymbol{D}^n$. More explicitly, we discuss the flabbiness of the sheaf $\mathscr{R}$, a duality between the sheaves $\mathscr{R}$ and $\mathscr{A}_{dec}$ of rapidly decreasing real analytic functions on $\boldsymbol{D}^n$ and Fourier transforms of modified Fourier hyperfunctions. (See Definition 1.1.5 and 1.1.6 in the present paper for the definitions of the sheaves $\mathscr{R}$ and $\mathscr{A}_{dec}$.)

The plan of the present paper is as follows:

We give the definition of modified Fourier hyperfunctions in §1.1. We describe pricisely the properties of the sheaf $\mathscr{R}$ of modified Fourier hyperfunctions in §1.2.

We need three preparations to show the flabbiness of the sheaf $\mathscr{R}$ and a duality between two sheaves $\mathscr{R}$ and $\mathscr{A}_{dec}$ on $\boldsymbol{D}^n$. One of them is the vanishing theorems of cohomology groups with values in the sheaves $\mathscr{O}_{inc}$ of holomorphic functions with the infra-exponential growth and $\mathscr{O}_{dec}$ of holomorphic functions with some exponential decay condition. Those theorems are prepared in our previous paper Saburi [40]. We review those theorems in §2.1. Next one is the Grauert theorem for the sheaf $\mathscr{O}_{inc}$ (Theorem 2.2.2). The last one is an approximation theorem for the sheaf $\mathscr{A}_{dec}$ (Theorem 2.3.1). We prove these theorems in §§2.2 and 2.3 respectively.

Under the preparations in §2, we prove the flabbiness of the sheaf $\mathscr{R}$ (Theorem 3.2.2) and a duality theorem between the sheaves $\mathscr{R}$ and $\mathscr{A}_{dec}$ (Corollary 3.2.3) in §3.

In §4, we discuss the Fourier transforms of modified Fourier hyperfunctions. First we review the Fourier transformations of the space $\mathscr{A}_{dec}(\boldsymbol{D}^n)$ of global sections of the sheaf $\mathscr{A}_{dec}$ and the space $\mathscr{R}(\boldsymbol{D}^n)$ of global sections of the sheaf $\mathscr{R}$ (Theorem 4.1.1 and p-251). Next we give a presentation of a modified Fourier hyperfunction as "a difference of boundary values of holomorphic functions from a comrlex domain to the real domain $\boldsymbol{D}^n$" (4.3.2). We also give a presentation of the Fourier transform of a modified Fourier hyperfunction as "a difference of boundary values of holomorphic functions from a complex domain to the real domain $\boldsymbol{D}^n$" (Definition 4.2.2). Finally we discuss the Fourier-Carleman-Leray-Sato transformation for the modified Fourier hyperfunctions in §4.3.

Here we clarify the connection between the present work and Nagamachi-Mugibayashi [33]. In [33], they introduced the modified Fourier hyperfunctions as analytic functionals and established their fundamental theory. On the other hand, we attempt to give its foundation by using the cohomologial theory of the sheaves $\mathscr{O}_{inc}$ and $\mathscr{O}_{dec}$ in the present paper. We also notice that the contents of the present paper were discussed in our previous work Saburi [38]. Saburi [38] includes some mistakes. We give improvements for them in the present paper. (See Remark at the end of §2.3 and Remark to the proof of Proposition 4.2.1.)

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## § 1. The definition of modified Fourier hyperfunctions and their fundamental properties.

### 1.1. Definitions and notations.

In this section we give the definition of modified Fourier hyperfunctions.

We denote by $\boldsymbol{D}^k$ the radial compactification of $\boldsymbol{R}^k$. (For the definition of the radial compactification of $\boldsymbol{R}^k$, see Definition 1.1.1 in Kawai [17].) We identify $\boldsymbol{C}^n$ with $\boldsymbol{R}^{2n}$, and denote its radial compactification by $\boldsymbol{Q}^n$. Then the closure of $\boldsymbol{R}^n$ in $\boldsymbol{Q}^n$ is nothing but $\boldsymbol{D}^n$.

DEFINITION 1.1.1. For an $\alpha \in \boldsymbol{R}$ and for an open set $U$ in $\boldsymbol{C}^n$, we denote by $\mathcal{O}^\alpha(U)$ the space of holomorphic functions defined as follows:

$$\mathcal{O}^\alpha(U)=\left\{f \in \mathcal{O}(U);\ \|f\|_{\alpha,U}=\sup_{z\in U} |f(z)|e^{-\alpha|z|}<\infty\right\}.$$

$\mathcal{O}^\alpha(U)$ is a Banach space with the norm $\|\cdot\|_{\alpha,U}$.

DEFINITION 1.1.2. (The sheaf of slowly increasing holomorphic functions) We denote by $\mathcal{O}_{inc}$ the sheaf on $\boldsymbol{Q}^n$ whose section module $\mathcal{O}_{inc}(W)$ over an open set $W$ in $\boldsymbol{Q}^n$ is given by the following:

$$\mathcal{O}_{inc}(W)=\{f \in O(W\cap \boldsymbol{C}^n);\ \|f\|_{\varepsilon,W'\cap \boldsymbol{C}^n}<\infty \text{ for any } W'\subset\subset W \text{ and any } \varepsilon>0\},$$

where the notation $W'\subset\subset W$ means that $W'$ is a relatively compact (open) subset of $W$.

If $0<\varepsilon''<\varepsilon'$ and $W'\subset\subset W''$, then the natural restriction mapping: $\mathcal{O}^{\varepsilon''}(W'')\hookrightarrow\mathcal{O}^{\varepsilon'}(W')$ is compact. Hence the space $\mathcal{O}_{inc}(W)$ is a Fréchet-Schwartz space with the seminorms $\{\|\cdot\|_{\varepsilon,W\cap \boldsymbol{C}^n}\}$.

DEFINITION 1.1.3. (The sheaf of rapidly decreasing holomorphic functions) We denote by $\mathcal{O}_{dec}$ the sheaf on $\boldsymbol{Q}^n$ whose section module $\mathcal{O}_{dec}(W)$ over and an open set $W$ in $\boldsymbol{Q}^n$ is given by the following:

$$\mathcal{O}_{dec}(W)=\{f \in \mathcal{O}(W\cap \boldsymbol{C}^n);\ \text{for any } W'\subset\subset W \text{ there exists an } \varepsilon>0 \text{ such that } \|f\|_{-\varepsilon,W'\cap \boldsymbol{C}^n}<\infty\}.$$

REMARK. The restrictions of the sheaves $\mathcal{O}_{inc}$ and $\mathcal{O}_{dec}$ to $\boldsymbol{C}^n$ coincide with the sheaf $\mathscr{O}$ of holomorphic functions on $C^n$.

DEFINITION 1.1.4. (The topology of $\mathscr{O}_{dec}(K)$) Let $K$ be a compact set in $Q^n$. Then we equip $\mathscr{O}_{dec}(K)$ with the following locally convex inductive limit topology of Banach spaces:

$$\mathscr{O}_{dec}(K) = \lim_{\substack{\varepsilon>0, W \supset K}} \operatorname{ind} \mathscr{O}^{-\varepsilon}(W) ,$$

where $W$ runs all open sets in $Q^n$ containing $K$.

With this topology $\mathscr{O}_{dec}(K)$ is a dual Fréchet-Schwartz space.

DEFINITION 1.1.5. (The sheaf of rapidly decreasing real analytic functions on $D^n$) We denote by $\mathscr{A}_{dec}$ the restriction of the sheaf $\mathscr{O}_{dec}$ to $D^n$.

REMARK. The restriction of the sheaf $\mathscr{A}_{dec}$ to $R^n$ coincides with the sheaf $\mathscr{A}$ of real analytic fuuntions on $R^n$.

DEFINITION 1.1.6. (The (pre)sheaf of modified Fourier hyperfunctions) We denote by $\mathscr{R}$ the presheaf on $D^n$ whose section module $\mathscr{R}(\Omega)$ over an open set $\Omega$ in $D^n$ is the $n$-th relative (or local) cohomology group with values in the sheaf $\mathscr{O}_{inc}$ on $Q^n$ with supports on $\Omega$:

$$\mathscr{R}(\Omega) = \mathrm{H}^n_\Omega(\tilde{\Omega}; \mathscr{O}_{inc}) ,$$

where $\tilde{\Omega}$ is an arbitrary open set in $Q^n$ containing $\Omega$ as a closed subset. We call an element in $\mathscr{R}(\Omega)$ a modified Fourier hyperfunction on $\Omega$. For a compact set $K$ in $D^n$, we denote especially by $\mathscr{R}[K]$ the space of all modified Fourier hyperfunctions on $D^n$ whose supports are contained in $K$.

**1.2.** Properties of $\mathscr{R}$.

Now we go on to describle the fundamental properties of modified Fourier hyperfunctions, which are proved in later sections.

( i ) The presheaf $\mathscr{R}$ constitutes a flabby sheaf on $D^n$. The restriction of $\mathscr{R}$ to $R^n$ coincides with the sheaf $\mathscr{B}$ of hyperfunctions on $R^n$.

(ii) Let $K$ be a compact set in $D^n$. Then $\mathscr{R}[K]$ is linearly topologically isomorphic to the strong dual space $\mathscr{A}'_{dec}(K)$ of $\mathscr{A}_{dec}(K)$.

(iii) The Fourier transformation $\mathscr{F}$ gives a linear topological automorphism of $\mathscr{A}_{dec}(D^n)$. Therefore we can define a Fourier transformation $\mathscr{F}_d$ of $\mathscr{R}(D^n)$ as the dual transformation of the Fourier transformation of $\mathscr{A}_{dec}(D^n)$.

On the other hand we can define another Fourier transformation of $\mathscr{R}(D^n)$ as follows. Let $f$ be a modified Fourier hyperfunction on $D^n$.

We decompose $f$ as $\sum_{1\leq j\leq 2^n} f_j$, with $\operatorname{supp} f_j$ contained in the closure of $j$-th orthant $\Gamma_j$ in $\boldsymbol{R}^n$. After this decomposition we define the Fourier transform $\mathscr{F}f$ of $f$ as the cohomology class $\{\langle f_j, e^{\sqrt{-1}\zeta\cdot z}\rangle\} \in \mathrm{H}^n_{D^n}(\boldsymbol{Q}^n; \mathscr{O}_{inc})$, where $\langle f_j, e^{\sqrt{-1}\zeta\cdot z}\rangle$ is defined by the duality given above, as far as $e^{\sqrt{-1}\zeta\cdot z} \in \mathscr{A}_{dec}(\bar{\Gamma}_j)$.

It will be proved that the two definitions of the Fourier transformation of $\mathscr{R}(\boldsymbol{D}^n)$ given above coincide.

## § 2. The sheaves $\mathscr{O}_{inc}$ and $\mathscr{O}_{dec}$.

**2.1.** Vanishing theorems of cohomology groups with values in the sheaves $\mathscr{O}_{inc}$ and $\mathscr{O}_{dec}$.

In this section we review soft resolutions of the sheaves $\mathscr{O}_{inc}$ and $\mathscr{O}_{dec}$, and vanishing theorems of cohomology groups with values in these sheaves. For the details and the proof of the theorems in this section, see Saburi [40].

First we mention soft resolution of the sheaves $\mathscr{O}_{inc}$ and $\mathscr{O}_{dec}$. We denote by $\mathscr{L}^{2,(p,q)}_{loc}$ the sheaf of differential forms of type $(p, q)$ on $\boldsymbol{C}^n$ whose coefficients are locally square summable functions.

DEFINITION 2.1.1. We denote by $\mathscr{X}^{(p,q)}$ the sheaf on $\boldsymbol{Q}^n$ whose section module $\mathscr{X}^{(p,q)}(W)$ over an open set $W$ in $\boldsymbol{Q}^n$ is given by the following:

$$\mathscr{X}^{(p,q)}(W)=\left\{f \in \mathscr{L}^{2,(p,q)}_{loc}(W\cap \boldsymbol{C}^n);\ \int_{K\cap \boldsymbol{C}^n} |f|^2 e^{-\varepsilon|z|} d\lambda \right. \\ \left. \text{for any } K\subset W \text{ and any } \varepsilon>0\right\},$$

where $d\lambda$ is the Lebesque measure on $\boldsymbol{C}^n \cong \boldsymbol{R}^{2n}$. Moreover we denote by $\mathscr{X}^{1,(p,q)}$ the sheaf on $\boldsymbol{Q}^n$ whose section module $\mathscr{X}^{1,(p,q)}(W)$ over an open set $W$ in $\boldsymbol{Q}^n$ is given by the following:

$$\mathscr{X}^{1,(p,q)}(W)=\{f \in \mathscr{X}^{(p,q)}(W);\ \bar{\partial} f \in \mathscr{X}^{(p,q+1)}(W)\},$$

where $\bar{\partial} f$ is defined in the sense of distributions.

DEFINITION 2.1.2. We denote by $\mathscr{Y}^{(p,q)}$ the sheaf on $\boldsymbol{Q}^n$ whose section module $\mathscr{Y}^{(p,q)}(W)$ over an open set $W$ in $\boldsymbol{Q}^n$ is given by the following:

$$\mathscr{Y}^{(p,q)}(W)=\left\{f \in \mathscr{L}^{2,(p,q)}_{loc}(W);\ \text{for any } K\subset W \text{ there} \right. \\ \left. \text{exists an } \varepsilon>0 \text{ such that } \int_{K\cap \boldsymbol{C}^n} |f|^2 e^{\varepsilon|z|} d\lambda<\infty\right\}.$$

Moreover we denote by $\mathscr{Y}^{1,(p,q)}$ the sheaf on $\boldsymbol{Q}^n$ whose section module $\mathscr{Y}^{1,(p,q)}(W)$ over an open set $W$ in $\boldsymbol{Q}^n$ is given by the following:

$$\mathscr{Y}^{1,(p,q)}(W)=\{f\in\mathscr{Y}^{(p,q)}(W);\ f\in\mathscr{Y}^{(p,q+1)}(W)\}\ .$$

Here we note that $\mathscr{X}^{(p,q)}(W)$ is a Fréchet-Kômura space. The strong dual space of $\mathscr{X}^{(p,q)}(W)$ is represented by a dual Fréchet-Kômura space $\mathscr{Y}^{(n-p,n-q)}_{comp}(W)$. (In the terminology of Komatsu [18], Fréchet-Kômura spaces are referred as $FS^*$ spaces and dual Fréchet-Kômura spaces are referred as $DFS^*$ spaces.) The pairing between these two spaces is given by the following:

$$\int_{W\cap C^n} f\wedge g \qquad \text{for}\quad (f,g)\in\mathscr{X}^{(p,q)}(W)\times\mathscr{Y}^{(n-p,n-q)}(W)\ .$$

PROPOSITION 2.1.1. *We have the following soft resolutions for the sheaves $\mathscr{O}_{inc}$ and $\mathscr{O}_{dec}$ on $\boldsymbol{Q}^n$:*

$$0\longrightarrow\mathscr{O}_{inc}\longrightarrow\mathscr{X}^{1,(0,0)}\xrightarrow{\bar\partial}\mathscr{X}^{1,(0,1)}\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}\mathscr{X}^{1,(0,n)}\longrightarrow 0\ ,$$
$$0\longrightarrow\mathscr{O}_{dec}\longrightarrow\mathscr{Y}^{1,(0,0)}\xrightarrow{\bar\partial}\mathscr{Y}^{1,(0,1)}\xrightarrow{\bar\partial}\cdots\xrightarrow{\bar\partial}\mathscr{Y}^{1,(0,n)}\longrightarrow 0\ .$$

Next we mention vanishing theorems of cohomology groups with values in the sheaves $\mathscr{O}_{inc}$ and $\mathscr{O}_{dec}$.

We give two definitions for domains in $\boldsymbol{Q}^n$.

DEFINITION 2.1.3. We call an open set $W$ in $\boldsymbol{Q}^n$ to be *acute* if it satisfies the following condition:

$$\sup_{z\in W\cap C^n}|\operatorname{Im} z|/(|\operatorname{Re} z|+A)<1 \qquad \text{for some}\quad A>0\ .$$

DEFINITION 2.1.4. We call an open set $V$ in $\boldsymbol{Q}^n$ to be $\mathscr{O}_{inc}$*-pseudoconvex* if it is acute and if there exists a strictly plurisubharmonic $C^\infty$ function $p$ on $V\cap C^n$ which satisfies the following condition (P):

$$(\mathrm{P})\begin{cases}(\mathrm{i})\ \ \{z\in V\cap C^n;\ p(z)<c\}\Subset V & \text{for any}\quad c\in\boldsymbol{R}\ ,\\ (\mathrm{ii})\ \ \sup_{z\in K\cap C^n} p(z)<\infty & \text{for any}\quad K\Subset V\ .\end{cases}$$

Using the terminology given above, we describe vanishing theorems of cohomology groups with values in the sheaves $\mathscr{O}_{inc}$ and $\mathscr{O}_{dec}$:

THEOREM 2.1.2. (The Cartan theorem B for the sheaf $\mathscr{O}_{inc}$. Cf. Exposé 18 in Cartan [3].) *Let $V$ be an $\mathscr{O}_{inc}$-pseudoconvex open set in $\boldsymbol{Q}^n$. Then we have*

$$H^q(V; \mathcal{O}_{inc})=0 \qquad (q\geqq 1).$$

THEOREM 2.1.3. (The Cartan theorem B for the sheaf $\mathcal{O}_{dec}$. Cf. Exposé 18 in Cartan [3].) *Let $K$ be a compact set in $\boldsymbol{Q}^n$. Suppose that $K$ has a fundamental system of neighborhoods consisting of $\mathcal{O}_{inc}$-pseudoconvex open sets in $\boldsymbol{Q}^n$. Then we have:*

$$H^q(K; \mathcal{O}_{dec})=0 \qquad (q\geqq 1).$$

THEOREM 2.1.4. (The Malgrange theorem for the sheaf $\mathcal{O}_{inc}$. Cf. Malgrange [22].) *Let $W$ be an acute open set in $\boldsymbol{Q}^n$. Then we have*

$$H^n(W; \mathcal{O}_{inc})=0 .$$

Here we give examples of $\mathcal{O}_{inc}$-pseudoconvex open sets in $\boldsymbol{Q}^n$, which play important roles in §4.

EXAMPLE. For $\delta>0$, we put

$$U_\delta=\{z\in \boldsymbol{C}^n; |\mathrm{Im}\, z|^2<\delta^2(|\mathrm{Re}\, z|^2+1)\}, \qquad V_\delta=\mathring{\bar{U}}_\delta .$$

If $0<\delta<1$, then $V_\delta$ is an $\mathcal{O}_{inc}$-pseudoconvex open set in $\boldsymbol{Q}^n$. We put $p_\delta(z)=-1/(|\mathrm{Im}\, z|^2-\delta^2(|\mathrm{Re}\, z|^2+1))$. Then $p_\delta$ gives a strictly plurisubharmonic $C^\infty$ function on $V\cap \boldsymbol{C}^n$ satisfying the condition (P) in Definition 2.1.4.

We give another example. We put

$$U_j=\{z\in \boldsymbol{C}^n; \mathrm{Im}\, z_j\neq 0\}, \quad T_j=\{z\infty\in S_\infty^{2n-1}; z\in U_j\},$$
$$V_\delta^j=(U_j\cup T_j)\cap V_\delta \qquad (j=1,\cdots,n).$$

If $0<\delta<1$, then $V_\delta^j$ is an $\mathcal{O}_{inc}$-pseudoconvex open set in $\boldsymbol{Q}^n$ $(j=1,\cdots,n)$. We put $r_j(z)=(1+\sum_{k\neq j}|z_k|^2+|\mathrm{Re}\, z_j|^2)/|\mathrm{Im}\, z_j|^2$ and $p_\delta^j(z)=r_j(z)+p_\delta(z)$. Then $p_\delta^j$ gives a strictly plurisubharmonic $C^\infty$ function on $V_\delta^j\cap \boldsymbol{C}^n$ satisfying the condition (P) in Definition 2.1.4.

**2.2.** The Grauert theorem for sheaf $\mathcal{O}_{inc}$.

In this section we see that there exist sufficiently many $\mathcal{O}_{inc}$-pseudoconvex open sets. That is, for any open set $\Omega$ in $\boldsymbol{D}^n$, there exists a fundamental system of neighborhoods of $\Omega$ consisting of $\mathcal{O}_{inc}$-pseudoconvex open sets in $\boldsymbol{Q}^n$.

We need some preparations.

We note that there exists a diffeomorphism $\varpi$ of $\boldsymbol{D}^k$ onto $B$ (=the closed unit ball in $\boldsymbol{R}^k$ centered at the origin):

$$(\varpi) \quad \varpi(y)=\begin{cases} x/|x| & \text{if } \ y=x\infty\in S_\infty^{2k-1} \\ x/\sqrt{|x|^2+1} & \text{if } \ y=x\in \boldsymbol{R}^k . \end{cases}$$

Then we have the following

LEMMA 2.2.1. *Let $W$ be an open set in $D^k$ and $a \in C^\infty(\varpi(W))$. Then we have the following estimation for the derivatives of $\varpi^* a$, the pullback of $a$ by $\varpi$:*

$$(2.2.1) \qquad \left|\frac{\partial}{\partial x_i}\varpi^* a(x)\right| \leqq A|\nabla a|(\varpi(x))/(|x|+1)$$

$$(2.2.2) \qquad \left|\frac{\partial^2}{\partial x_i \partial x_j}\varpi^* a(x)\right| \leqq B[|\nabla^2 a|(\varpi(x))+|\nabla a|(\varpi(x))]/(|x|^2+1)$$

$$x \in W \cap R^k\,, \quad (i, j=1, \cdots, k)\,.$$

*Here $A$ and $B$ are constants independent of $a$ and $x$, and, for a $C^\infty$ function $b$ on $R^k$, we put*

$$|\nabla b|(x)=\sup_{1\leq i\leq k}\left|\frac{\partial}{\partial x_i}b(x)\right|\,, \qquad |\nabla^2 b|(x)=\sup_{1\leq i,j\leq k}\left|\frac{\partial^2}{\partial x_i \partial x_j}b(x)\right|\,.$$

PROOF. This is an immediate consequence of a direct calculation. Q.E.D.

Now we go on to the Grauert theorem for the sheaf $\mathcal{O}_{inc}$:

THEOREM 2.2.2 (Cf. §3 in Grauert [6]). *Let $\Omega$ be an open set in $D^n$ and $W$ an open neighborhood of $\Omega$ in $Q^n$. Then there exists an $\mathcal{O}_{inc}$-pseudoconvex open set $V$ in $Q^n$ such that $\Omega \subset V \subset W$ and $V \cap D^n = \Omega$.*

PROOF. We identify $Q^n$ with $D^{2n}$, and denote again by $\varpi$ the diffeomorphism of $Q^n$ onto $B(=B^{2n})$ defined as in ($\varpi$). We denote by $\pi_R$ the natural projection of $C^n$ to $R^n=\{z \in C^n; \operatorname{Im} z=0\}$. Put $W'=W \cap \overline{\pi_R^{-1}(\mathring{\Omega} \cap C^n)}$. Then we have $\overline{\pi_R(\mathring{W'} \cap C^n)}=\Omega$ and $W' \cap D^n=\Omega$. Without loss of generality, we can assume that $W'$ is acute. Here we rewrite $W'$ by $W$.

*a*) We choose a real valued $C^\infty$ function $\gamma$ on $\varpi(W) \cap \mathring{B}$ satisfying the following conditions:

i) $\{z \in \varpi(W) \cap \mathring{B};\ \gamma(z)<c\} \subset\subset \varpi(W)$ for any $c \in R$,

ii) $M_k=\sup_{z \in K \cap \mathring{B}}\{\gamma(z), |\nabla\gamma|(z), |\nabla^2\gamma|(z)\}<\infty$ for any $K \subset\subset \varpi(W)$.

We can take an exhaustion $\{K_\nu\}$ of $W$ consisting of compact subsets of $W$ which satisfies the following conditions:

iii) $K_\nu \subset \mathring{K}_{\nu+1}$,

iv) $\overline{\pi_R(K_\nu \cap C^n)} \subset \overline{\pi_R(\mathring{K}_{\nu+1} \cap C^n)}$.

We also choose a positive valued $C^\infty$ function $a$ on $\varpi(\Omega) \cap \mathring{B}$ satisfying the following conditions:

v) $\inf\{a(x);\ x \in \varpi[\pi_R(K_{\nu+1} \cap C^n) \backslash \pi_R(K_\nu \cap C^n)]\} > CM_{\varpi(K_{\nu+1})}$ $(\nu = 1, 2, \cdots)$,

vi) $\{x \in \varpi(\Omega);\ a(x) < c\} \subset \varpi(\Omega)$ for any $c \in R$,

vii) $N_L = \sup_{x \in L \cap \bar{B}}\{a(x), |\nabla a|(x), |\nabla^2 a|(x)\} < \infty$ for any $L \subset \varpi(\Omega)$.

We will specify later the constant $C$ in v) (which does not depend on $K_\nu$). We put

$$p_1(z) = \frac{|\mathrm{Im}\, z|^2}{|\mathrm{Re}\, z|^2 + 1} a(\varpi(\mathrm{Re}\, z)) + \gamma(\varpi(z)) \qquad z \in W \cap C^n .$$

Then $p_1$ is $C^\infty$ on $W \cap C^n$. Moreover there exists an open set $W'$ in $Q^n$ such that $\Omega \subset W' \subset W$ and $p_1$ is strictly plurisubharmonic on $W' \cap C^n$. We will find this open set $W'$ by an explicit calculation of the Hermitian quadratic form:

$$ {}^t\bar{w}\left[\frac{\partial^2}{\partial z_i \partial \bar{z}_j} p_1(z)\right] w \qquad z \in W \cap C^n,\ w \in C^n .$$

We denote $z = x + \sqrt{-1}y (x, y \in R^n)$. We note that the following estimations hold:

$$\sup_{1 \le i \le n} \left|\frac{\partial}{\partial x_j} \frac{1}{|x|^2+1}\right| < \frac{C_1}{(|x|+1)^3} \qquad x \in R^n . \tag{2.2.3}$$

$$\sup_{1 \le i, j \le n} \left|\frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{|x|^2+1}\right| < \frac{C_2}{(|x|+1)^4} \qquad x \in R^n . \tag{2.2.4}$$

Using (2.2.1), (2.2.2), (2.2.3) and (2.2.4), we have the following estimation from bellow:

$$\begin{aligned} {}^t\bar{w}\left[\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\left(\frac{|y|^2}{|x|^2+1} a(\varpi(x))\right)\right] w &\ge \frac{1}{2} \frac{a(\varpi(x))}{|x|^2+1} |w|^2 \\ &\quad - n\left[\frac{|y|^2}{(|x|+1)^4}\{B'(|\nabla a|(\varpi(x)) + |\nabla^2 a|(\varpi(x))) + C_2' a(\varpi(x))\}\right. \\ &\quad \left. + \frac{|y|}{(|x|+1)^3}\{A'|\nabla a|(\varpi(x)) + C_1' a(\varpi(x))\}\right]|w|^2 \qquad z \in W \cap C^n,\ w \in C^n . \end{aligned}$$

Hence, if we put $U_\varepsilon = \{z \in C^n;\ |\mathrm{Im}\, z| < \varepsilon(|\mathrm{Re}\, z| + 1)\}$, we have the following estimation from below:

$$\begin{aligned} {}^t\bar{w}\left[\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\left(\frac{|y|^2}{|x|^2+1} a(\varpi(x))\right)\right] w &\ge \frac{1}{2} \frac{a(\varpi(x))}{|x|^2+1} |w|^2 \\ &\quad - \frac{n}{|x|^2+1} \varepsilon C' [a(\varpi(x)) + |\nabla a|(\varpi(x)) + |\nabla^2 a|(\varpi(x))]|w|^2 \quad z \in W \cap U_\varepsilon,\ w \in C^n , \end{aligned}$$

where $C'$ is a constant which does not depend on $z$ and $a$.

Thus applying (2, 2, 2) also to $[\partial^2/\partial z_i \partial \bar{z}_j \gamma(z)]$, we have

$$
\begin{aligned}
{}^t\bar{w}\left[\frac{\partial^2}{\partial z_i \partial \bar{z}_j} p_1(z)\right] w \geqq & \frac{1}{2} \frac{a(\varpi(x))}{|x|^2+1}|w|^2 \\
& -\frac{n}{|x|^2+1}\Big[\varepsilon C'\{a(\varpi(x))+|\nabla a|(\varpi(x))+|\nabla^2 a|(\varpi(x))\} \\
& +B''\{|\nabla\gamma|(\varpi(z))+|\nabla^2\gamma|(\varpi(z))\}\Big]|w|^2 \\
\geqq & \frac{1}{2} \frac{a(\varpi(x))}{|x|^2+1}|w|^2-\frac{n}{|x|^2+1}[3\varepsilon C' N_{\varpi(K)}+2B' M_{\varpi(k)}]|w|^2 \\
& z \in K \cap U_\varepsilon,\ w \in \boldsymbol{C}^n \quad (K \subset W) .
\end{aligned}
$$

Therefore, if we put $C=3nB''$ ($C$ is the constant in v) not yet fixed) and $\varepsilon_\nu = B'' M_{\varpi(K_\nu)}/C' N_{\varpi(\pi_{\boldsymbol{R}}(K_\nu))}$ $(\nu=1, 2, \cdots)$, we have

$$
\begin{aligned}
4\,{}^t\bar{w}\left[\frac{\partial^2}{\partial z_i \partial \bar{z}_j} p_1(z)\right] w & \geqq \frac{|w|^2}{|x|^2+1}[6nB'' M_{\varpi(K_\nu)}-3nB'' M_{\varpi(K_\nu)}-2nB'' M_{\varpi(K_\nu)}] \\
& = nB'' M_{\varpi(K_\nu)} \frac{|w|^2}{|x|^2+1}>0 \\
& z \in (K_\nu \backslash K_{\nu-1}) \cap U_{\varepsilon_\nu},\ w \in \boldsymbol{C}^n \quad (\nu=1, 2, \cdots, K_0=\phi) .
\end{aligned}
$$

Hence if we put $W'=W \cap \overline{(\cup_\nu (U_{\varepsilon_\nu} \cap (K_\nu \backslash K_{\nu-1})))}^{\circ}$, $p_1(z)$ is strictly plurisubharmonic on $W' \cap \boldsymbol{C}^n$. It is clear that $W'$ satisfies $W' \subset W$ and $W' \cap \boldsymbol{D}^n = \Omega$.

b) Next, for $w \in \boldsymbol{R}^n$ and positive number $\delta$, put

$$
q_{w,\delta}(z)=\max\left\{0,\ 2\frac{\delta|\operatorname{Im} z|^2-|\operatorname{Re}(z-w)|^2}{|\operatorname{Im} z|^2}\right\},
$$

Then $q_{w,\delta}$ is a plurisubharmonic function of $\boldsymbol{C}^n$. We can choose a sequence $\{w_\nu\} \subset \Omega \backslash S_\infty^{2n-1}$ and a sequence of positive numbers $\{\delta_\nu\} \subset \boldsymbol{R}$ so that

viii) $q(z)=\sum_\nu q_\nu(z)$ is a locally finite sum,

ix) $q(z)>1$ on $(\partial W' \backslash \partial_{\boldsymbol{D}^n} \Omega) \cap \boldsymbol{C}^n$,

x) $V=W' \cap \overline{\{z \in \boldsymbol{C}^n;\ q(z)<1\}}^{\circ}$ contains $\Omega$,

where $q_\nu$ is a suitable mollification of $q_{w_\nu,\delta_\nu}$ and $\partial_{\boldsymbol{D}^n}$ denotes the boundary of $\Omega$ in $\boldsymbol{D}^n$.

c) We put

$$
p=p_1+1/(1-q) .
$$

Then $p$ is a strictly plurisubharmonic $C^\infty$ function on $V \cap \boldsymbol{C}^n$ which satisfies the condition (P) in the definition of $\mathcal{O}_{inc}$-pseudoconvexity (Definition 2.1.4.). $V$ is what we need. Q.E.D.

**2.3.** An approximation theorem in $\mathscr{A}_{dec}$.

In this section we prove an approximation theorem in the sheaf $\mathscr{A}_{dec}$. The essential part of the proof of this theorem relies, as in Kawai [16], [17], on the theory of $L^2$ estimates for the $\bar{\partial}$ operator in Hörmander [8], [9] and the theory of topological spaces such as Fréchet-Schwartz spaces, dual Fréchet-Schwartz spaces, Fréchet-Kômura spaces and dual Fréchet-Kômura spaces in Komatsu [18], ]19], [20].

THEOREM 2.3.1. *Let $K$ be a compact set in $\boldsymbol{D}^n$. Then $\mathscr{A}_{dec}(\boldsymbol{D}^n)$ is dense in $\mathscr{A}_{dec}(K)$.*

For the proof of the theorem, we need some preparations.

DEFINITION 2.3.1. For $\alpha \in \boldsymbol{R}$, we denote $\mathscr{O}^{2;\alpha}_{loc}$ the sheaf on $\boldsymbol{Q}^n$ whose section module $\mathscr{O}^{2;\alpha}_{loc}(W)$ over an open set $W$ in $\boldsymbol{Q}^n$ is given by the following:

$$\mathscr{O}^{2;\alpha}_{loc}(W)=\left\{f \in \mathscr{O}(W\cap \boldsymbol{C}^n);\ |||f|||^2_{\alpha,K\cap \boldsymbol{C}^n}=\int_{K\cap \boldsymbol{C}^n}|f(z)|^2e^{-\alpha|z|}d\lambda<\infty \quad \text{for any} \quad K\subset W\right\},$$

where $d\lambda$ is the Lebesgue measure on $\boldsymbol{C}^n\cong \boldsymbol{R}^{2n}$. We also denote by $\mathscr{L}^{2;\alpha}_{loc}$ the sheaf on $\boldsymbol{Q}^n$ whose section module $\mathscr{L}^{2;\alpha}_{loc}(W)$ over an open set $W$ in $\boldsymbol{Q}^n$ is given by the following:

$$\mathscr{L}^{2;\alpha}_{loc}(W)=\{f \in \mathscr{L}^2_{loc}(W\cap \boldsymbol{C}^n);\ |||f|||_{\alpha,K\cap \boldsymbol{C}^n}<\infty \quad \text{for any} \quad K\subset W\},$$

where $\mathscr{L}^2_{loc}$ denotes the sheaf of locally square summable functions on $\boldsymbol{C}^n\cong \boldsymbol{R}^{2n}$.

For any open set $W$ in $\boldsymbol{Q}^n$, $\mathscr{O}^{2;\alpha}_{loc}(W)$ is a Fréchet-Schwartz space, and $\mathscr{L}^{2;\alpha}_{loc}(W)$ is a Fréchet-Kômura space with the seminorms $\{|||\cdot|||_{\alpha,K\cap \boldsymbol{C}^n}\}$. $\mathscr{O}^{2;\alpha}_{loc}(W)$ is a closed subspace of $\mathscr{L}^{2;\alpha}_{loc}(W)$. The strong dual space of $\mathscr{L}^{2;\alpha}_{loc}(W)$ is represented by the dual Fréchet-Kômura space

$$\mathscr{L}^{2,-\alpha}_{loc,comp}(W)=\{f \in \mathscr{L}^{2;-\alpha}_{loc}(W);\ \operatorname{supp} f\subset W\}.$$

We note that for any $\alpha$ and $\alpha'$ satisfying $\alpha'<\alpha$, $\mathscr{O}^{2;\alpha'}_{loc}(W)\subset \mathscr{O}^{2;\alpha}_{loc}(W)$ and $\mathscr{L}^{2;\alpha'}_{loc}(W)\subset \mathscr{L}^{2;\alpha}_{loc}(W)$ hold.

LEMMA 2.3.2. *Let $K$ be a compact set in $\boldsymbol{Q}^n$, $\{W_j\}$ be a decreasing approximating sequence of $K$ consisting of open sets in $\boldsymbol{Q}^n$ such that $W_{j+1}\subset W_j$. Then we have the following linear topological isomorphism:*

$$\mathscr{O}_{dec}(K)=\lim_{\substack{\longrightarrow \\ j}} \operatorname{ind} \mathscr{O}^{2;-1/j}_{loc}(W). \tag{2.3.1}$$

PROOF. Consider the following injective sequence:

$$\cdots \xrightarrow{\rho} \mathcal{O}^{-1/j}(W_j \cap C^n) \xrightarrow{\rho} \mathcal{O}^{2,-1/j}_{loc}(W_j)$$
$$\xrightarrow{\rho} \mathcal{O}^{-1/2j}(W_{2j} \cap C^n) \xrightarrow{\rho} \mathcal{O}^{2,-1/2j}_{loc}(W_{2j}) \xrightarrow{\rho} \cdots ,$$

where $\rho$ is the natural restriction mapping. Then we find the isomorphism (2.3.1) by virtue of the continuity of $\rho$. Q.E.D.

LEMMA 2.3.3. *Let $W$ be an acute open set in $Q^n$. Then for any $\alpha'$ and $\alpha \in R$ $(\alpha'<\alpha)$, $\mathcal{O}^{2,\alpha'}_{loc}(W)$ is dense in $\mathcal{O}^{2,\alpha}_{loc}(W)$.*

PROOF. Let $f \in \mathcal{O}^{2,\alpha}_{loc}(W)$. Then since $W$ is acute, $f_\nu(z) = f(z)\exp(-(1/\nu)z^2) \in \mathcal{O}^{2,\alpha'}_{loc}(W)$ hold for any $\nu \in N$, where $z^2 = z_1^2 + \cdots + z_n^2$. Moreover by the Lebesgue dominated convergence theorem, we have

$$|||f_\nu - f|||_{\alpha, K\cap C^n} \longrightarrow 0 \qquad (\nu \longrightarrow \infty)$$

for all $K \subset W$. This shows the lemma. Q.E.D.

We need another

LEMMA 2.3.4. *Let $K$ be a compact set in $D^n$. Then there exists a decreasing approximating sequence $\{V_j\}$ of $D^n$ consisting of $\mathcal{O}_{inc}$-pseudoconvex open sets in $Q^n$ and a decreasing approximating sequence $\{W_j\}$ of $K$ consisting of open sets in $Q^n$ satisfying the conditions*:

i)
$$\begin{array}{l} V_1 \supset V_2 \supset \cdots \supset D^n \\ \cup \quad \cup \\ W_1 \supset W_2 \supset \cdots \supset K . \end{array}$$

ii) *For any $j$ and for any $L \subset W_j$, there exist an open set $U$ such that $L \subset U \subset W_j$ and a strictly plurisubharmonic $C^\infty$ function $\vartheta$ on $V_j \cap C^n$ such that*

$$\vartheta < 0 \quad \text{on} \quad L \cap C^n, \ \vartheta > 0 \quad \text{near} \quad \partial U \cap C^n \quad \text{and} \quad \sup_{z \in V_j \cap C^n} \vartheta(z) < \infty .$$

PROOF. First we construct $V_j$ and $W_j$. We put

$$U_j = \{z \in C^n; \ |\mathrm{Im}\, z|^2 < 2^{-j}(|\mathrm{Re}\, z|^2 + 1)\} , \qquad V_j = \mathring{\bar{U}}_j .$$

We note that $V_j$ is pseudoconvex.

We say that an open set $W$ in $Q^n$ is of type (E), if there exist open sets $\{\Omega_k\}_{k=1}^\infty$ in $Q^n$ such that

$$W = \bigcap_k \Omega_k , \qquad \text{where} \quad \Omega_k = \mathring{\bar{\omega}}_k \quad \text{and}$$

$$\omega_k=\{z\in C^n;\ |\mathrm{Re}\, z\ -a_k|^2-|\mathrm{Im}\, z|^2>B_k,\ |\mathrm{Im}\, z|^2<A_k(|\mathrm{Re}\, z|^2+1)\}\ ,$$

where $B_k$ and $A_k$ are positive numbers and $a_k\in R^n$. Since $K$ is a compact set in $D^n$, there exists a decreasing approximating sequence $\{W_j\}$ of $K$ consisting of open sets of type (E) in $Q^n$ satisfying the condition i).

Next, for any given compact subset $L$ of $W_j$, we find an open set $U$ in $Q^n$ and a strictly plurisubharmonic $C^\infty$ function $\vartheta$ on $V_j\cap C^n$ satisfying the condition ii). Here we fix the index $j$ of $W_j$ and $V_j$, and rewrite $W_j$ and $V_j$ as $W$ and $V$ respectively, and put $d=2^{-j}$.

Since $W$ is of type (E), it can be written as

$$W=\bigcap_k \Omega_k\ ,\qquad \text{where}\quad \Omega_k=\mathring{\bar{\omega}}_k\quad \text{and}$$

$$\omega_k=\{z\in C^n;\ |\mathrm{Re}\, z-a_k|^2-|\mathrm{Im}\, z|^2>B_k,\ |\mathrm{Im}\, z|^2<A_k(|\mathrm{Re}\, z|^2+1)\}\ .$$

Here we note that we can choose $A_k$'s as $A_k=A$ (i.e. independent of $k$). Then, for any compact set $L$ in $W$, there exists an open set $W'$ of type (E) in $Q^n$ such that

$$L\subset W'\subset W\ ,$$

$$W'=\bigcap_k \Omega'_k\ ,\qquad \text{where}\quad \Omega'_k=\mathring{\bar{\omega}}'_k\quad \text{and}$$

$$\omega'_k=\{z\in C^n;\ |\mathrm{Re}\, z-a_k|^2-|\mathrm{Im}\, z|^2>B'_k,\ |\mathrm{Im}\, z|^2<A'(|\mathrm{Re}\, z|^2+1)\}\ ,$$

Here we note $B'_k>B_k>0$, $A'<A<d$, $\mathrm{dist}(\partial L\cap C^n, \partial W'\cap C^n)>0$ and $\mathrm{dist}(\partial W'\cap C^n, \partial W\cap C^n)>0$. We take positive $\varepsilon$ which satisfies $d+2\varepsilon<1$ and is smaller than
$(1/4)\mathrm{dist}(\partial L\cap C^n, \partial W'\cap C^n)$ and $(1/4)\,\mathrm{dist}\,(\partial W'\cap C^n, \partial W\cap C^n)$.

Here we put

$$\psi_k(z)=\frac{1}{h_k(z)+\alpha_k}-\gamma_k\ ,\qquad \varphi_k=\chi*\psi_k\ ,$$

where we put

$$h_k(z)=|\mathrm{Re}\, z-a_k|^2-|\mathrm{Im}\, z|^2\ ,$$

$$\alpha_k=\frac{d+2\varepsilon}{1-(d+2\varepsilon)}|a_k|^2+(d+2\varepsilon)+1\quad \text{and}\quad \gamma_k=\frac{1}{B'_k+\alpha_k}\ ,$$

and $\chi$ is a positively valued molifier with the support contained in the $\varepsilon$-ball in $C^n$ centered at the origin. We note that the following inequalities and equality hold:

$$(*)\qquad \begin{cases} h_k(z)+\alpha_k \geqq 1 & \text{if} \quad z \in U_{d+2\varepsilon}\,, \\ \dfrac{\partial^2}{\partial z_i \partial \bar{z}_j}\psi_k(z) = \dfrac{2(z_i-a_i)(\bar{z}_j-a_j)}{(h_k(z)+\alpha_k)^3}\,, \end{cases}$$

$$(**)\qquad \begin{cases} \psi_k(z)<0 & \text{if} \quad |\mathrm{Re}\, z-a_k|^2-|\mathrm{Im}\, z|^2 > B'_k\,, \\ \psi_k(z)\geqq 0 & \text{if} \quad z\in U_{d+2\varepsilon} \text{ and } |\mathrm{Re}\, z-a_k|^2-|\mathrm{Im}\, z|^2 \leqq B'_k\,, \end{cases}$$

where we put $U_{d+2\varepsilon}=\{z\in \boldsymbol{C}^n;\ |\mathrm{Im}\, z|^2<(d+2\varepsilon)(|\mathrm{Re}\, z|^2+1)\}$. From $(*)$, we find that $\psi_k$ is $C^\infty$ and plurisubharmonic on a neighborhood of $V\cap \boldsymbol{C}^n$. Moreover we can find that $\varphi_k$ is $C^\infty$ and strictly plurisubharmonic on a neighborhood of $V\cap \boldsymbol{C}^n$, and $\varphi_k<1$ on there.

Then, if neccesary, by a slight modification of the choice $a_k$'s, we can make the function

$$\psi(z)=\max\left\{\sup_k \varphi_k(z),\ \frac{|\mathrm{Im}\, z|^2}{A'(|\mathrm{Re}\, z|^2+1)}-1\right\}$$

satisfy the assumption of Lemma A.1 in Appendix on a neighborhood of $V\cap \boldsymbol{C}^n$. Thus by Lemma A.1, the function $\vartheta=\chi*\psi$ defines a strictly plurisubharmonic $C^\infty$ function on $V\cap \boldsymbol{C}^n$. Now we put

$$U=\overset{\circ}{\overline{W'_{3\varepsilon}}}\,, \qquad \text{where} \quad W'_{3\varepsilon}=\{z\in \boldsymbol{C}^n;\ \mathrm{dist}(z,\ W')<3\varepsilon\}\,.$$

Then, from $(**)$ and the definitions of functions $\varphi_k$ and $\psi$, we find that the function $\vartheta$ and the open set $U$ satisfies the condition ii). Q.E.D.

Now we go on to the proof of Theorem 2.3.1:

PROOF OF THEOREM 2.3.1. *a*) By Lemma 2.3.4, there exist a decreasing approximating sequence $\{V_j\}$ of $\boldsymbol{D}^n$ consisting $\mathcal{O}_{inc}$-pseudoconvex open sets in $\boldsymbol{Q}^n$ and a decreasing approximating sequence $\{W_j\}$ of $K$ consisting of open sets in $\boldsymbol{Q}^n$ satisfying the conditions i) and ii) in Lemma 2.3.4.

*b*) Since $\mathcal{O}_{dec}(K)$ is a dual Fréchet-Schwartz space it is sufficient to show the following statement $(*)$ by Theorem 6′ in Komatsu [18] and Lemma 2.3.2:

$(*)$ When $j$ is sufficiently large, $\mathcal{O}^{2,-2/j}_{loc}(V_j)$ is dense in $\mathcal{O}^{2,-1/j}_{loc}(W_j)$.

In what follows, we assume that $j$ is sufficiently large, and rewrite as $V=V_j$, $W=W_j$ and $\varepsilon=1/j$.

Using the terminology of the Hahn-Banach theorem, we can interpret $(*)$ as follows:

"If $\mu\in[\mathcal{O}^{2,-\varepsilon}_{loc}(W)]'$ vanishes on $\mathcal{O}^{2,-2\varepsilon}_{loc}(V)$, then $\mu=0\in[\mathcal{O}^{2,-\varepsilon}_{loc}(W)]'$."

Since $\mathcal{O}^{2,-\varepsilon}_{loc}(W)$ is a subspace of $\mathcal{L}^{2,-\varepsilon}_{loc}(W)$, there exists a $u \in \mathcal{L}^{2,\varepsilon}_{loc,comp}(W)$ such that

$$\langle \mu, v\rangle = \int_{W\cap C^n} v\bar{u}d\lambda \qquad v \in \mathcal{O}^{2,-\varepsilon}_{loc}(W)$$

again by the Hahn-Banach theorem, where $\bar{u}$ denotes the complex conjugate of $u$. Therefore it is sufficient to show that $u$ is orthogonal to $\mathcal{O}^{2,-\varepsilon}_{loc}(W)$.

*c*) Put $L=\overline{\operatorname{supp} u}$. Then by Lemma 2.3.4, there exist a strictly plurisubharmonic $C^\infty$ function $\vartheta$ on $V\cap C^n$ and an open neighborhood $U$ of $L$ in $W$ such that

$$\vartheta<0 \ \text{ on } \ L\cap C^n\,, \qquad \vartheta>0 \ \text{ on } \ \partial U\cap C^n \ \text{ and } \sup_{z\in V\cap C^n} \vartheta(z)<\infty\,.$$

We put $\vartheta^+(z)=\max\{\theta(z), 0\}$.

*d*) We put $g_{4\varepsilon}(z)=\cosh(4\varepsilon\sqrt{z^2})$ $(z^2=z_1^2+\cdots+z_n^2)$. Then by Lemma 3.1.5 in Saburi [40], we have the following estimation for sufficiently small $\varepsilon>0$:

$$\exp(-4\varepsilon|z|)<|g_{4\varepsilon}(z)|^{-1}<C\exp(-3\varepsilon|z|) \qquad (z\in V\cap C^n)\,,$$

where $C$ is a constant independent of $z$. Hence we have $u/\bar{g}_{4\varepsilon} \in L^{2,-2\varepsilon}_{loc,comp}(W)$.

On the other hand, for a positive $\alpha$, we put

$$\begin{aligned}&\mathcal{O}^2(V\in C^n;\ \varepsilon|z|+2\log(|z|^2+1)+\alpha\vartheta^+(z))\\ &\quad=\left\{v\cap\mathcal{O}(V\cap C^n);\ \int_{V\cap C^n}|v|^2\exp(-\varepsilon|z|-2\log(|z|^2+1)-\alpha\vartheta^+(z))d\lambda<\infty\right\}.\end{aligned}$$

Then we have $v/g_{4\varepsilon}\in\mathcal{O}^{2,-2\varepsilon}_{loc}(V)$, for any $v\in\mathcal{O}(V\cap C^n;\ \varepsilon|z|+2\log(|z|^2+1)+\alpha\vartheta^+(z))$ and any $\alpha>0$.

Hence by the assumption for $\mu$, we have

$$\int_{W\cap C^n} v\,\overline{(u/\bar{g}_{4\varepsilon})}d\lambda=\int_{W\cap C^n}(v/g_{4\varepsilon})ud\lambda=\langle\mu,\ v/g_{4\varepsilon}\rangle=0$$

for any $v\in\mathcal{O}^2(V\cap C^n;\ \varepsilon|z|+2\log(|z|^2+1)+\alpha\vartheta^+(z))$ and any $\alpha>0$. Therefore by Proposition 2.3.2 in Hörmander [8], there exists $f\in L^2_{(0,1)}(V\cap C^n;\ -\varepsilon|z|)$ such that

$$\mathfrak{d}f=u/\bar{g}_{4\varepsilon},\ \operatorname{supp} f\subset\{z\in V\cap C^n;\ \vartheta(z)\leqq 0\}\,,$$

where we followed the notation in Hörmander [8], [9]:

$$L^2_{(0,1)}(V\cap C^n;\ -\varepsilon|z|)=\left\{f\in\mathcal{L}^{2,(0,1)}_{loc}(V\cap C^n);\ \int_{V\cap C^n}|f(z)|^2e^{\varepsilon|z|}d\lambda<\infty\right\}.$$

*e*) We choose $\chi \in C^1(V\cap C^n)$ $(0\leqq\chi\leqq 1)$ such that

$$\chi=1 \quad \text{on} \quad L\cap C^n, \qquad \chi=0 \quad \text{on} \quad C^n\backslash(U\cup N),$$
$$\operatorname{supp}(\bar\partial\chi)\subset N \quad \text{and} \quad \sup(\bar\partial\chi)<\infty,$$

where $N$ is a neighborhood of $\partial U$ in $W$ which does not meet $L$. Then for each $v\in\mathcal{O}^{2,-4\varepsilon}_{loc}(W)$, we have

$$\chi v g_{4\varepsilon}\in L^2(V\cap C^n;\ \varepsilon|z|+2\log(|z|^2+1)),$$
$$\bar\partial(\chi v g_{4\varepsilon})\in L^2_{(0,1)}(V\cap C^n;\ \varepsilon|z|) \quad \text{and} \quad \operatorname{supp}(\bar\partial(\chi v g_{4\varepsilon}))\subset N.$$

Hence we have

$$\langle\mu, v\rangle=\int_{W\cap C^n} v\bar u d\lambda=\int_{V\cap C^n} v g_{4\varepsilon}\overline{(u/\bar g_{4\varepsilon})}d\lambda=\int_{V\cap C^n} v g_{4\varepsilon}\overline{\bar\partial f}d\lambda=\int_{V\cap C^n}\bar\partial(\chi v g_{4\varepsilon})\cdot\bar f d\lambda=0$$

for any $v\in\mathcal{O}^{2,-4\varepsilon}_{loc}(W)$. Here we recall that $\mathcal{O}^{2,-4\varepsilon}_{loc}(W)$ is dense in $\mathcal{O}^{2,-\varepsilon}_{loc}(W)$ by Lemma 2.3.3. Hence we have also

$$\langle\mu, v\rangle=0 \qquad \text{for any} \quad v\in\mathcal{O}^{2,-\varepsilon}_{loc}(W).$$

Thus we have $\mu=0\in[\mathcal{O}^{2,-\varepsilon}_{loc}(W)]'$. Q.E.D.

REMARK. To prove the theorem in this section, we followed mostly to the proof of Theorem 2.2.1 in Kawai [17]. But, by preparing Lemmas 2.3.2 and 2.3.3, we attempt to improve the original proof of Kawai's. Here we note that there is a mistake (or misprint?) in Kawai's proof. That is, the claim (b)-(iv) for the strictly plurisubharmonic function $\vartheta(z)$ on $U_j\cap C^n$ in that proof:

$$\text{(b)-(iv)} \qquad \sup_{z\in L\cap C^n}\vartheta(z)<\infty \qquad \text{for any} \quad L\Subset\Omega_j$$

is not sufficient for the proof of the theorem. (Here we followed Kawai's notations) It should be replaced by

$$\text{(b)-(iv)}' \qquad \sup_{z\in L\cap C^n}\vartheta(z)<\infty \qquad \text{for any} \quad L\Subset U_j.$$

Because, as his proof relied on Proposition 2.3.2 in Hörmander [8], the required condition that the function $\vartheta(z)$ should satisfy is as follows:

$$C=\bigcup_{\lambda=1}^{\infty}\{v\in\mathcal{L}^2_{loc}(U_j\cap C^n;\ \lambda\vartheta^+-\delta'\|z\|+2\log(1+|z|^2));\ \bar\partial v=0\}$$
$$\subset B=\left\{u\in O(U_j\cap C^n);\ \int_{L\cap C^n}|u|^2e^{\delta\|z\|}dV<\infty,\ {}^{\forall}L\Subset U_j\right\}.$$

(Here we also followed Kawai's notations.) It is obvious that a sufficient condition for the above condition is not the claim (b)-(iv), but (b)-(iv)'. But fortunately, since the function $\vartheta(z)$ constructed there satisfies the claim (b)-(iv)', the proof can be saved.

In connection with the above problem, Nagamachi-Mugibayashi [33] made a same mistake as Kawai's. In Appendix in [33], they tried to prove the approximation theorem in the sheaf $\underset{\approx}{\mathcal{O}}$ following to the proof of Theorem 2.2.1 in Kawai [17]. Here, because of the same reason as above, we also have to say that the claim

$$\text{(b)-(iv)} \qquad \sup_{z\in L\cap C^n} \vartheta(z)<\infty \qquad \text{for any} \quad L\subset\Omega_p$$

for the strictly plurisubharmonic function $\vartheta(z)$ on $U_p^n\cap C^n$ should be replaced by the claim

$$\text{(b)-(iv)}' \qquad \sup_{z\in L\cap C^n} \vartheta(z)<\infty \qquad \text{for any} \quad L\subset U_p^n\,.$$

(Here we followed the notations of Nagamachi-Mugibayashi [33].) Further we note that the function $\vartheta(z)$ constructed there does not satisfies the claim (b)-(iv)' in general. Because, in their case, the function $\sigma(z)$ (defined there) does not satisfy the claim (b)-(iv)' in general.

$$\sigma(z)=O(|z|^2) \qquad \text{on} \quad U_p^n\backslash\Omega_p\,.$$

We confess that when we wrote Saburi [38], we were not aware of the above mistake in Nagamachi-Mugibayashi [33]. So, in Saburi [38], we referred the approximation theorem in Nagamachi-Muguibayashi [33]. But we can now rectify the mistake by showing Lemma 2.3.4 in the present paper.

## § 3. Duality theorems and pure codimensionality of $D^n$ with respect to the sheaf $\mathcal{O}_{inc}$.

In this section we show the fundamental properties of the sheaf $\mathcal{R}$ of modified Fourier hyperfunctions. These are stated in Theorem 3.2.2 and its Corollary. Here, as in §2.3, we use the theory of Fréchet-Schwartz spaces, dual Fréchet-Schwartz spaces, Fréchet-Kômura spaces and dual Fréchet-Kômura spaces in Komatsu [18], [19], [20]. We also use the general theory of local (or relative) cohomology groups. As to the general theory of local (or relative) cohomology groups, we refer to Godement [5], Grothendieck [7] and Morimoto [26].

### 3.1. Martineau-Harvey duality for the sheaves $\mathcal{O}_{inc}$ and $\mathcal{O}_{dec}$.

In this section we treat the Serre duality and the Martineau-Harvey duality for the sheaves $\mathcal{O}_{inc}$ and $\mathcal{O}_{dec}$.

First we start with the Serre duality:

THEOREM 3.1.1 *Let $V$ be an open set in $\boldsymbol{Q}^n$. Assume* $\mathrm{H}^q(V; \mathcal{O}_{inc})= 0 (q \geqq 1)$. *Then we have the following linear topological isomorphism:*

$$[\mathrm{H}^j(V; \mathcal{O}_{inc})]' \cong H^{n-j}_{comp}(V; \mathcal{O}_{dec}) . \tag{3.1.1}$$

REMARK. The above theorem also holds under the assumption $\dim \mathrm{H}^q(V; \mathcal{O}_{inc}) < \infty$ $(q \geqq 1)$.

PROOF OF THEOREM 3.1.1. Consider the following dual complexes:

$$(3.1.2) \quad 0 \longrightarrow \mathscr{X}^{(0,0)}(V) \xrightarrow{\bar\partial} \mathscr{X}^{(0,1)}(V) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{X}^{(0,n)}(V) \longrightarrow 0$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$(3.1.2)' \quad 0 \longleftarrow \mathscr{Y}^{(n,0)}_{comp}(V) \xleftarrow[-\bar\partial]{} \mathscr{Y}^{(n,n-1)}_{comp}(V) \xleftarrow[-\bar\partial]{} \cdots \xleftarrow[-\bar\partial]{} \mathscr{Y}^{(n,0)}_{comp}(V) \longleftarrow 0 .$$

We recall that the $j$-th cohomology group of the complex (3.1.2) is isomorphic to $\mathrm{H}^j(V; \mathcal{O}_{inc})$, and the $j$-th cohomology group of the complex (3.1.2)′ is isomorphic to $\mathrm{H}^j_{comp}(V; \mathcal{O}_{dec})$. Here we note that $\mathscr{X}^{(p,q)}(V)$ are Fréchet-Kômura spaces, and $\bar\partial$ is a densely defined closed operator of those spaces. Hence by the assumption $\mathrm{H}^q(V; \mathcal{O}_{inc})=0$, we find that $\bar\partial$ has a closed range. Thus by the Serre-Komatsu duality theorem (Theorem 19 in Komatsu [18], p-381), we have the isomorphism (3.1.1). Q.E.D.

COROLLARY 3.1.2. *Let $V$ be an acute open set in $\boldsymbol{Q}^n$, $K$ a compact set in $V$. Suppose*

$$\mathrm{H}^q(V; \mathcal{O}_{inc})=0 \qquad (q \geqq 1) ,$$
$$\mathrm{H}^q(K; \mathcal{O}_{dec})=0 \qquad (q \geqq 1) ,$$

*then we have the following linear topological isomorphism:*

$$[H^q(V\backslash K; \mathcal{O}_{inc})]' \cong H^{n-q}_{comp}(V\backslash K; \mathcal{O}_{dec}) . \tag{3.1.3}$$

PROOF. *a*) Similarly to the proof of Theorem 3.1.1, consider the following dual complexes:

(3.1.4)
$$0 \longrightarrow \mathscr{X}^{(0,0)}(V\backslash K) \xrightarrow{\bar\partial_1} \mathscr{X}^{(0,1)}(V\backslash K) \xrightarrow{\bar\partial_2} \cdots \xrightarrow{\bar\partial_n} \mathscr{X}^{(0,n)}(V\backslash K) \longrightarrow 0$$

(3.1.4)′
$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$0 \longleftarrow \mathscr{Y}^{(0,n)}_{comp}(V\backslash K) \xleftarrow[-\bar\partial_n]{} \mathscr{Y}^{(0,n-1)}_{comp}(V\backslash K) \xleftarrow[-\bar\partial_{n-1}]{} \cdots \xleftarrow[-\bar\partial_1]{} \mathscr{Y}^{(0,0)}_{comp}(V\backslash K) \longleftarrow 0 .$$

It is sufficient to show that $-\bar{\partial}$ have closed ranges in (3.1.4)$'$.

We recall the following exact sequence:

$$\begin{aligned}(3.1.5)\qquad 0 &\longrightarrow H^0_{comp}(V\backslash K;\ \mathcal{O}_{dec}) \longrightarrow H^0_{comp}(V;\ \mathcal{O}_{dec}) \longrightarrow H^0(K;\ \mathcal{O}_{dec})\\ &\longrightarrow H^1_{comp}(V\backslash K;\ \mathcal{O}_{dec}) \longrightarrow H^1_{comp}(V;\ \mathcal{O}_{dec}) \longrightarrow H^1(K;\ \mathcal{O}_{dec})\\ &\longrightarrow \cdots\\ &\longrightarrow H^n_{comp}(V\backslash K;\ \mathcal{O}_{dec}) \longrightarrow H^n_{comp}(V;\ \mathcal{O}_{dec}) \longrightarrow H^n(K;\ \mathcal{O}_{dec})\\ &\longrightarrow 0\,.\end{aligned}$$

Here we list up the vanishing terms in (3.1.5): By the assumption;

$$(3.1.6)\qquad H^q(K;\ \mathcal{O}_{dec})=0 \qquad (q\geqq 1)\,.$$

By the assumption and Theorem 3.1.1;

$$(3.1.7)\qquad H^q_{comp}(V;\ \mathcal{O}_{dec})=0 \qquad (0\leqq q\leqq n-1)\,.$$

By the uniqueness of analytic continuation;

$$(3.1.8)\qquad H^0_{comp}(V\backslash K;\ \mathcal{O}_{dec})=0\,.$$

Therefore we have the following isomorphisms:

$$(3.1.9)\qquad H^0(K;\ \mathcal{O}_{dec})\cong H^1_{comp}(V\backslash K;\ \mathcal{O}_{dec})\,,$$

$$(3.1.10)\qquad H^q_{comp}(V\backslash K;\ \mathcal{O}_{dec})=0 \qquad (2\leqq q\leqq n-1)\,,$$

$$(3.1.11)\qquad H^n_{comp}(V\backslash K;\ \mathcal{O}_{dec})=H^n_{comp}(V;\ \mathcal{O}_{dec})\,.$$

Hence by (3.1.10), we have the closedness of $\mathrm{Im}(-\bar{\partial}_q)$ $(2\leqq q\leqq n-1)$.

*b*) We will prove the closedness of $\mathrm{Im}(-\bar{\partial}_q)$ $(q=1, n)$.

First we will see the closedness of $\mathrm{Im}(-\bar{\partial}_1)$. Since $V$ is acute, $V\backslash K$ is also acute. Then we have

$$H^n(V\backslash K;\ \mathcal{O}_{inc})=0$$

by the Malgrange theorem for the sheaf $\mathcal{O}_{inc}$ (Theorem 2.1.4). Hence we have the closedness $\mathrm{Im}\,\bar{\partial}_n$ in (3.1.4). Then by the Serre-Komatsu duality theorem (Theorem 19 in Komatsu [18] p-381), we have the closedness of $\mathrm{Im}(-\bar{\partial}_1)$.

Next we will see the closedness of $\mathrm{Im}(-\bar{\partial}_n)$. Consider the following commutative diagram:

$$\begin{array}{ccc} 0 \longleftarrow \mathscr{Y}_{comp}^{(0,n)}(V\backslash K) & \overset{-\bar{\partial}_n^{V\backslash K}}{\longleftarrow} & \mathscr{Y}_{comp}^{(0,n-1)}(V\backslash K) \\ \downarrow \iota & & \downarrow \iota \\ 0 \longleftarrow \mathscr{Y}_{comp}^{(0,n)}(V) & \underset{-\bar{\partial}_n^V}{\longleftarrow} & \mathscr{Y}_{comp}^{(0,n-1)}(V) , \end{array}$$

where $\iota$ denotes the natural inclusion mapping. We note that $\iota$ is continuous. By the assumption $\mathrm{H}^1(V; \mathcal{O}_{inc})=0$, we have the closedness of $\mathrm{Im}\,\bar{\partial}_1^V$. Hence we have the closedness of $\mathrm{Im}(-\bar{\partial}_n^V)$ by the Serre-Komatsu duality theorem. On the other hand, we have $\mathrm{Im}(-\bar{\partial}_n^{V\backslash K})=\iota^{-1}(\mathrm{Im}(-\bar{\partial}_n^V))$ by (3.1.11). Thus by the continuity of $\iota$, we have the closedness of $\mathrm{Im}(-\bar{\partial}_n^{V\backslash K})$. Q.E.D.

Now we go on to the Martineau-Harvey duality theorem for the sheaves $\mathcal{O}_{inc}$ and $\mathcal{O}_{dec}$:

THEOREM 3.1.3. *Let $K$ be a compact set in $\boldsymbol{Q}^n$. Suppose that $K$ has a fundamental system of neighborhoods consisting of $\mathcal{O}_{inc}$-pseudoconvex open sets in $\boldsymbol{Q}^n$. Then we have the following linear topological isomorphism:*

$$(3.1.12) \qquad \begin{aligned} &H_K^q(V; \mathcal{O}_{inc})=0 \qquad (q\neq n) \\ &H_K^n(V; \mathcal{O}_{inc})\cong\mathcal{O}'_{dec}(K) , \end{aligned}$$

*where $V$ is any open neighborhood of $K$.*

PROOF. *a*) By the excision theorem, we may assume that $V$ is $\mathcal{O}_{inc}$-pseudoconvex, because $K$ has a fundamental system of neighborhoods consisting of $\mathcal{O}_{inc}$-pseudoconvex open sets in $\boldsymbol{Q}^n$. Then by the Cartan theorem B for the sheaves $\mathcal{O}_{inc}$ and $\mathcal{O}_{dec}$ (Theorems 2.1.2 and 2.1.3), we have

$$(3.1.13) \qquad \mathrm{H}^q(V; \mathcal{O}_{inc})=\mathrm{H}^q(K; \mathcal{O}_{dec})=0 \qquad (q\geqq 1) .$$

Hence $K$ and $V$ satisfies assumptions in Theorem 3.1.1 and Corollary 3.1.2.

*b*) Consider the following exact sequenc:

$$(3.1.14) \qquad \begin{aligned} 0 &\longrightarrow \mathrm{H}_K^0(V; \mathcal{O}_{inc}) \longrightarrow \mathrm{H}^0(V; \mathcal{O}_{inc}) \longrightarrow \mathrm{H}^0(V\backslash K; \mathcal{O}_{inc}) \\ &\longrightarrow \mathrm{H}_K^1(V; \mathcal{O}_{inc}) \longrightarrow \mathrm{H}^1(V; \mathcal{O}_{inc}) \longrightarrow \mathrm{H}^1(V\backslash K; \mathcal{O}_{inc}) \\ &\longrightarrow \cdots \\ &\longrightarrow \mathrm{H}_K^n(V; \mathcal{O}_{inc}) \longrightarrow \mathrm{H}^n(V; \mathcal{O}_{inc}) \longrightarrow \mathrm{H}^n(V\backslash K; \mathcal{O}_{inc}) \\ &\longrightarrow 0 . \end{aligned}$$

Here we list up the vanishing terms in (3.1.14): By the uniqueness of the analytic continuation;

$$\mathrm{H}^0_K(V;\ \mathcal{O}_{inc})=0\ . \tag{3.1.15}$$

By (3.1.13)

$$\mathrm{H}^q(V;\ \mathcal{O}_{inc})=0 \qquad (q\geqq 1)\ . \tag{3.1.16}$$

Hence we have

$$\mathrm{H}^1_K(V;\ \mathcal{O}_{inc})\cong\mathcal{O}_{inc}(V\backslash K)/\mathcal{O}_{inc}(V) \tag{3.1.17}$$

$$\mathrm{H}^q_K(V;\ \mathcal{O}_{inc})\cong\mathrm{H}^{q-1}(V\backslash K;\ \mathcal{O}_{inc}) \qquad (q\geqq 2) \tag{3.1.18}$$

*c*) Now we will show (3.1.12) in the case $q=1$. By Corollary 3.1.2, (3.1.11) and Theorem 3.1.1, we have the following isomorphism:

$$\begin{aligned}\mathcal{O}'_{inc}(V\backslash K)&=[\mathrm{H}^0(V\backslash K;\ \mathcal{O}_{inc})]'\\ &\cong\mathrm{H}^n_{comp}(V\backslash K;\ \mathcal{O}_{dec})\cong\mathrm{H}^n_{comp}(V;\ \mathcal{O}_{dec})\\ &\cong[\mathrm{H}^0(V;\ \mathcal{O}_{inc})]'=\mathcal{O}'_{inc}(V)\ .\end{aligned} \tag{3.1.19}$$

Recall that $\mathcal{O}_{inc}(V)$ and $\mathcal{O}_{inc}(V\backslash K)$ are Fréchet-Schwartz spaces, and so reflexive. Then by (3.1.19), we have

$$\mathcal{O}_{inc}(V\backslash K)\cong\mathcal{O}_{inc}(V)\ . \tag{3.1.20}$$

Hence by (3.1.17) and (3.1.20), we have

$$\mathrm{H}^1(V;\ \mathcal{O}_{inc})=0\ .$$

*d*) Next we will show (3.1.12) in the case $2\leqq q\leqq n-1$. If $2\leqq q\leqq n-1$, we have by (3.1.18), Corollary 3.1.2 and (3.1.10)

$$[\mathrm{H}^q_K(V;\ \mathcal{O}_{inc})]'\cong[\mathrm{H}^{q-1}(V\backslash K;\ \mathcal{O}_{inc})]'\cong\mathrm{H}^{n-(q-1)}_{comp}(V\backslash K;\ \mathcal{O}_{dec})=0\ .$$

Hence we have

$$\mathrm{H}^q(V;\ \mathcal{O}_{inc})=0 \qquad (2\leqq q\leqq n-1)\ .$$

*e*) At last we will show (3.1.12) in the case $q=n$. By (3.1.18), Corollary 3.1.2 and (3.1.19), we have

$$\begin{aligned}[\mathrm{H}^n_K(V;\ \mathcal{O}_{inc})]'&\cong[\mathrm{H}^{n-1}(V\backslash K;\ \mathcal{O}_{inc})]'\\ &\cong\mathrm{H}^1_{comp}(V\backslash K;\ \mathcal{O}_{dec})\cong\mathrm{H}^0_{comp}(K;\ \mathcal{O}_{dec})=\mathcal{O}_{dec}(K)\ .\end{aligned} \tag{3.1.21}$$

Recall that $\mathcal{O}_{dec}(K)$ is a dual Fréchet-Schwartz space and so reflexive.

Then by (3.1.21), we have

$$\mathrm{H}^n_K(V;\ \mathcal{O}_{inc})\cong\mathcal{O}'_{dec}(K)\ .$$

Thus we have completed the proof. Q.E.D.

REMARK. We can regard (3.1.20) as the Hartogs phenomenon in the case of the sheaf $\mathcal{O}_{inc}$.

**3.2** Pure codimensionality of $D^n$ with respect to the sheaf $\mathcal{O}_{inc}$.

In this section we treat the pure $n$-codimensionality of $\boldsymbol{D}^n$ with respect to the sheaf $\mathcal{O}_{inc}$ on $\boldsymbol{Q}^n$. Moreover we show that the presheaf $\mathscr{R}$ of modified Fourier hyperfunctions of $\boldsymbol{D}^n$ constitutes a flabby sheaf.

THEOREM 3.2.1. *$\boldsymbol{D}^n$ is pure $n$-codimensional with respect to the sheaf $\mathcal{O}_{inc}$ on $\boldsymbol{Q}^n$.*

PROOF. Put

$$U=\left\{z\in\boldsymbol{C}^n;\ |\mathrm{Im}\,z|^2<\frac{1}{2}\,|\mathrm{Re}\,z|^2+1\right\}\,,\qquad V=\mathring{\bar{U}}\,.$$

By the excision theorem, it is sufficient to show

$$\mathrm{H}^q_\Omega(V\backslash\partial_{D^n}\Omega;\ \mathcal{O}_{inc})=0\qquad(q\neq n)\tag{3.2.1}$$

for any open set $\Omega$ in $\boldsymbol{D}^n$. Here $\partial_{D^n}\Omega$ denotes the boundary of $\Omega$ in $\boldsymbol{D}^n$. In what follows, we rewrite $\partial_{D^n}\Omega$ by $\partial\Omega$.

We first show

$$\mathrm{H}^q_\Omega(V\backslash\partial\Omega;\ \mathcal{O}_{inc})=0\qquad(q\neq n-1,\ n)\ .\tag{3.2.1$'$}$$

Consider the following exact sequence:

$$\begin{aligned}0&\longrightarrow \mathrm{H}^0_{\partial\Omega}(V;\ \mathcal{O}_{inc})\longrightarrow \mathrm{H}^0_{\bar{\Omega}}(V;\ \mathcal{O}_{inc})\longrightarrow \mathrm{H}^0_\Omega(V\backslash\partial\Omega;\ \mathcal{O}_{inc})\\&\longrightarrow \mathrm{H}^1_{\partial\Omega}(V;\ \mathcal{O}_{inc})\longrightarrow \mathrm{H}^1_{\bar{\Omega}}(V;\ \mathcal{O}_{inc})\longrightarrow \mathrm{H}^1_\Omega(V\backslash\partial\Omega;\ \mathcal{O}_{inc})\\&\longrightarrow\cdots\\&\longrightarrow \mathrm{H}^n_{\partial\Omega}(V;\ \mathcal{O}_{inc})\longrightarrow \mathrm{H}^n_{\bar{\Omega}}(V;\ \mathcal{O}_{inc})\longrightarrow \mathrm{H}^n_\Omega(V\backslash\partial\Omega;\ \mathcal{O}_{inc})\\&\longrightarrow 0\ .\end{aligned}\tag{3.2.2}$$

Here we list up the vanishing terms in (3.2.2): By the Grauert theorem for the sheaf $\mathcal{O}_{inc}$ (Theorem 2.2.2), there exists a fundamental system of neighborhoods consisting of $\mathcal{O}_{inc}$-pseudoconvex open sets in $\boldsymbol{Q}^n$ for any compact set in $\boldsymbol{D}^n$. Hence we have

$$\mathrm{H}^q_{\partial\Omega}(V;\, \mathcal{O}_{inc}) = \mathrm{H}^q_{\bar{\Omega}}(V;\, \mathcal{O}_{inc}) = 0 \qquad (q \neq n) \tag{3.2.3}$$

by the Martineau-Harvey duality theorem for the sheaves $\mathcal{O}_{inc}$ and $\mathcal{O}_{dec}$.

Hence by the exactness of (3.2.2), we have

$$\mathrm{H}^q_{\Omega}(V \backslash \partial\Omega;\, \mathcal{O}_{inc}) = 0 \qquad (q \neq n-1,\, n)\ . \tag{3.2.1'}$$

Next we show

$$\mathrm{H}^n_{\Omega}(V \backslash \partial\Omega;\, \mathcal{O}_{inc}) = 0\ . \tag{3.2.1''}$$

By Theorem 2.3.1 and the Martineau-Harvey duality theorem, the mapping

$$\begin{array}{ccc} j\colon \mathrm{H}^n_{\partial\Omega}(V;\, \mathcal{O}_{inc}) & \longrightarrow & \mathrm{H}^n_{\bar{\Omega}}(V;\, \mathcal{O}_{inc}) \\ \wr\| & & \|\wr \\ \mathscr{A}'_{dec}(\partial\Omega) & \longrightarrow & \mathscr{A}'_{dec}(\bar{\Omega}) \end{array}$$

is injective. Then the exactness of the following sequence

$$0 \longrightarrow \mathrm{H}^{n-1}_{\Omega}(V \backslash \partial\Omega;\, \mathcal{O}_{inc}) \longrightarrow \mathscr{A}'_{dec}(\partial\Omega) \longrightarrow \mathscr{A}'_{dec}(\bar{\Omega})$$

implies (3.2.1''). Q.E.D.

THEOREM 3.2.2. *The presheaf $\mathscr{R}$ of modified Fourier hyperfunctions on $\boldsymbol{D}^n$ constitutes a flabby sheaf.*

PROOF. By the Malgrange theorem for the sheaf $\mathcal{O}_{inc}$ (Theorem 2.1.4), the restriction of the sheaf $\mathcal{O}_{inc,\boldsymbol{Q}^n}$ to an aucute open set in $\boldsymbol{Q}^n$ has its flabby dimension not greater than $n$. On the other hand $\boldsymbol{D}^n$ is pure $n$-codimensional with respect to the sheaf $\mathcal{O}_{inc,\boldsymbol{Q}^n}$ by Theorem 3.2.1. Thus we find that the presheaf $\mathscr{R}$ constitutes a flabby sheaf on $\boldsymbol{D}^n$. (See Crollary 4.9.5 in Morimoto [26], for example.) Q.E.D.

COROLLARY 3.2.3. *For a compact set $K$ in $\boldsymbol{D}^n$, we have the following linear topological isomorphism:*

$$\mathscr{R}[K] \xrightarrow{\ \sim\ } \mathscr{A}'_{dec}(K)\ ,$$

*where $\mathscr{R}[K]$ is the space of all modified Fourier hyperfunctions on $\boldsymbol{D}^n$ supported by $K$.*

PROOF. This is a consequence of Theorem 3.2.2, the Martineau-Harvey duality theorem for the sheaves $\mathcal{O}_{inc}$ and $\mathcal{O}_{dec}$ (Theorem 3.1.3) and the Grauert theorem for the sheaf $\mathcal{O}_{inc}$ (Theorem 2.2.2). Q.E.D.

## § 4. Fourier transformation of modified Fourier hyperfunctions.

In this section we treat the Fourier transformations of $\mathscr{A}_{dec}(\boldsymbol{D}^n)$ and $\mathscr{R}(\boldsymbol{D}^n)$.

### 4.1. The Fourier transformation of $\mathscr{A}_{dec}(\boldsymbol{D}^n)$.

In this section we review the Fourier transformation of $\mathscr{O}_{dec}(\boldsymbol{D}^n)$ developed by Nagamachi-Mugibayashi [33].

For $z, \zeta \in \boldsymbol{C}^n$, we write $z = x + \sqrt{-1}y$, $\zeta = \xi + \sqrt{-1}\eta$ ($x, y, \xi, \eta \in \boldsymbol{R}^n$), and $z \cdot \zeta = z_1\zeta_1 + \cdots + z_n\zeta_n$.

DEFINITION 4.1.1. (Fourier transformation of $\mathscr{A}_{dec}(\boldsymbol{D}^n)$). For $\varphi \in \mathscr{A}_{dec}(\boldsymbol{D}^n)$ we define its Fourier transform $\mathscr{F}\varphi$ by

$$\mathscr{F}\varphi(\zeta) = \int_{\boldsymbol{R}^n} \varphi(x) e^{\sqrt{-1}\zeta \cdot x} dx \,, \tag{4.1.1}$$

where $\zeta$ moves a complex neighborhood of $\boldsymbol{R}^n$ (determined by $\varphi$).

THEOREM 4.1.1. (Proposition 3.2 in Nagamachi-Mugibayashi [33]) *The Fourier transformation of $\mathscr{A}_{dec}(\boldsymbol{D}^n)$ is a linear topological automorphism.*

For the proof of the theorem, we need following

PROPOSITION 4.1.2. *For $\varepsilon > 0$, we put*

$$U_\varepsilon = \{z = x + \sqrt{-1}y \in \boldsymbol{C}^n;\ |y| < \varepsilon(|x|+1)\} \,.$$

*Then the Fourier transformation $\mathscr{F}$ is a bounded linear operator of $\mathscr{O}^{-\varepsilon}(U_\varepsilon)$ ($\subset \mathscr{A}_{dec}(\boldsymbol{D}^n)$) into $\mathscr{O}^{-\varepsilon'}(U_{\varepsilon'})$ for any $\varepsilon'$ ($0 < \varepsilon' < \varepsilon/\sqrt{\varepsilon^2+1}$).*

PROOF. First we note that, for $\varphi \in \mathscr{O}^{-\varepsilon}(U_\varepsilon)$,

$$|\varphi(z) e^{\sqrt{-1}\zeta \cdot z}| \leqq \|\varphi\|_{\varepsilon, U_\varepsilon} \exp(-\varepsilon|z| - x\cdot\eta - y\cdot\xi) \qquad (z \in U_{\varepsilon'},\ \zeta \in \boldsymbol{C}^n) \tag{4.1.2}$$

holds. Hence we have especially

$$|\varphi(x) e^{\sqrt{-1}\zeta \cdot x}| \leqq \|\varphi\|_{\varepsilon, U_\varepsilon} \exp(-\varepsilon|x| - x\cdot\eta) \qquad (x \in \boldsymbol{R}^n,\ \zeta \in \boldsymbol{C}^n) \,. \tag{4.1.2$'$}$$

From (4.1.2′), we have $\mathscr{F}\varphi \in \mathscr{O}(U'_\varepsilon)$ ($U'_\varepsilon = \{\zeta \in \boldsymbol{C}^n;\ |\eta| < \varepsilon\}$) for $\varphi \in \mathscr{O}^{-\varepsilon}(U_\varepsilon)$.

Next we consider to extend the domain of holomorphy of $\mathscr{F}\varphi$, and to obtain the boundedness of $\mathscr{F}$.

For $\delta > 0$, $\beta$ ($0 < \beta < 1$) and $a \in \boldsymbol{R}^n$ ($|a| = 1$), we put

$$S_{a,\delta} = \{z = x + \sqrt{-1}y \in \boldsymbol{C}^n;\ x \in \boldsymbol{R}^n,\ y = \delta(|x|+1)a\} \,,$$
$$U_{a,\beta,\delta} = \{\zeta = \xi + \sqrt{-1}\eta \in \boldsymbol{C}^n;\ a\cdot\xi > \beta|\xi|,\ |\eta| < \delta(|\xi|+1)\} \,.$$

Then for $\varphi \in \mathcal{O}^{-\varepsilon}(U_\varepsilon)$, we define

$$\mathcal{F}_{a,\delta}\varphi(\zeta)=\int_{S_{a,\delta}} \varphi(z)e^{\sqrt{-1}\zeta\cdot z}dz \qquad (0<\delta<\varepsilon)\ .$$

Then using (4.1.2), we have

$$(4.1.3) \qquad \mathcal{F}_{a,\delta}\varphi \in \mathcal{O}(U_{a,\beta,\delta'})\ ,$$

$$(4.1.4) \qquad |\mathcal{F}_{a,\delta}\varphi(\zeta)| \leqq C\|\varphi\|_{\varepsilon,U_\varepsilon} e^{-\delta'|\xi|} \qquad (\zeta \in U_{a,\beta,\delta'})\ ,$$

where positive numbers $\delta'$ and $\beta$ $(0<\delta'<\delta,\ 0<\beta<1)$ satisfy $\delta'=\delta\beta$, and $C$ is a constant independent of $a$, $\delta$, $\delta'$ and $\beta$.

From (4.1.4) we have

$$(4.1.5) \qquad |\mathcal{F}_{a,\delta}\varphi(\zeta)| \leqq C\|\varphi\|_{\varepsilon,U_\varepsilon} e^{-(\delta'/\sqrt{\delta'^2+1})} \qquad (\zeta \in U_{a,\beta,\delta'})\ .$$

On the other hand we have

$$(4.1.6) \qquad \mathcal{F}_{a,\delta}\varphi(\zeta)=\mathcal{F}\varphi(\zeta) \qquad (\zeta \in U'_\varepsilon \cap U_{a,\beta,\delta'})$$

by the $\bar\partial$ closedness of $\varphi e^{\sqrt{-1}z\cdot\zeta}dz$. Here we note

$$U_\delta=\{\zeta=\xi+\sqrt{-1}\eta \in \boldsymbol{C}^n;\ \xi=0,\ |\eta|<\delta\} \cup \Big(\bigcup_{a\in S^{n-1},\,\delta'=\delta\beta} U_{a,\beta,\delta'}\Big)\ .$$

Thus, from (4.1.3), (4.1.5) and (4.1.6), we find that the Fourier transformation gives a bounded operator of $O^{-\varepsilon}(U_\varepsilon)$ into $O^{-\varepsilon'}(U_{\varepsilon'})$ $(0<\varepsilon'<\varepsilon/\sqrt{\varepsilon^2+1})$.

Q.E.D.

PROOF OF THEOREM 4.1.1. We note that $\mathcal{O}_{dec}$ is a dual Fréchet-Schwartz space. Then we find that the Fourier transformation gives a closed linear operator of $\mathcal{A}_{dec}(D^n)$ into itself by Proposition 4.1.2 and Theorem 6′ in Komatsu [18]. Hence, also by Theorem 6′ in Komatsu [18], we find the continuity of $\mathcal{F}$.

Next we define

$$\bar{\mathcal{F}}\varphi(z)=\Big(\frac{1}{2\pi}\Big)^n \int_{R^n} \varphi(\xi)e^{-\sqrt{-1}z\cdot\zeta}d\zeta \qquad (\varphi \in \mathcal{A}_{dec}(\boldsymbol{D}^n))\ .$$

Then $\bar{\mathcal{F}}$ gives also a continuous linear operator of $\mathcal{A}_{dec}(\boldsymbol{D}^n)$ into itself, and satisfies $\bar{\mathcal{F}}\mathcal{F}=$identity.

These show that $\mathcal{F}$ gives a continuous linear automorphism of $\mathcal{A}_{dec}(\boldsymbol{D}^n)$.

Q.E.D.

## 4.2. The Fourier transformation of $\mathcal{R}(\boldsymbol{D}^n)$.

In this section we treat the Fourier transformation of $\mathcal{R}(\boldsymbol{D}^n)$, and give an explicit presentation of the pairing between $\mathcal{R}(\boldsymbol{D}^n)=H^n_{\boldsymbol{D}^n}(\boldsymbol{Q}^n;\ \mathcal{O}_{inc})$

and $\mathscr{A}_{dec}(\boldsymbol{D}^n)$.

We have shown in §3.2 that $\mathscr{R}(\boldsymbol{D}^n)$ is isomorphic to the strong dual space $\mathscr{A}'_{dec}(\boldsymbol{D}^n)$ of $\mathscr{A}_{dec}(\boldsymbol{D}^n)$ (Corollary 3.2.3). Hence using Theorem 4.1.1, we can immediately define a Fourier transformation of $\mathscr{R}(\boldsymbol{D}^n)$:

DEFINITION 4.2.1. We denote by $\mathscr{F}_d$ and $\bar{\mathscr{F}}_d$ the dual operators of Fourier transformation $\mathscr{F}$ and the inverse Fourier transformation $\bar{\mathscr{F}}$ of $\mathscr{A}_{dec}(\boldsymbol{D}^n)$ respectively:

$$\langle \mathscr{F}_d f, \varphi\rangle = \langle f, \mathscr{F}\varphi\rangle \qquad (f, \varphi) \in \mathscr{R}(\boldsymbol{D}^n)\times \mathscr{A}_{dec}(\boldsymbol{D}^n)\ , \tag{4.2.1}$$

$$\langle \bar{\mathscr{F}}_d f, \varphi\rangle = \langle f, \bar{\mathscr{F}}\varphi\rangle \qquad (f, \varphi) \in \mathscr{R}(\boldsymbol{D}^n)\times \mathscr{A}_{dec}(\boldsymbol{D}^n)\ . \tag{4.2.2}$$

We call $\mathscr{F}_d$ and $\bar{\mathscr{F}}_d$ the Fourier transformation and the inverse Fourier transformation of $\mathscr{R}(\boldsymbol{D}^n)$ respectively.

$\mathscr{F}_d$ and $\bar{\mathscr{F}}_d$ are linear topological automorphisms of $\mathscr{R}(\boldsymbol{D}^n)$, and $\bar{\mathscr{F}}_d = \mathscr{F}_d^{-1}$ hold.

Next we define a mapping of $\mathscr{R}(\boldsymbol{D}^n) = \mathrm{H}^n_{\boldsymbol{D}^n}(\boldsymbol{Q}^n; \mathscr{O}_{inc})$ to $\mathscr{A}'_{dec}(\boldsymbol{D}^n)$ which is a linear topological isomorphism.

First we present $\mathrm{H}^n_{\boldsymbol{D}^n}(\boldsymbol{Q}^n; \mathscr{O}_{inc})$ as a cohomology group of a covering. By the excision theorem, we have $\mathrm{H}^n_{\boldsymbol{D}^n}(\boldsymbol{Q}^n; \mathscr{O}_{inc}) = \mathrm{H}^n_{\boldsymbol{D}^n}(V; \mathscr{O}_{inc})$ for any open neighborhood $V$ of $\boldsymbol{D}^n$ in $\boldsymbol{Q}^n$. We put

$$U = \left\{z \in \boldsymbol{C}^n;\ |\mathrm{Im}\, z| < \frac{1}{2}(|\mathrm{Re}\, z|^2 + 1)^{1/2}\right\}, \quad V = \mathring{\bar{U}}, \quad V_0 = V,$$

$$V_j = \{z \in V;\ \mathrm{Im}\, z_j \neq 0 \ \text{ if } \ z \in \boldsymbol{C}^n,\ \mathrm{Im}\, w_j \neq 0 \ \text{ if } \ z = w\infty \in S^{2n-1}_\infty\} \quad (j = 1, \cdots, n)\ ,$$

$$\mathscr{V} = \{V_j\}_{j=0}^n \quad \text{and} \quad \mathscr{V}' = \{V_j\}_{j=1}^n\ .$$

$V$ is an open neighborhood of $\boldsymbol{D}^n$ in $\boldsymbol{Q}^n$. $\mathscr{V}$ is an open covering of $V$. $\mathscr{V}'$ is an open covering of $V \backslash \boldsymbol{D}^n$. $V_j$ $(j = 0, 1, \cdots, n)$ are $\mathscr{O}_{inc}$-pseudoconvex. Since any finite intersection of $\mathscr{O}_{inc}$-pseudoconvex open sets is also $\mathscr{O}_{inc}$-pseudoconvex, we have

$$\mathrm{H}^q\left(\bigcap_{k=1}^m V_{j_k};\ \mathscr{O}_{inc}\right) = 0 \qquad (q \geqq 1,\ 1 \leqq m \leqq n,\ V_{j_k} \in \mathscr{V})$$

by the Cartan theorem B for the sheaf $\mathscr{O}_{inc}$ (Theorem 2.1.2). Hence by the Leray theorem for local cohomology groups (see for example, Morimoto [26]), we have

$$\mathrm{H}^n_{\boldsymbol{D}^n}(\boldsymbol{Q}^n; \mathscr{O}_{inc}) \cong \mathrm{H}^n_{\boldsymbol{D}^n}(V; \mathscr{O}_{inc}) \cong \mathrm{H}^n(\mathscr{V}, \mathscr{V}'; \mathscr{O}_{inc})\ .$$

Then again by the Cartan theorem B for the sheaf $\mathscr{O}_{inc}$, we have

$$H^1(\mathscr{V}, \mathscr{V}'; \mathscr{O}_{inc}) \cong Z^0(\mathscr{V}'; \mathscr{O}_{inc})/Z^0(\mathscr{V}; \mathscr{O}_{inc}) \qquad (n=1)\,,$$
$$H^n(\mathscr{V}, \mathscr{V}'; \mathscr{O}_{inc}) \cong H^{n-1}(\mathscr{V}'; \mathscr{O}_{inc}) \qquad (n\geqq 2)\,.$$

Since the number of elements of $\mathscr{V}'$ is $n$, we have $C^n(\mathscr{V}'; \mathscr{O}_{inc})=0$. Hence we have $Z^{n-1}(\mathscr{V}'; \mathscr{O}_{inc})=C^{n-1}(\mathscr{V}'; \mathscr{O}_{inc})$ and

$$H^{n-1}(\mathscr{V}'; \mathscr{O}_{inc})=C^{n-1}(\mathscr{V}'; \mathscr{O}_{inc})/\delta C^{n-2}(\mathscr{V}'; \mathscr{O}_{inc}) \qquad (n\geqq 2)\,.$$

We denote by $\Gamma_p$ the $p$-th orthant in $\boldsymbol{R}^n$, and put

$$V_p = V\cap(\overline{\boldsymbol{R}^n\times\sqrt{-1}\mathring{\Gamma}_p})\,,$$

then an element $f$ in $C^{n-1}(\mathscr{V}'; \mathscr{O}_{inc})$ is a $2^n$-tuple of holomorphic functions: $f=\{f_p\}_{p=1}^{2^n}$ $(f_p\in\mathscr{O}_{inc}(V_p))$.

Thus we have found that $H^n_{D^n}(\boldsymbol{Q}^n; \mathscr{O}_{inc})$ is presented as a cohomology group of a covering:

$$H^n_{D^n}(\boldsymbol{Q}^n; \mathscr{O}_{inc})=\begin{cases}\mathscr{O}_{inc}(V\backslash D)/\mathscr{O}_{inc}(V) & (n=1)\\ H^{n-1}(\mathscr{V}'; \mathscr{O}_{inc}) & (n\geqq 2)\,,\end{cases}$$

and $f\in H^n_{D^n}(\boldsymbol{Q}^n; \mathscr{O}_{inc})$ is presented by $2^n$-tuple of holomorphic functions:

$$f=[\{f_p\}_{p=1}^{2^n}] \qquad (f_p\in\mathscr{O}_{inc}(V_p))\,.$$

Next we go on to define a pairing between $H^n_{D^n}(\boldsymbol{Q}^n; \mathscr{O}_{inc})$ and $\mathscr{A}_{dec}(\boldsymbol{D}^n)$:

DEFINITION 4.2.2. Let $a^{(p)}$ be a unit vector in $\Gamma_p$, the $p$-th orthant in $\boldsymbol{R}^n$ $(p=1, \cdots, 2^n)$. Then for a $\delta>0$, we put

$$S_{a^{(p)},\delta}=\{z=x+\sqrt{-1}y\in\boldsymbol{C}^n;\, x\in\boldsymbol{R}^n,\, y=\delta(|x|^2+1)^{1/2}a^{(p)}\} \qquad (p=1, \cdots, 2^n)\,.$$

We define a linear mapping $\iota$: $H^n_{D^n}(Q^n; \mathscr{O}_{inc})\to\mathscr{A}'_{dec}(\boldsymbol{D}^n)$ as

$$\text{(4.2.3)}\quad \langle\iota f, \varphi\rangle=\sum_{p=1}^{2^n}\prod_{k=1}^{n}\operatorname{sgn}(a_k^{(p)})\int_{S_{a^{(p)},\delta}} f_p(z)\varphi(z)dz$$
$$(f, \varphi)\in H^n_{D^n}(\boldsymbol{Q}^n; \mathscr{O}_{inc})\times\mathscr{A}_{dec}(\boldsymbol{D}^n)\,, \qquad f=[\{f_p\}]\,,$$

where $\delta$ is a sufficiently small positive number determined by $\varphi$.

The convergence of the integral (4.2.3) and the continuity of $\iota f$ follow from the growth conditions of the sheaves $\mathscr{O}_{inc}$ and $\mathscr{O}_{dec}$. The integral (4.2.3) is independent of a choice of $\{a^{(p)}\}$ and $\delta$ by virture of $\bar\partial$ closedness of $f_p\varphi dz$ $(p=1, \cdots, 2^n)$. We can also check that $\iota f=0$ holds for $f=0\in H^n_{D^n}(\boldsymbol{Q}^n; \mathscr{O}_{inc})$ by using of the $\bar\partial$ closedness of $f_p\varphi dz$ $(p=1, \cdots, 2^n)$. Hence $\iota f$ is independent of the choice of a representative $\{f_p\}$ of $f$. Thus we have found that $\iota$ is well defined.

Next we will find that $\iota$ is isomorphic. To prove this, we first show the injectivity of $\iota$, and next we construct $\iota^{-1}$.

PROPOSITION 4.2.1. *$\iota$: $\mathrm{H}^n_{D^n}(\boldsymbol{Q}^n; \mathcal{O}_{inc})\to \mathcal{A}'_{dec}(\boldsymbol{D}^n)$ defined by (4.2.3) is injective.*

PROOF. We show in the case $n=1$. (The proof goes similarly when $n\geqq 2$.)

Since $\mathrm{H}^1_D(\boldsymbol{Q}; \mathcal{O}_{inc})=\mathcal{O}_{inc}(V\backslash D)/\mathcal{O}_{inc}(V)$, an element $f$ in $\mathrm{H}^1_D(Q; \mathcal{O}_{inc})$ is presented by an element $F$ in $\mathcal{O}_{inc}(V\backslash D)$: $f=[F]$.

Let $\iota f=0$ and $f=[F]$. To prove the injectivity $\iota$, it is sufficient to show $F\in\mathcal{O}_{inc}(V)$.

By Lemma A.2 in Appendix, we can find $J\in\mathcal{O}_{inc}(\boldsymbol{Q})$ such that

$$(4.2.4)\qquad |J(z)|\geqq C_\delta(1+|z|^2)(1+|F(z)|)$$
$$\left(\delta(|\mathrm{Re}\, z|^2+1)^{1/2}<|\mathrm{Im}\, z|<\left(\frac{1}{2}-\delta\right)(|\mathrm{Re}\, z|^2+1)^{1/2}\right)$$

holds, where $C_\delta$ is a positive constant and $0<\delta<1/4$. We put $\gamma_\delta=S_{1,\delta}\cup S_{-1,\delta}$. For $t\in V\backslash D$, we choose $\delta$ $(0<\delta<1/4)$ so that $\delta(|\mathrm{Re}\, t|^2+1)^{1/2}<|\mathrm{Im}\, t|<((1/2)-\delta)(|\mathrm{Re}\, t|^2+1)^{1/2}$ holds (see FIGURE 1). Then by (4.2.4) and the Cauchy integral formula, we have

$$(4.2.5)\quad F(t)=\frac{J(t)}{2\pi\sqrt{-1}}\int_{\gamma_\delta-\gamma_{1/2-\delta}}\frac{F(z)dz}{(t-z)J(z)}$$
$$=\frac{J(t)}{2\pi\sqrt{-1}}\int_{\gamma_\delta}\frac{F(z)dz}{(t-z)J(z)}-\frac{J(t)}{2\pi\sqrt{-1}}\int_{\gamma_{1/2-\delta}}\frac{F(z)dz}{(t-z)J(z)}=\Phi_1(t)-\Phi_2(t)\ .$$

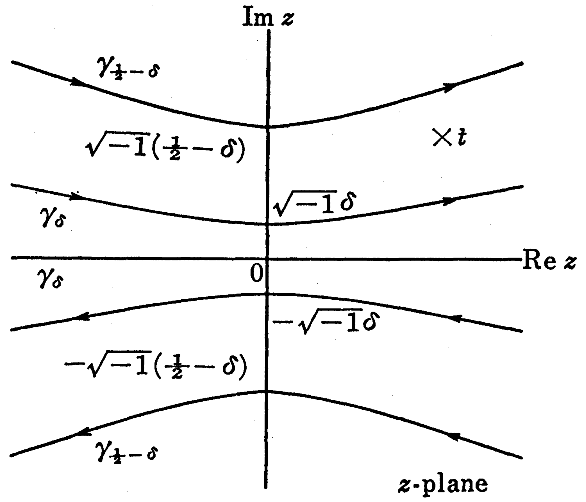


FIGURE 1

Here we are going to show $\Phi_1=0$. Note $|\mathrm{Im}\, t|>\delta(|\mathrm{Re}\, t|^2+1)^{1/2}$ holds and recall the estimation (4.2.4). Then we find $\exp(-(1/\nu)z^2)/(t-z)J(z)\in\mathscr{A}_{dec}(D)$ as a function of $z$ for any $\nu\in N$. Hence by the assumption $\iota f=0$, we have

$$0=\langle\iota f, \exp(-(1/\nu)z^2)/(t-z)J(z)\rangle=\int_{\gamma_\delta}\frac{\exp(-(1/\nu)z^2)F(z)}{(t-z)J(z)}dz \qquad (\nu=1, 2, \cdots)\,.$$

Then by the Lebesgue dominated convergence theorem, we have

$$0=\int_{\gamma_\delta}\frac{\exp(-(1/\nu)z^2)F(z)}{(t-z)J(z)}dz \longrightarrow \int_{\gamma_\delta}\frac{F(z)dz}{(t-z)J(z)} \qquad (\nu\longrightarrow\infty)\,.$$

Thus we have proved $\Phi_1(t)\equiv 0$. Moreover we have $F(t)=-\Phi_2(t)$. Hence for the proof of the proposition, it is sufficient to show $\Phi_2\in\mathscr{O}_{inc}(\mathrm{V})$.

We can easily find the holomorphy of $\Phi_2(t)$ on the domain

$$\left\{t\in C;\ |\mathrm{Im}\, t|<\left(\frac{1}{2}-\delta\right)(|\mathrm{Re}\, t|^2+1)^{1/2}\right\}\,.$$

Since the estimation (4.2.4) holds we can shift the path $\gamma_{1/2-\delta}$ of the integral $\Phi_2(t)$ by moving $\delta$ in the interval $(0, 1/4)$. Then we can find the holomorphy of $\Phi_2(t)$ on the whole of $U$.

The rest is to show that $\Phi_2(t)$ has the infra-exponential growth on any domain of a form $K\cap C$ $(K\subset V)$. This is an immediate consequence of the fact $J\in\mathscr{O}_{inc}(Q)$ and the estimation (4.2.4). Q.E.D.

REMARK TO THE PROOF OF PROPOSITION 4.2.1. In Saburi [38], to show $F\in\mathscr{O}_{inc}(V)$, we used the function $\exp(-(t-z)^2)$ (instead of $J(t)/J(z)$) as a damping function in the Cauchy integral (4.2.5):

$$(4.2.5)' \qquad F(t)=\frac{1}{2\pi\sqrt{-1}}\int_{\gamma_\delta-\gamma_{1/2-\delta}}\frac{\exp(-(t-z)^2)f(z)}{(t-z)}dz\,.$$

But Professor Nagamachi told us that it is hard to show the infra-exponential growth of the function $F$ on any domain of a form $K\cap C$ $(K\subset V)$. In fact we had made a mistake in estimating the function $F$ by using the integral (4.2.5)′ in Saburi [38]. Afterward Professor Kaneko showed us the existence of holomorphic function $J$ with infra-exponential growth such as in (4.2.4). Without his suggestion we could not obtain the results in this section. Here the author expresses his sincere gratitude to Professor Keneko for his helpful suggestion.

Next we construct $\iota^{-1}$. It is constructed through defining another Fourier transformation $\mathscr{F}$ of $\mathscr{R}(\boldsymbol{D}^n)$. $\mathscr{F}$ and $\mathscr{F}_d$ coincide through $\iota$. We will prove this fact and the isomorphy of $\iota$ at the same time.

We need some preparations.

LEMMA 4.2.2. (Lemma 4.15 in Nagamachi-Mugibayashi [33].) *As to the decomposition of supports of elements of $\mathscr{A}'_{dec}(\boldsymbol{D}^n)$, we have the following exact sequence:*

$$\text{(4.2.6.)} \qquad \bigoplus_{1\leq p,q\leq 2^n}{}' \mathscr{A}'_{dec}(\bar{\Gamma}_p\cap\bar{\Gamma}_q) \xrightarrow{\ \alpha\ } \bigoplus_p \mathscr{A}'_{dec}(\bar{\Gamma}_p) \xrightarrow{\ \beta\ } \mathscr{A}'_{dec}(\boldsymbol{D}^n) \longrightarrow 0\ ,$$

*where $\Gamma_p$ is the p-th orthant in $\boldsymbol{R}^n$, and we put*

$$\bigoplus_{1\leq p,q\leq 2^n}{}' \mathscr{A}'_{dec}(\bar{\Gamma}_p\cap\bar{\Gamma}_q)=\{\{\mu_{p,q}\}_{p,q=1}^{2^n};\ \mu_{p,q}\in\mathscr{A}'_{dec}(\bar{\Gamma}_p\cap\bar{\Gamma}_q),\ \mu_{p,q}+\mu_{q,p}=0\}\ ,$$

*and the mappings $\alpha$ and $\beta$ are defined as follows:*

$$\alpha\colon \{\mu_{p,q}\}_{p,q=1}^{2^n}\longmapsto\Big\{\sum_{q=1}^{2^n}\mu_{p,q}\Big\}_{p=1}^{2^n}\ ,\qquad \beta\colon\{\mu_p\}_{p=1}^{2^n}\longmapsto\sum_{p=1}^{2^n}\mu_p\ .$$

PROOF. We note that $\mathscr{A}'_{dec}(K)\cong\mathscr{R}[K]$ holds for each compact set $K$ in $\boldsymbol{D}^n$ by Corollary 3.2.3 and that $\mathscr{R}$ is flabby by Theorem 3.2.2. Then by the Meyer-Vietoris theorem for flabby sheaves (See for example, Morimoto [26], p-208.), we have the exactness of the sequence (4.2.6).

LEMMA 4.2.3. *Let $\Gamma$ be a proper cone in $\boldsymbol{R}^n$ with the vertex at the origin. For $\mu\in\mathscr{A}'_{dec}(\bar{\Gamma})$, we put $\mathscr{F}\mu(\zeta)=\langle\mu_z, e^{\sqrt{-1}\zeta\cdot z}\rangle$ $(\zeta\cdot z=z_1\zeta_1+\cdots+z_n\zeta_n)$. Then $\mathscr{F}\mu$ defines a holomorphic functihn on $R^n\times\sqrt{-1}\Gamma^\circ$, where we put $\Gamma^\circ=\{\eta\in\boldsymbol{R}^n;\ \langle\eta, x\rangle>0$ for any $x\in\Gamma\}$. Moreover we have $F\in\mathscr{O}_{inc}(\overline{\boldsymbol{R}^n\times\sqrt{-1}\Gamma^\circ})$.*

PROOF. We note that, for any $\zeta\in\boldsymbol{R}^n\times\sqrt{-1}\Gamma^\circ$, $e^{\sqrt{-1}\zeta\cdot z}$ defines an element in $\mathscr{O}_{dec}(\bar{\Gamma})$ as a function of $z$. Then we have the lemma by the definition of $\mathscr{A}'_{dec}(\bar{\Gamma})$. Q.E.D.

Using Lemmas 4.2.2 and 4.2.3, we can define another Fourier transformation of $\mathscr{A}'_{dec}(\boldsymbol{D}^n)$:

DEFINITION 4.2.2. For a $\mu\in\mathscr{R}(\boldsymbol{D}^n)$ $(\cong\mathscr{A}'_{dec}(\boldsymbol{D}^n))$, using the decomposition $\mu=\sum\mu_p$ $(\mu_p\in\mathscr{A}'_{dec}(\bar{\Gamma}_p))$ in Lemma 4.2.2, we put

$$\mathscr{F}\mu=[\{\mathscr{F}\mu_p\}]\in \mathrm{H}^n_{D^n}(V;\ \mathscr{O}_{inc})\ ,$$

$$\mathscr{F}\mu_p(\zeta)=\prod_{k=1}^n \mathrm{sgn}(b_k^{(p)})\langle\mu_{p,z}, e^{\sqrt{-1}\zeta\cdot z}\rangle\in\mathscr{O}_{inc}(W_p)\ ,$$

where we put

$$W_p=\overline{R^n\times\sqrt{-1}\Gamma_p^{\circ\vee}} \quad (\bar{\Gamma}_p^{\circ}=\bar{\Gamma}_p),\ V=\left\{z\in C^n;\ |\mathrm{Im}\,z|<\frac{1}{2}(|\mathrm{Re}\,z|^2+1)^{1/2}\right\},$$

and $b^{(p)}$ is a unit vector in $\Gamma_p$. We call $\mathscr{F}\mu$ and $\mathscr{F}\mu_p$ the Fourier transform of $\mu$ and $\mu_p$ respectively.

By Lemmas 4.2.2 and 4.2.3 we can check that $\mathscr{F}\mu$ does not depend on the decomposition $\mu=\sum\mu_p$ $(\mu_p\in\mathscr{A}'_{dec}(\bar{\Gamma}_p))$. Hence the above definition of the Fourier transformation $\mathscr{F}$ of $\mathscr{R}(D^n)$ is well defined.

Now we construct $\iota^{-1}$:

DEFINITION 4.2.3. We define a linear operator $\kappa$ of $\mathscr{A}'_{dec}(D^n)$ into $\mathrm{H}^n_{D^n}(V;\mathscr{O}_{inc})$ by $\kappa=\mathscr{F}\circ\bar{\mathscr{F}}_d$.

We will show $\kappa=\iota^{-1}$:

PROPOSITION 4.2.4. *The linear operator*

$$\iota\circ\kappa\colon \mathscr{A}'_{dec}(D^n)\longrightarrow\mathscr{A}'_{dec}(D^n)$$

*is the identity.*

PROOF. We have to show the following equality:

$$\langle\iota\circ\kappa(\mu),\varphi\rangle=\langle\mu,\varphi\rangle \qquad \text{for any} \quad (\mu,\varphi)\in\mathscr{A}'_{dec}(D^n)\times\mathscr{A}_{dec}(D^n)\,.$$

We put $\nu=\bar{F}_d\mu$, and decompose $\nu$ as $\nu=\sum\nu_p$ $(\nu_p\in\mathscr{A}'_{dec}(\bar{\Gamma}_p))$, where $\Gamma_p$ is the $p$-th orthant in $R^n$. Then we have

$$\begin{aligned}\langle\iota\circ\kappa(\mu),\varphi\rangle&=\langle\iota\circ\mathscr{F}\circ\overline{\mathscr{F}_d\mu},\varphi\rangle\\&=\sum_{p=1}^{2^n}\prod_{k=1}^{n}\mathrm{sgn}(a_k^{(p)})\int_{S_{a(p),\delta}}\mathscr{F}\nu_p(\zeta)\varphi(\zeta)d\zeta\\&=\sum_{p=1}^{2^n}\prod_{k=1}^{n}\mathrm{sgn}(a_k^{(p)})\int_{S_{a(p),\delta}}\prod_{j=1}^{n}\mathrm{sgn}(a_j^{(p)})\langle\nu_{p,z},\mathrm{e}^{\sqrt{-1}\zeta\cdot z}\rangle\varphi(\zeta)d\zeta\\&=\sum_{p=1}^{2^n}\int_{S_{a(p),\delta}}\langle\nu_{p,z},\mathrm{e}^{\sqrt{-1}\zeta\cdot z}\varphi(\zeta)\rangle d\zeta\,.\end{aligned}$$

Here we note that the integral $\int_{S_{a(p),\delta}}\mathrm{e}^{\sqrt{-1}\zeta\cdot z}\varphi(\zeta)d\zeta$ converges in the topology of $\mathscr{A}_{dec}(\bar{\Gamma}_p)(p=1,\cdots,2^n)$ for any $\varphi\in\mathscr{A}_{dec}(D^n)$. Hence we have

$$\begin{aligned}\langle\iota\circ\kappa(\mu),\varphi\rangle&=\sum_{p=1}^{2^n}\int_{S_{a(p),\delta}}\langle\nu_{p,z},\mathrm{e}^{\sqrt{-1}\zeta\cdot z}\varphi(\zeta)\rangle d\zeta\\&=\sum_{p=1}^{2^n}\left\langle\nu_{p,z},\int_{S_{a(p),\delta}}\mathrm{e}^{\sqrt{-1}\zeta\cdot z}\varphi(\zeta)d\zeta\right\rangle\\&=\langle\nu,\mathscr{F}\varphi\rangle=\langle\bar{\mathscr{F}}_d\mu,\mathscr{F}\varphi\rangle=\langle\mu,\varphi\rangle\,.\end{aligned}$$

Q.E.D.

Now we have

THEOREM 4.2.5. *$\iota$ is a linear isomorphism of* $H^n_{D^n}(V; \mathcal{O}_{inc})$ *onto* $\mathcal{A}'_{dec}(\boldsymbol{D}^n)$.

PROOF. This is an immediate consequence of Propositions 4.2.1 and 4.2.4. Q.E.D.

REMARK. As a consequence of above results, we have $\mathcal{F}_d = \iota \circ \mathcal{F}$.

**4.3.** Fourier-Carleman-Leray-Sato transformation.

In this section we treat Fourier transforms of modified Fourier hyperfunctions supported by a proper cone.

Let $\Gamma$ be a proper closed cone in $\boldsymbol{R}^n$ with the vertex at the origin and $f \in \mathcal{R}[\Gamma]$. Then the function $J(\zeta) = \langle f_z, e^{\sqrt{-1}\zeta\cdot z}\rangle$ defines an element in $\mathcal{O}_{inc}(\overline{\boldsymbol{R}^n \times \sqrt{-1}\mathring{\Gamma}^\circ})$ by Lemma 4.2.3. Considering a suitable covering of $V \backslash D$ ($V = \mathring{\bar{U}}$, $U = \{z \in \boldsymbol{C}^n;\ |\mathrm{Im}\, z| < (1/2)(|\mathrm{Re}\, z|^2+1)^{1/2}\}$), we find that $J$ defines an element $[J]$ in $H^n_{D^n}(V; \mathcal{O}_{inc}) = \mathcal{R}(\boldsymbol{D}^n)$. Conversely we have the following.

THEOREM 4.3.1. *Let $\Gamma$ be a proper convex closed cone in $\boldsymbol{R}^n$ with the vertex at the origin and $J \in \mathcal{O}_{inc}(\overline{\boldsymbol{R}^n \times \sqrt{-1}\mathring{\Gamma}^\circ})$. Then we have $g = \bar{\mathcal{F}}_d \circ \iota[J] \in \mathcal{R}[\Gamma]$.*

REMARK. The Fourier transformation gives an linear automorphism of $\mathcal{R}(\boldsymbol{D}^n)$. Hence by Lemma 4.2.3 and Theorem 4.3.1, we find that if $\Gamma$ is a cone such as in Theorem 4.3.1, then $\mathcal{F}$ gives an isomorphism of $\mathcal{R}[\Gamma]$ onto $\mathcal{O}_{inc}(\overline{\boldsymbol{R}^n \times \sqrt{-1}\mathring{\Gamma}^\circ})$.

PROOF OF THEOREM 4.3.1. First we prove in the case $n=1$. In this case, $\Gamma$ is the closed half line $\mathcal{R}_+$ in $\mathcal{R}$. Since $\mathcal{A}_{dec}(\boldsymbol{D}^n)$ is dense in $\mathcal{A}_{dec}(\Gamma)$ (Theorem 2.3.1), it is sufficient to show the following inequality for any $\varepsilon$ $(0 < \varepsilon \ll 1)$.

$$(4.3.1) \qquad |\langle g, \varphi\rangle| \leqq C\|\varphi\|_{\varepsilon,\Gamma_\varepsilon} = C \sup_{z \in \Gamma_\varepsilon} |\varphi(z)| e^{\varepsilon|z|} \qquad \text{for any} \quad \varphi \in \mathcal{O}^{-\varepsilon}(U_\varepsilon)\,,$$

where

$$U_\varepsilon = \{z = x + \sqrt{-1}y;\ |y| < \varepsilon(|x|+1)\}\,,$$
$$\Gamma_\varepsilon = \{z \in U_\varepsilon;\ \mathrm{Re}\, z \geqq -\varepsilon\}$$

and $C$ is a constant independent of $\varphi$.

To have the estimation (4.3.1), we decompose the integral:

$$\text{(4.3.2)}\qquad \begin{aligned}\langle g, \varphi\rangle &= \langle \iota[J], \mathscr{F}\varphi\rangle = \int_{A_\delta} J(\zeta)\mathscr{F}\varphi(\zeta)d\zeta \\ &= \int_{A_\delta^-} J(\zeta)\mathscr{F}\varphi(\zeta)d\zeta + \int_{A_\delta^+} J(\zeta)\mathscr{F}\varphi(\zeta)d\zeta \\ &= I_\delta^- + I_\delta^+ ,\end{aligned}$$

where $\delta$ is a positive number with $0<\delta<\varepsilon$, and we put

$$\begin{aligned} &A_\delta = A_\delta^- + A_\delta^+ , \\ &A_\delta^- = \{\zeta = \xi + \sqrt{-1}\eta;\ \eta = \delta(|\xi|+1),\ \xi \leqq 0\} , \\ &A_\delta^+ = \{\zeta = \xi + \sqrt{-1}\eta;\ \eta = \delta(|\xi|+1),\ \xi \geqq 0\} . \end{aligned}$$

(See FIGURE 2.) Moreover we take a positive $\delta'$ $(0<\delta'<\varepsilon)$, and put

$$\begin{aligned} &B_\delta^- = B_{\delta-}^- + B_{\delta+}^- , \\ &B_{\delta-}^- = \{z = x + \sqrt{-1}y;\ y = \delta(x+2\delta'-1),\ x \leqq -\delta'\} , \\ &B_{\delta+}^- = \{z = x + \sqrt{-1}y;\ y = -\delta(x+1),\ x \geqq -\delta'\} , \\ &B_\delta^+ = B_{\delta-}^+ + B_{\delta+}^+ , \\ &B_{\delta-}^+ = \{z = x + \sqrt{-1}y;\ y = -\delta(x+2\delta'-1),\ x \leqq -\delta'\} , \\ &B_{\delta+}^+ = \{z = x + \sqrt{-1}y;\ y = \delta(x+1),\ x \geqq -\delta'\} . \end{aligned}$$

(See FIGURE 3.) We decompose again the integral (4.3.2) as follows:

$$\text{(4.3.3}^-\text{)}\qquad \begin{aligned} I_\delta^- &= \frac{1}{2\pi}\int_{A_\delta^-} J(\zeta)\int_{B_\delta^+} \varphi(z)e^{-\sqrt{-1}\zeta\cdot z}dzd\zeta \\ &= \frac{1}{2\pi}\int_{A_\delta^-} J(\zeta)\int_{B_{\delta-}^+} \varphi(z)e^{-\sqrt{-1}\zeta\cdot z}dzd\zeta + \frac{1}{2\pi}\int_{A_\delta^-} J(\zeta)\int_{B_{\delta+}^+} \varphi(z)e^{-\sqrt{-1}\zeta\cdot z}dzd\zeta \\ &= J_-^- + J_+^- , \end{aligned}$$

$$\text{(4.3.3.}^+\text{)}\qquad \begin{aligned} I_\delta^+ &= \frac{1}{2\pi}\int_{A_\delta^+} J(\zeta)\int_{B_\delta^-} \varphi(z)e^{-\sqrt{-1}\zeta\cdot z}dzd\zeta \\ &= \frac{1}{2\pi}\int_{A_\delta^+} J(\zeta)\int_{B_{\delta-}^-} \varphi(z)dzd\zeta + \frac{1}{2\pi}\int_{A_\delta^+} J(\zeta)\int_{B_{\delta+}^-} \varphi(z)e^{-\sqrt{-1}\zeta\cdot z}dzd\zeta \\ &= J_-^+ + J_+^+ . \end{aligned}$$

Here we estimate the integrands. Since $F \in \mathscr{O}_{inc}(\overline{\boldsymbol{R}\times\sqrt{-1}\dot{\boldsymbol{R}}_+})$ $(\Gamma^\circ = \Gamma = \boldsymbol{R}_+)$ and $\varphi \in \mathscr{O}^{-\varepsilon}(U_\varepsilon)$, we have

$$\text{(4.3.4)}\qquad |J(\zeta)\varphi(z)e^{-\sqrt{-1}\zeta\cdot z}| \leqq \|J\|_{\varepsilon_1, E_\delta}\|\varphi\|_{\varepsilon,\Gamma_\varepsilon} \exp(-\varepsilon|z| + \varepsilon_1|\zeta| + x\eta + y\xi) \qquad (\zeta, z) \in E_\delta \times \Gamma_\varepsilon ,$$

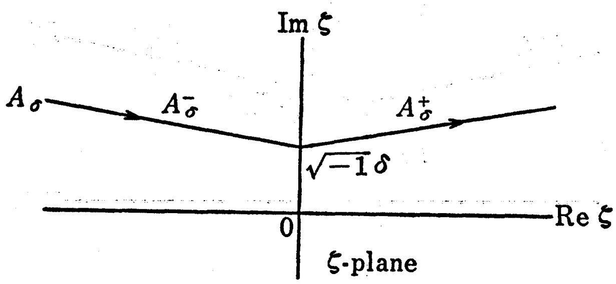


FIGURE 2

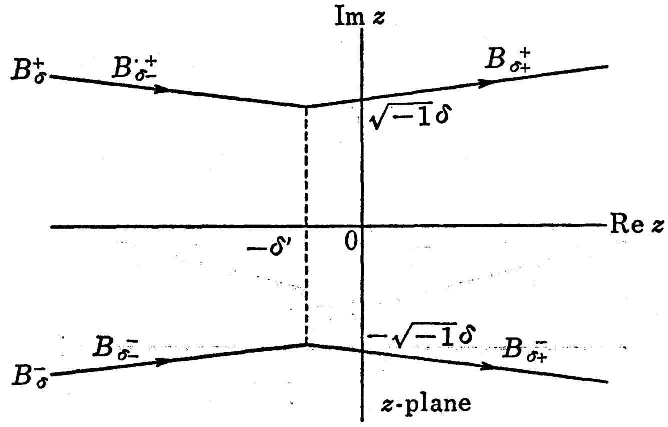


FIGURE 3

where $\varepsilon_1$ is an arbitrary positive number, and we put

$$E_\delta=\{\zeta=\xi+\sqrt{-1}\eta;\ \xi\in R,\ |\eta|\geqq\delta(|\xi|+1)\}\ .$$

Since $\varepsilon_1$ is arbitrary, we take $\varepsilon_1$ as $0<\varepsilon_1<\delta/(\delta+1)$. Then we have immediately from (4.3.4)

$$|J_+^-|+|J_+^+|\leqq C(\varepsilon,\ \delta)\|\varphi\|_{\varepsilon,\Gamma_\varepsilon}\ . \tag{4.3.5}$$

Therefore we have to estimate the rests $J_-^-$ and $J_-^+$. To obtain the estimation for those integrals, we deform their paths. We put

$$\alpha_\delta=\{\zeta=\xi+\sqrt{-1}\eta;\ \xi=0,\ \eta>\delta\}\ ,$$
$$\beta_{\delta,\delta'}=\{z=x+\sqrt{-1}y;\ x=-\delta',\ |y|\leqq\delta(-\delta'+1)\}\ .$$

Then, thanks to the estimation (4.3.4), we can deform the integrals $J_-^-$ and $J_-^+$ as follows (See Figure 4 and 5.):

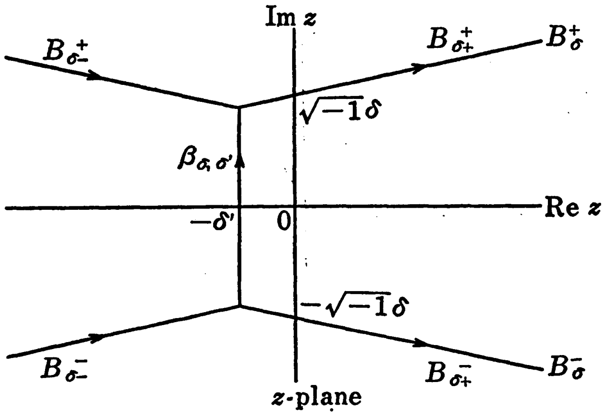


FIGURE 4

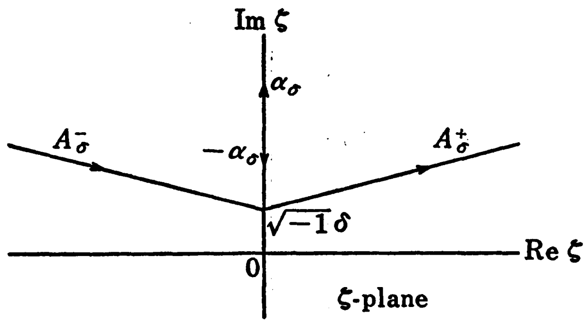


FIGURE 5

(4.3.6)
$$\begin{aligned} J_{-}^{-}+J_{-}^{+} &= \frac{1}{2\pi}\left(\int_{A_{\delta}^{-}\times B_{\delta-}^{+}}+\int_{A_{\delta}^{+}\times B_{\delta-}^{-}}\right) \\ &= \frac{1}{2\pi}\left(\int_{-\alpha_{\delta}\times B_{\delta-}^{+}}+\int_{\alpha_{\delta}\times B_{\delta-}^{-}}\right) \\ &= \frac{1}{2\pi}\int_{\alpha_{\delta}\times(B_{\delta-}^{-}-B_{\delta-}^{+})} \\ &= \frac{1}{2\pi}\int_{\alpha_{\delta}\times(-\beta_{\delta,\delta'})}\ . \end{aligned}$$

Since $\beta_{\delta,\delta'}$ is contained in $\Gamma_{\varepsilon}$, we have from (4.3.4) and (4.3.6)

(4.3.7)
$$|J_{-}^{-}+J_{-}^{+}|\leqq B\|\varphi\|_{\varepsilon,\Gamma_{\varepsilon}}\ .$$

Since $C(\varepsilon, \delta)$ in (4.3.5) and $B$ in (4.3.7) do not depend on $\varphi$, we have the estimation (4.3.1). Thus we have proved the theorem in the case $n=1$.

Next we go on to the proof in the case $n\geqq 2$. For simplicity, we prove only in the case $n=2$.

Since $\Gamma$ is a closed convex cone, we have

$$\Gamma = \bigcap_{\xi \in \Gamma^\circ \cap S^{n-1}} H_\xi , \qquad H_\xi = \{x \in \boldsymbol{R}^2;\ x\cdot\xi > 0\} .$$

Hence it is sufficient to show that $g \in \mathscr{A}'_{dec}(\bar{H}_\xi) \cong \mathscr{R}[\bar{H}_\xi]$ holds for all $\xi \in \Gamma^\circ \cap S^1$.

We take an $e \in \Gamma^\circ \cap S^1$ and fix it. Similarly to the case $n=1$, it is sufficient to show the following inequality:

$$|\langle g, \varphi\rangle| \leqq A\|\varphi\|_{\varepsilon, \tilde{H}_\varepsilon} \qquad (\varphi \in O^{-\varepsilon}(U_\varepsilon)) \tag{4.3.1}$$

for any $\varepsilon > 0$, where $A$ is a constant independent of $\varphi$, and we put

$$\begin{aligned} &U_\varepsilon = \{z = x + \sqrt{-1}y \in \boldsymbol{C}^2;\ |y| < \varepsilon(|x|+1)\} , \\ &\tilde{H}_\varepsilon = \{z = x + \sqrt{-1}y \in U_\varepsilon;\ x \in H_\varepsilon - \varepsilon e\} , \\ &H_\varepsilon = \{x \in \boldsymbol{R}^2;\ x\cdot e > -\varepsilon|x|\} . \end{aligned}$$

First we prepare the paths of the integral $\langle g, \varphi\rangle$. We denote the closed $j$-th qadrant by $\Gamma_j$. We take $e^j \in -\Gamma_j \cap S^1$ $(j=1, 2, 3, 4)$ such that

$$e^j\cdot\xi \leqq -\frac{1}{\sqrt{2}}|\xi| \qquad \text{for any} \quad \xi \in \Gamma_j \ (j=1, 2, 3, 4) .$$

Then, for $\delta > 0$, we put

$$\begin{aligned} &S_\delta = \bigcup_{j=1}^{4} S_\delta^j , \quad \text{where} \\ &S_\delta^j = \{\zeta = \xi + \sqrt{-1}\eta \in \boldsymbol{C}^2;\ \xi \in \Gamma_j,\ \eta = \delta(|\xi|+1)e\} , \\ &T_\delta^j = T_{\delta-}^{\,j} \cup T_{\delta+}^{\,j} , \quad \text{where} \\ &T_{\delta-}^{\,j} = \{z = x + \sqrt{-1}y \in \boldsymbol{C}^2;\ x \in H_\delta^c,\ y = \delta(|x|+1)e^j\} , \\ &T_{\delta+}^{\,j} = \{z = x + \sqrt{-1}y \in \boldsymbol{C}^2;\ x \in H_\delta,\ y = \delta(|x|+1)e^j\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j=1, 2, 3, 4) , \end{aligned}$$

where $H_\delta^c$ is the complement of $H_\delta$. We choose a sufficiently small $\delta > 0$ and decompose the integral $\langle g, \varphi\rangle$ as follows:

$$\begin{aligned} (2\pi)^2\langle g, \varphi\rangle &= \int_{S_\delta} J(\zeta) \int_{R^2} \varphi(z) e^{-\sqrt{-1}\zeta\cdot z} dz d\zeta \\ &= \sum_{j=1}^{4} \int_{S_\delta^j} J(\zeta) \int_{R^2} \varphi(z) e^{-\sqrt{-1}\zeta\cdot z} dz d\zeta \\ &= \sum_{j=1}^{4} \int_{S_\delta^j} J(\zeta) \int_{T_\delta^j} \varphi(z) e^{-\sqrt{-1}\zeta\cdot(z-\delta e)} dz d\zeta \end{aligned}$$

$$=\sum_{j=1}^{4}\int_{S_\delta^j}J(\zeta)\int_{T_{\delta-}^j}\varphi(z-\delta e)e^{-\sqrt{-1}\zeta\cdot(z-\delta e)}dzd\zeta$$
$$+\sum_{j=1}^{4}\int_{S_\delta^j}J(\zeta)\int_{T_{\delta+}^j}\varphi(z-\delta e)e^{-\sqrt{-1}\zeta\cdot(z-\delta e)}dzd\zeta$$
$$=\sum_{j=1}^{4}J_-^j+\sum_{j=1}^{4}J_+^j .$$

Now we are going to estimate the integral $\langle g, \varphi\rangle$. Since $J\in \mathscr{O}_{inc}(\overline{R^2\times\sqrt{-1}\mathring{\Gamma}^{\circ}})$ and $\varphi\in\mathscr{O}^{-\varepsilon}(U_\varepsilon)$, we have the following estimation for the integrand:

$$(4.3.8)\qquad |J(\zeta)\varphi(z)e^{-\sqrt{-1}\zeta\cdot z}|\leqq\|J\|_{\varepsilon,\tilde{H}_\varepsilon}\exp(-\varepsilon|z|+\varepsilon_1|\zeta|+x\cdot\eta+y\cdot\xi)$$
$$(\zeta, z)\in E_\delta\times\tilde{H}_\varepsilon ,$$

where $\varepsilon_1$ is an arbitrary positive number, and we put

$$E_\delta=\{\zeta=\xi+\sqrt{-1}\eta\in C^2;\ |\eta|\geqq\delta(|\xi|+1),\ \eta=\lambda e,\ \lambda\geqq\delta\}$$

Since $\varepsilon_1$ is arbitrary, we assume that $\varepsilon_1$ is sufficiently small in what follows.

We note $(T_{\delta+}^j-\delta e)\subset\tilde{H}_\varepsilon$ for a sufficiently small $\delta>0$. Then using the estimation (4.3.8), we have the estimation (4.3.1) for terms $J_+^j$ ($j=1, 2, 3, 4$). Hence we have to estimate the rests $J_-^j$ ($j=1, 2, 3, 4$).

Similarly to the case $n=1$, we deform the paths of the integrals. We note that the following estimation holds:

$$(4.3.9)\quad \mathrm{Re}[-\sqrt{-1}\zeta\cdot(z-\delta e)]$$
$$=\mathrm{Re}[-\sqrt{-1}(\xi+\sqrt{-1}\nu\delta(|\xi|+1)e\cdot(x-\delta e+\sqrt{-1}\mu\delta(|x|+1)e^j)]$$
$$=(x-\delta e)\cdot\nu\delta(|\xi|+1)e+\mu\delta(|x|+1)e^j\cdot\xi$$
$$\leqq-\nu\delta|x|\,|\xi|-\nu\delta^2(|\xi|+1)-(\mu\delta/\sqrt{2})(|x|+1)|\xi|$$
$$\leqq-\delta'|\zeta|$$
$$\text{for}\quad (\zeta, z)\in S_{\nu\delta}^j\times T_{\mu\delta}^j\ (\nu\geqq1,\ 0\leqq\mu\leqq1)$$

for a sufficiently small $\delta'$ ($j=1, 2, 3, 4$). Then, thanks to the estimations (4.3.8) and (4.3.9), we can justify the following deformation of the integrals $J_-^j$ ($j=1, 2, 3, 4$):

$$J_-^1+J_-^2=\int_{S_\delta^1\times T_{\delta-}^1}+\int_{S_\delta^2\times T_{\delta-}^1}=\int_{\alpha_\delta^1\times T_{\delta-}^1}+\int_{-\alpha_\delta^1\times T_{\delta-}^2}$$
$$=\int_{\alpha_\delta^1\times(T_{\delta-}^1-T_{\delta-}^2)}=\int_{\alpha_\delta^1\times(-\beta_\delta^1)} ,$$

$$J_{-}^{3}+J_{-}^{4}=\int_{S_{\delta}^{3}\times T_{\delta-}^{3}}+\int_{S_{\delta}^{4}\times T_{\delta-}^{4}}=\int_{-\alpha_{\delta}^{2}\times T_{\delta-}^{3}}+\int_{\alpha_{\delta}^{2}\times T_{\delta-}^{4}}$$
$$=\int_{\alpha_{\delta}^{2}\times(T_{\delta-}^{4}-T_{\delta-}^{3})}=\int_{\alpha_{\delta}^{2}\times(-\beta_{\delta}^{2})},$$

where we put

$$\alpha_{\delta}^{1}=\{\zeta=\xi+\sqrt{-1}\eta\in C^{2};\ \xi_1=0,\ \xi_2\geqq 0,\ \eta=\lambda(|\xi|+1)e,\ \lambda\geqq\delta\},$$
$$\alpha_{\delta}^{2}=\{\zeta=\xi+\sqrt{-1}\eta\in C^{2};\ \xi_1=0,\ \xi_2\leqq 0,\ \eta=\lambda(|\xi|+1)e,\ \lambda\geqq\delta\},$$
$$\beta_{\delta}^{1}=\{z-\delta e\in C^{2};\ z=x+\sqrt{-1}y,\ x\in\partial H_{\delta},\ y=\lambda(|x|+1)e^{j}\quad (j=1\ \text{or}\ 2),\ 0\leqq\lambda\leqq\delta\},$$
$$\beta_{\delta}^{2}=\{z-\delta e\in C^{2};\ z=x+\sqrt{-1}y,\ x\in\partial H_{\delta},\ y=\lambda(|x|+1)e^{j}\quad (j=3\ \text{or}\ 4),\ 0\leqq\lambda\leqq\delta\}.$$

We note that $\beta_{\delta}^{1}$ and $\beta_{\delta}^{2}$ are contained in $\tilde{H}_{\varepsilon}$. Then again by the estimations (4.3.8) and (4.3.9), we have the estimation (4.3.1) for the integrals $J_{-}^{1}+J_{-}^{2}$ and $J_{-}^{3}+J_{-}^{4}$. Q.E.D.

## Appendix.

Here we prove some lemmas which we used in the text.

LEMMA A.1. *Let $W$ be an open set in $C^n$, $\psi$ a locally summable plurisubharmonic function on $W$. Suppose that $\psi$ satisfies the following conditions:*

i) *$\psi$ is of class $C^2$ and strictly plurisubharmonic on an open subset $U$ of $W$ such that the Lebesgue measure of $F=W\backslash U$ is zero.*

ii) *If we extend the second derivatives $\partial^2\psi/\partial z_j\partial\bar{z}_k$ of $\psi$ on $U$ to $W$ as*

$$\psi_{jk}(z)=\begin{cases}\dfrac{\partial^2\psi}{\partial z_j\partial\bar{z}_k}(z) & \text{if } z\in U\\ 0 & \text{if } z\in F,\end{cases}$$

*then $\psi_{jk}$ defines a $\mathscr{L}^1_{loc}$ function on $W$.*

*Then, for any non-negative $C^\infty$ function $\chi$ with the support contained in the $\varepsilon$-ball in $C^n$ centered at the origin, the convolution $\chi*\psi$ defines a strictly plurisubharmonic function on $W_\varepsilon=\{z\in W;\ dist(z, \partial W)>\varepsilon\}$.*

PROOF. Consider the distributional derivatives $\partial^2\psi/\partial z_j\partial\bar{z}_k$ on $W$ $(1\leqq j, k\leqq n)$. Since $\psi$ is plurisubharmonic on $W$, we find that the Hermitian matrix $[(\partial^2/\partial z_j\partial\bar{z}_k)(\chi*\psi)(z)]_{j,k=1}^{n}$ is positive semidefinite on $W_\varepsilon$ for any $\chi\in D(B_\varepsilon)$, where $B_\varepsilon$ is the $\varepsilon$-ball in $C^n$ centered at the origin.

Next we put

$$T_{jk}=\frac{\partial^2\psi}{\partial z_j\partial\bar{z}_k}-\psi_{jk} .$$

We note that the supports of $T_{jk}$ are contained in $F$, and that the following equation holds:

$$\frac{\partial^2}{\partial z_j\partial\bar{z}_k}(\chi*\psi)=\chi*\psi_{jk}+\chi*T_{jk} .$$

From the assumptions i) and ii) for $\psi$, we find that the Hermitian matrix $[\chi*\psi_{jk}(z)]_{j,k=1}^n$ is positive definite on $W_\varepsilon$. Hence it is sufficient to show the positive semidefiniteness of the Hermitian matrix $[\chi*T_{jk}(z)]_{j,k=1}^n$ on $W_\varepsilon$. We will prove this by contradiction.

Suppose there exist $w\in W_\varepsilon$ and $\alpha\in C^n$ such that

$$\sum_{j,k=1}^n \chi*T_{jk}(w)\alpha_j\bar{\alpha}_k<0 .$$

We fix these $w$ and $\alpha$.

Since the Lebesgue measure of $F$ is zero, for any $\delta>0$, there exists an open neighborhood $U_\delta$ of $F$ in $W$ such that the Lebesgue measure of $U_\delta$ is smaller than $\delta$. We take a $C^\infty$ function $\rho_\delta$ on $W$ such that

$$0\leqq\rho_\delta\leqq 1 , \quad \rho_\delta=1 \text{ on a neighborhood of } F \text{ and } \operatorname{supp}\rho_\delta\subset U_\delta .$$

We put $\sigma_\delta(z)=\rho_\delta(w-z)$. Then, since the supports of $T_{jk}$ are contained in $F$, we have

$$\begin{aligned}(\sigma_\delta\chi)*T_{jk}(w)&=\langle\sigma_\delta(w-z)\chi(w-z),\ T_{jk}(z)\rangle\\&=\langle\rho_\delta(z)\chi(w-z),\ T_{jk}(z)\rangle=\langle\chi(w-z),\ T_{jk}(z)\rangle\\&=\chi*T_{jk}(w) .\end{aligned}$$

On the other hand, since $\psi_{jk}$ are $\mathscr{L}^1_{loc}$ functions, the integrals $(\sigma_\delta\chi)*\psi_{jk}(w)$ tend to zero as $\delta$ tends to zero. Hence there exists a $\delta_0>0$ such that

$$\sum_{j,k=1}^n (\sigma_{\delta_0}\chi)*\psi_{jk}(w)\alpha_j\bar{\alpha}_k<-\frac{1}{2}\sum_{j,k=1}^n \chi*T_{jk}(w)\alpha_j\bar{\alpha}_k .$$

Then we have

$$\sum_{j,k=1}^n \frac{\partial^2}{\partial z_j\partial\bar{z}_k}((\sigma_{\delta_0}\chi)*\psi)(w)\alpha_j\bar{\alpha}_k$$

$$=\sum_{j,k}(\sigma_{\delta_0}\chi)*\psi_{jk}(w)\alpha_j\bar{\alpha}_k+\sum_{j,k}(\sigma_{\delta_0}\chi)*T_{jk}(w)\alpha_j\bar{\alpha}_k$$
$$<-\frac{1}{2}\sum_{j,k}\chi*T_{jk}(w)\alpha_j\bar{\alpha}_k+\sum_{j,k}\chi*T_{jk}(w)\alpha_j\bar{\alpha}_k$$
$$=\frac{1}{2}\sum_{j,k}\chi*T_{jk}(w)\alpha_j\bar{\alpha}_k<0\ .$$

This contradicts to the positive semidefiniteness of the Hermitian matrix $[(\partial^2/\partial z_j\partial\bar{z}_k)((\alpha_{\delta_0}\chi)*\psi)(z)]$ on $W_\varepsilon$. Q.E.D.

LEMMA A.2. *For $\delta>0$, we put*

$$V_\delta=\mathring{\bar{U}}_\delta\ ,\qquad \text{where}\quad U_\delta=\{z\in C^n;\ |\mathrm{Im}\, z|^2<\delta^2(|\mathrm{Re}\, z|^2+1)\}\ .$$

*Then for any $f\in\mathcal{O}_{inc}(V_\delta)$ and any $\delta$ $(0<\delta<1/2)$, there exists an entire function $J$ of infra-exponential type such that*

$$|J(z)|\geqq(1+|z|^{n+1})|f(z)|\qquad (z\in U_\delta)\ .$$

PROOF. This is an immediate consequence of the following two lemmas:

LEMMA A.3. (Proposition 8.1.6 in Kaneko [15].) *Let $\varphi$ be a monotone increasing function on the half line $[1,\infty)$ with $\varphi>1$ and $\lim_{t\to\infty}\varphi(t)=\infty$. Then the infinite product*

$$J(\zeta)=\prod_{k=1}^{\infty}\left(1+\frac{\zeta^2}{(k\varphi(k))^2}\right)\qquad (\zeta\in C^n,\ \zeta^2=\zeta_1^2+\cdots+\zeta_n^2) \tag{A.1}$$

*defines a entire function on $C^n$ of infra-exponential type. Moreover we have the following estimation from below:*

$$|J(\zeta)|\geqq C\exp\left(\left(\log\frac{3}{2}\right)\frac{|\zeta|}{\varphi(|\zeta|+1)}\right)\qquad \left(|\mathrm{Im}\,\zeta|\leqq\max\left\{\frac{1}{\sqrt{3}}|\mathrm{Re}\,\zeta|,\ 1\right\}\right)\ . \tag{A.2}$$

LEMMA A.4. (Lemma 8.1.7 in Kaneko [15].) *For any given countable family $\{f_k\}$ of continuous positive functions on the half line $[0,\infty)$ with the infra-exponential growth, we can find a monotone increasing function $\varphi$ on the half line $[1,\infty)$ such that $\varphi>1$, $\lim_{t\to\infty}\varphi(t)=\infty$ and*

$$f_k(t)\geqq C_k\exp\left(\frac{t}{\varphi(t+1)}\right)\qquad (t\geqq 0,\ k=1,2,\cdots)\ ,$$

*where $C_k$ are constants independent of $t$.*

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