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On Homeomorphisms with Markov Partitions

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Introduction

Let (X, d) be a compact metric space and f be a homeomorphism from X onto itself. f is called *expansive* if there exists $e>0$ such that $d(f^n(x), f^n(y))\leqq e$ for all $n\in \boldsymbol{Z}$ implies $x=y$. The number e is called an *expansive constant* of f. A sequence $\{x_i\}_{i\in \boldsymbol{Z}}$ of X is a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1})<\delta$ for all $i\in \boldsymbol{Z}$. We say that $x\in X$ *ε-traces* a sequence $\{x_i\}_{i\in \boldsymbol{Z}}$ of X if $d(f^i(x), x_i)<\varepsilon$ for all $i\in \boldsymbol{Z}$. f is called to have the *pseudo orbit tracing property* (abbrev. P. O. T. P.) if for any $\varepsilon>0$ there exists $\delta>0$ such that every δ-pseudo orbit of f is ε-traced by a point of X. These properties of f are independent of metrics for X compatible with original topology.

A typical example of expansive homeomorphisms with P. O. T. P. is given in [1, 2]; it is shown that expansive group automorphisms of solenoidal groups have P. O. T. P. though in general group automorphisms with P. O. T. P. are not expansive. Another example is obtained from an *expanding map* $g: X\to X$, that is, g is an onto open map and there are $\delta>0$ and $\lambda>1$ such that $d(x, y)<\delta$ implies $d(g(x), g(y))\geqq\lambda d(x, y)$. Such maps become homeomorphisms through inverse limit and it is known that these homeomorphisms are expansive and have P. O. T. P.. As these examples show, we can construct easily examples of expansive homeomorphisms with P. O. T. P. and they form a larger class than that of Anosov diffeomorphisms. It was posed as a problem in topological dynamics whether every expansive homeomorphism of a torus satisfying P. O. T. P. is topologically conjugate to a toral automorphism, and the technique in this paper is important to solve this problem ([6]).

Ja. G. Sinai constructed in [9] Markov partitions for Anosov diffeomorphisms of compact C^∞ manifolds. After that R. Bowen [3, 5] constructed the same partitions for basic sets of Axiom A diffeomorphisms

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by following the Sinai's paper. In [8], D. Ruelle stated that Markov partitions could be constructed for certain homeomorphisms of compact metric spaces in a topological setting.

The purpose of this paper is to construct Markov partitions for expansive homeomorphisms with P. O. T. P. by following the ideas of R. Bowen. More precisely we can state the result as follows.

THEOREM. *If $f: X \to X$ is an expansive homeomorphism with* P.O.T.P., *then X has Markov partitions with arbitrary small diameter.*

A Markov partition of (X, f) induces a subshift (Σ, σ) of finite type and a surjective continuous map $\pi: \Sigma \to X$ such that $f \circ \pi = \pi \circ \sigma$ where σ is the shift homeomorphism (p. 84 of [5]). Then from the same proof as in [4] it follows that there is a positive number d such that $\mathrm{card}(\pi^{-1}(x)) \leqq d$ for all $x \in X$.

If the number N_m of the fixed points of f^m for $m>0$ is finite, the zeta function ζ_f of f is the formal power series defined by $\zeta_f(z) = \exp(\sum_{m=1}^{\infty} (N_m/m) z^m)$. We have the following corollary by the same proof as in [7].

COROLLARY. *If $f: X \to X$ is as in Theorem, then the zeta function of f is rational.*

§ 1. Definitions and lemmas.

Let $f: X \to X$ be as in Theorem and $e>0$ be an expansive constant of f. Put and fix $\varepsilon_0 = e/4$. For any $x \in X$ and $\varepsilon > 0$, we set

$$W_\varepsilon^s(x) = \{y \in X | d(f^n(x), f^n(y)) \leqq \varepsilon,\ n \geqq 0\},$$
$$W_\varepsilon^u(x) = \{y \in X | d(f^n(x), f^n(y)) \leqq \varepsilon,\ n \leqq 0\}.$$

LEMMA 1. *There exists $0 < \delta_0 < \varepsilon_0$ such that $W_{\varepsilon_0}^s(x) \cap W_{\varepsilon_0}^u(y)$ consists of one point for any $x, y \in X$ with $d(x, y) < \delta_0$.*

PROOF. Since f has P. O. T. P., there exists $0 < \delta_0 < \varepsilon_0/2$ such that any δ_0-pseudo orbit is $\varepsilon_0/2$-traced by a point of X. For $x, y \in X$ we assume $d(x, y) < \delta_0$. Then a sequence $\{x_i\}_{i \in Z}$ of X defined by $x_i = f^i(x)$ for $i \geqq 0$ and $x_i = f^i(y)$ for $i < 0$ is δ_0-pseudo orbit of f. Hence there is $z \in X$ such that $d(f^i(z), x_i) < \varepsilon_0/2$ for $i \in \boldsymbol{Z}$. Then $z \in W_{\varepsilon_0/2}^s(x) \subset W_{\varepsilon_0}^s(x)$ and $z \in W_{\varepsilon_0}^u(y)$ since $d(x, y) < \varepsilon_0/2$. Therefore $W_{\varepsilon_0}^s(x) \cap W_{\varepsilon_0}^u(y)$ is non-empty. By expansiveness of f, the conclusion is obtained.

Let δ_0 be as in Lemma 1 and put $\Delta(\delta_0) = \{(x, y) \in X \times X | d(x, y) < \delta_0\}$.

Then we can define a map $[\ ,\ ]: \Delta(\delta_0) \to X$ by assigning $[x, y] \in W^s_{\varepsilon_0}(x) \cap W^u_{\varepsilon_0}(y)$ to $(x, y) \in \Delta(\delta_0)$.

LEMMA 2. $[\ ,\ ]: \Delta(\delta_0) \to X$ *is a continuous map satisfying* $[x, x] = x$ and

$$[[x, y], z] = [x, z],\ [x, [y, z]] = [x, z]$$

when the two sides of these relations are defined.

PROOF. We assume that a sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ of $\Delta(\delta_0)$ converges to $(x, y) \in \Delta(\delta_0)$. Put $z_n = [x_n, y_n]$. Since X is compact, there is a subsequence $\{z_{n_j}\}_{j=1}^{\infty}$ of $\{z_n\}_{n=1}^{\infty}$ such that $\{z_{n_j}\}_{j=1}^{\infty}$ converges to a point $z \in X$. By the definition of $[\ ,\ ]$, $d(f^i(x_{n_j}), f^i(z_{n_j})) \leqq \varepsilon_0$ for every $i \geqq 0$ and n_j. Hence $d(f^i(x), f^i(z)) \leqq \varepsilon_0$ and $z \in W^s_{\varepsilon_0}(x)$. Similarly, $z \in W^u_{\varepsilon_0}(y)$ and $z = [x, y]$. This shows that $\{z_n\}_{n=1}^{\infty}$ converges to $[x, y]$.

It is clear that $[x, x] = x$ for all $x \in X$. Since $[x, y] \in W^s_{\varepsilon_0}(x)$, $[[x, y], z] \in W^s_{2\varepsilon_0}(x) \cap W^u_{\varepsilon_0}(z)$. By expansiveness of f, $[[x, y], z] = [x, z]$. Similarly, $[x, [y, z]] = [x, z]$.

PROPOSITION 3 (Local product structure). *Let* $f: X \to X$ *be as in Theorem. Then under the above notations there exist* $0 < \delta_1 < \delta_0/2$ *and* $0 < \rho < \delta_1$ *such that for any* $x \in X$ *the sets*

$$W^u_{loc}(x) = \{y \in W^u_{\varepsilon_0}(x) | d(x, y) < \delta_1\},$$
$$W^s_{loc}(x) = \{y \in W^s_{\varepsilon_0}(x) | d(x, y) < \delta_1\},$$

and

$$N_x = [W^u_{loc}(x), W^s_{loc}(x)]$$

have the following properties;

(a) N_x *is an open subset of* X *and* $\mathrm{diam}(N_x) < \delta_0$,

(b) $[\ ,\ ]: W^u_{loc}(x) \times W^s_{loc}(x) \to N_x$ *is a homeomorphism,*

(c) $N_x \supset B_\rho(x)$ *where* $B_\rho(x) = \{y \in X | d(x, y) \leqq \rho\}$.

PROOF. The map $g_1: X \times \Delta(\delta_0) \to \boldsymbol{R}$ defined by $g_1(x, (y, z)) = d(x, [y, z])$ for $(x, (y, z)) \in X \times \Delta(\delta_0)$ is continuous and $g_1(x, (x, x)) = 0$ by Lemma 2. By compactness of X there is $0 < \delta_1 < \delta_0/2$ such that $\mathrm{diam}\{x, y, z\} < 2\delta_1$ implies $d(x, [y, z]) < \delta_0/3$. If $(y, z) \in W^u_{loc}(x) \times W^s_{loc}(x)$, then $d(x, [y, z]) < \delta_0/3$ and so $\mathrm{diam}(N_x) < \delta_0$. Suppose $w \in N_x$. Then there are $y \in W^u_{loc}(x)$ and $z \in W^s_{loc}(x)$ with $w = [y, z]$. Since $d(x, w) < \delta_0/3$, we can define the maps $P_u: B_{\delta_0/3}(w) \to W^u_{\varepsilon_0}(x)$ and $P_s: B_{\delta_0/3}(w) \to W^s_{\varepsilon_0}(x)$ by $P_u(v) = [v, x]$ and $P_s(v) = [x, v]$ for $v \in B_{\delta_0/3}(w)$. Then by Lemma 2, P_u and P_s are continuous, $P_u(w) = [y, x] = y$ and $P_s(w) = z$. Hence there is a neighborhood U of w in X such that

$P_u(U) \subset W^u_{loc}(x)$ and $P_s(U) \subset W^s_{loc}(x)$. If $v \in U$, then $v \in N_x$ since $v=[[v, x], [x, v]]$ by expansiveness. This implies that N_x is open in X and so a) holds.

Define $h: N_x \to W^u_{loc}(x) \times W^s_{loc}(x)$ by $h(w)=([w, x], [x, w])$ for $w \in N_x$. Clearly h is continuous by Lemma 2. In fact h is the inverse map of $[\ ,\ ]: W^u_{loc}(x) \times W^s_{loc}(x) \to N_x$. Therefore we have b).

The map $g_2: \Delta(\delta_0) \to \boldsymbol{R}$ defined by $g_2(x, y)=\mathrm{diam}\{x, [y, x], [x, y]\}$ $((x, y) \in \Delta(\delta_0))$ is continuous and $g_2(x, x)=0$ by Lemma 2. Hence by compactness of X there is $0<\rho<\delta_1$ such that $d(x, y) \leqq \rho$ implies $g_2(x, y)<\delta_1$. This shows that $[y, x] \in W^u_{loc}(x)$ and $[x, y] \in W^s_{loc}(x)$. Hence $y=[[y, x], [x, y]] \in N_x$ and the proof is completed.

Hereafter we fix the numbers δ_1 and ρ that are chosen in Proposition 3. Let $W^u_{loc}(x)$, $W^s_{loc}(x)$ and N_x be as in Proposition 3. We claim that for any $x \in X$, $N_x \ni y, z$ implies $[y, z] \in N_x$. Indeed, by the definition of N_x, there exist $u_1, u_2 \in W^u_{loc}(x)$ and $v_1, v_2 \in W^s_{loc}(x)$ such that $y=[u_1, v_1]$ and $z=[u_2, v_2]$. Hence we have $[y, z]=[[u_1, v_1], [u_2, v_2]]=[u_1, v_2] \in N_x$.

For $x, y \in X$ we put $D^u_{x,y}=W^u_{loc}(x) \cap N_y$ and $D^s_{x,y}=W^s_{loc}(x) \cap N_y$.

Lemma 4. *Foy $x, y \in X$ with $d(x, y) \leqq \rho$, $D^\sigma_{x,y}$ is an open neighborhood of x in $W^\sigma_{loc}(x)$ for $\sigma=u, s$ and the maps $[\ , y]: D^u_{x,y} \to D^u_{y,x}$, $[y,\ ]: D^s_{x,y} \to D^s_{y,x}$ are homeomorphisms.*

Proof. Since N_y is open in X by a) of Proposition 3, $D^\sigma_{x,y}$ is open in $W^\sigma_{loc}(x)$. Since $x \in B_\rho(y) \subset N_y$ by c) of Proposition 3, $x \in D^\sigma_{x,y}$ and so $D^\sigma_{x,y}$ is an open neighborhood of x in $W^\sigma_{loc}(x)$. If $z \in D^u_{x,y}$, then $[z, y] \in W^u_{loc}(y)$ since $z \in N_y$ and $[z, y] \in N_x$ since $z \in W^u_{loc}(x) \subset N_x$ and $y \in B_\rho(x) \subset N_x$. Hence $[z, y] \in D^u_{y,x}$. Similarly $z \in D^u_{y,x}$ implies $[z, x] \in D^u_{x,y}$. Hence $[D^u_{x,y}, y]=D^u_{y,x}$ and by Lemma 2 the map $[\ , y]: D^u_{x,y} \to D^u_{y,x}$ is a homeomorphism with the inverse map $[\ , x]: D^u_{y,x} \to D^u_{x,y}$. Similarly $[y, D^s_{x,y}]=D^s_{y,x}$ and the map $[y,\ ]: D^s_{x,y} \to D^s_{y,x}$ is a homeomorphism.

Lemma 5. *Foy any $x \in X$,*

(a) *$f(W^s_{loc}(x)) \cap W^s_{loc}(f(x))$ is open in $W^s_{loc}(f(x))$,*

(b) *$f^{-1}(W^u_{loc}(x)) \cap W^u_{loc}(f^{-1}(x))$ is open in $W^u_{loc}(f^{-1}(x))$.*

Proof. If $w \in f(N_x) \cap W^s_{loc}(f(x))$, then $f^{-1}(w) \in N_x$. So there exist $y \in W^u_{loc}(x)$ and $z \in W^s_{loc}(x)$ with $f^{-1}(w)=[y, z]$. Then $f^{-1}(w) \in W^u_{\varepsilon_0}(z)$ and $w \in W^s_{loc}(f(x)) \subset W^s_{2\varepsilon_0}(f(z))$. By expansiveness of f, $f^{-1}(w)=z$ and $w \in f(W^s_{loc}(x))$. Therefore we have $f(N_x) \cap W^s_{loc}(f(x))=f(W^s_{loc}(x)) \cap W^s_{loc}(f(x))$. Since N_x is open in X by Proposition 3, we have (a). Similarly, (b) holds.

A subset R of X is called a *rectangle* if $\mathrm{diam}(R) \leqq \rho$ and if $[x, y] \in R$

for $x, y \in R$. In this definition, we note that $[x, y] \in W^s_{loc}(x) \cap W^u_{loc}(y)$ since $\rho < \delta_1$. For $A \subset X$, $\mathrm{cl}(A)$ denotes the closure of A in X. It is clear that if R is a rectangle, then $\mathrm{cl}(R)$ is also a rectangle. When R is a rectangle and $x \in R$, we define $W^s(x, R)$ and $W^u(x, R)$ by $W^s_{loc}(x) \cap R$ and $W^u_{loc}(x) \cap R$ respectively. $\mathrm{int}\, W^s(x, R)$ and $\mathrm{int}\, W^u(x, R)$ denote the interior of $W^s(x, R)$ in $W^s_{loc}(x)$ and the interior of $W^u(x, R)$ in $W^u_{loc}(x)$ respectively. We put $\partial W^s(x, R) = W^s(x, R) \setminus \mathrm{int}\, W^s(x, R)$ and $\partial W^u(x, R) = W^u(x, R) \setminus \mathrm{int}\, W^u(x, R)$.

Hereafter R denotes a rectangle.

LEMMA 6. *For* $x, y \in R$,
(a) $R = [W^u(x, R), W^s(x, R)]$,
(b) $[\partial W^u(x, R), W^s(x, R)] = [\partial W^u(y, R), W^s(y, R)]$,
(c) $[W^u(x, R), \partial W^s(x, R)] = [W^u(y, R), \partial W^s(y, R)]$.

PROOF. First we show (a). Since R is a rectangle, $[W^u(x, R), W^s(x, R)] \subset R$. If $z \in R$, then $[z, x] \in R \cap W^u_{loc}(x) = W^u(x, R)$ and $[x, z] \in W^s(x, R)$. Hence $z = [[z, x], [x, z]] \in [W^u(x, R), W^s(x, R)]$. Next we show (b). Since $\mathrm{diam}(R) \leqq \rho$ and $y \in R$, $R \subset B_\rho(y) \subset N_y$ and so $W^u(x, R) \subset D^u_{x,y}$. Similarly, $W^u(y, R) \subset D^u_{y,x}$. Since $[W^u(x, R), y] = W^u(y, R)$, by Lemma 4 we have $[\partial W^u(x, R), y] = \partial W^u(y, R)$. Hence

$$[\partial W^u(x, R), W^s(x, R)] = [[\partial W^u(x, R), y], [y, W^s(x, R)]] = [\partial W^u(y, R), W^s(y, R)]\,.$$

In the similar way (c) holds.

We define $\partial^s R = [\partial W^u(x, R), W^s(x, R)]$ and $\partial^u R = [W^u(x, R), \partial W^s(x, R)]$. Then these do not depend on $x \in R$ by Lemma 6 and $\partial^s R, \partial^u R \subset R$ since R is a rectangle. $\mathrm{int}(R)$ denotes the interior of R in X. We put $\partial R = R \setminus \mathrm{int}(R)$.

LEMMA 7. *Under the above notations,*
(a) $\mathrm{int}(R) = [\mathrm{int}\, W^u(x, R), \mathrm{int}\, W^s(x, R)]$,
(b) $\partial R = \partial^s R \cup \partial^u R$.

PROOF. Since $R \subset N_x$, the interior of R in N_x is equal to $\mathrm{int}(R)$. Since $R = [W^u(x, R), W^s(x, R)]$ by (a) of Lemma 6, (a) follows from (b) of Proposition 3. By (b) of Proposition 3, we have $\partial R = R \setminus \mathrm{int}(R) = [W^u(x, R), W^s(x, R)] \setminus [\mathrm{int}\, W^u(x, R), \mathrm{int}\, W^s(x, R)] = \partial^s R \cup \partial^u R$.

Notice that $\mathrm{int}(R)$ is also a rectangle by (*a*) of Lemma 7.

LEMMA 8. *If* $\mathrm{int}(R) \neq \emptyset$ *and* $x \in \mathrm{int}(R)$, *then* $\mathrm{int}\, W^\sigma(x, R) = W^\sigma(x, \mathrm{int}(R))$ *for* $\sigma = u, s$.

Proof. Since int(R) is open in X, clearly $W^s(x, \text{int}(R))$ is open in $W^s_{loc}(x)$ and so $W^s(x, \text{int}(R)) \subset \text{int}\, W^s(x, R)$. Similarly, $W^u(x, \text{int}(R)) \subset \text{int}\, W^u(x, R)$. Hence $x \in \text{int}\, W^u(x, R)$. If $z \in \text{int}\, W^s(x, R)$, then $z = [x, z]$. By (a) of Lemma 7, $z \in \text{int}(R)$. Therefore $z \in W^s(x, \text{int}(R))$ and so we have $\text{int}\, W^s(x, R) = W^s(x, \text{int}(R))$. Similarly, we can show the lemma for $\sigma = u$.

Lemma 9. $\text{cl}\, W^\sigma(x, R) = W^\sigma(x, \text{cl}(R))$ *for* $\sigma = u, s$.

Proof. Since cl(R) is a rectangle,

$$[x, \text{cl}(R)] \subset \text{cl}(R) \cap W^s_{loc}(x) = W^s(x, \text{cl}(R)) .$$

If $z \in W^s(x, \text{cl}(R))$, then $z \in W^s_{loc}(x)$ and so $z = [x, z] \in [x, \text{cl}(R)]$. Hence $W^s(x, \text{cl}(R)) = [x, \text{cl}(R)]$. Since $[x, \text{cl}(R)]$ is closed in X by Lemma 2, we have $\text{cl}\, W^s(x, R) \subset W^s(x, \text{cl}(R))$. Similarly, $\text{cl}\, W^u(x, R) \subset W^u(x, \text{cl}(R))$. By (a) of Lemma 6, $R = [W^u(x, R), W^s(x, R)]$ and so R is contained in $[\text{cl}\, W^u(x, R), \text{cl}\, W^s(x, R)]$ which is closed in X by Lemma 2. Hence $\text{cl}(R) \subset [\text{cl}\, W^u(x, R), \text{cl}\, W^s(x, R)] \subset [W^u(x, \text{cl}(R)), W^s(x, \text{cl}(R))] = \text{cl}(R)$. By (b) of Proposition 3 we obtain the conclusion.

R is called *proper* if $R = \text{cl}(\text{int}(R))$.

Definition. A Markov partition is a finite cover $\{R_1, \cdots, R_m\}$ of X consisting of proper rectangles such that

(a) $\text{int}\, R_i \cap \text{int}\, R_j = \varnothing$ for $i \neq j$,

(b) $f W^s(x, R_i) \subset W^s(f(x), R_j)$ and
$f W^u(x, R_i) \supset W^u(f(x), R_j)$ when $x \in \text{int}\, R_i \cap f^{-1}(\text{int}\, R_j)$.

§ 2. Construction of rectangles.

In order to construct Markov partitions for expansive homeomorphisms with P. O. T. P., we state here the ideas stated by R. Bowen to construct Markov partitions for basic sets of Axiom A diffeomorphisms. Let $f: X \to X$ be as in Theorem and as before δ_1 and ρ be as in Proposition 3. Let β be a positive number with $\beta \leqq \rho/2$ such that $d(f(x), f(y)) < \delta_1$ and $d(f^{-1}(x), f^{-1}(y)) < \delta_1$ when $d(x, y) \leqq \beta$. Since f has P. O. T. P., let $0 < \alpha < \beta/2$ be a number such that any α-pseudo orbit is $\beta/2$-traced by a point of X. Choose $0 < \gamma < \alpha/2$ such that $d(f(x), f(y)) < \alpha/2$ when $d(x, y) < \gamma$. Let $P = \{P_1, \cdots, P_r\}$ be a γ-dense subset of X and $\Sigma(P) = \{(q_j) \in \prod_{-\infty}^{+\infty} P \mid d(f(q_j), q_{j+1}) < \alpha,\ j \in \boldsymbol{Z}\}$. Then for each $q \in \Sigma(P)$ there is a unique $\theta(q) \in X$ which $\beta/2$-traces q. Conversely for any $x \in X$ there exists $q \in \Sigma(P)$ with $x = \theta(q)$. We can find a map θ from $\Sigma(P)$ onto X

such that $f\circ\theta=\theta\circ\sigma$ where $\sigma: \Sigma(P)\to\Sigma(P)$ is the shift homeomorphism, i.e., $\sigma(q)_j=q_{j+1}$ for any $q\in\Sigma(P)$ and $j\in \boldsymbol{Z}$. Let $T_s=\{\theta(q)\mid q\in\Sigma(P), q_0=P_s\}(s=1, \cdots, r)$. Then $\operatorname{diam}(T_s)\leqq\beta$ and $T=\{T_1, \cdots, T_r\}$ is a covering of X. For fixed s with $1\leqq s\leqq r$, if $x, y\in T_s$, then there exist $q, q'\in\Sigma(P)$ such that $x=\theta(q)$, $y=\theta(q')$ and $q_0=q'_0=P_s$. Define $q^*=[q, q']\in\Sigma(P)$ by $q^*_j=q_j$ for $j\geqq 0$ and $q^*_j=q'_j$ for $j\leqq 0$. By the definition of θ, we have

$$d(f^j(\theta(q^*)), f^j(\theta(q)))<\beta \quad \text{for} \quad j\geqq 0\,,$$
$$d(f^j(\theta(q^*)), f^j(\theta(q')))<\beta \quad \text{for} \quad j\leqq 0\,.$$

Hence $[x, y]=[\theta(q), \theta(q')]=\theta(q^*)\in T_s$ and so T_s is a rectangle. Since f is expansive, clearly θ is continuous (cf. see p. 80 of [5]). Hence T_s $(1\leqq s\leqq r)$ are closed in X.

For $x\in X$, let $T(x)=\{T_j\in T\mid x\in T_j\}$ and $T^*(x)=\{T_k\in T\mid T_k\cap T_j\neq\emptyset$ for some $T_j\in T(x)\}$. Put $Z=X\setminus\bigcup_{j=1}^r\partial T_j$. Then Z is open dense in X. Let $Z^*=\{x\in X\mid W^s_{loc}(x)\cap\partial^s T_k=\emptyset,\ W^u_{loc}(x)\cap\partial^u T_k=\emptyset$ for every $T_k\in T^*(x)\}$. Then $Z^*\subset Z$ by (b) of Lemma 7.

Lemma 10. *Z^* is dense in X.*

Proof. For $x\in X$ we put $\partial^s_x=\bigcup_{T_k\in T^*(x)}\partial^s T_k$ and $\partial^u_x=\bigcup_{T_k\in T^*}(x)\partial^u T_k$. Since $\operatorname{diam}(T_k)\leqq\beta$, we have that $\bigcup_{T_k\in T^*(x)}T_k\subset B_\rho(x)\subset N_x$. Hence we have

$$\begin{aligned}[\partial^s_x, x]&=\bigcup_{T_k\in T^*(x)}[\partial^s T_k, x]\\&=\bigcup_{T_k\in T^*(x)}[\partial W^u(y_k, T_k), x]\subset W^u_{loc}(x)\end{aligned}$$

where $y_k\in T_k$. Since $W^u(y_k, T_k)\subset D^u_{y_k,x}$, by Lemma 4 $[\partial W^u(y_k, T_k), x]$ is nowhere dense in D^u_{x,y_k} and so is in $W^u_{loc}(x)$. Hence $[\partial^s_x, x]$ is nowhere dense in $W^u_{loc}(x)$. Similarly, $[x, \partial^u_x]$ is nowhere dense in $W^s_{loc}(x)$. If $x\in Z$, then there exists an open neighborhood $U_x(\subset N_x)$ of x in X such that for any $y\in U_x$, $T(y)=T(x)$ and so $\partial^\sigma_y=\partial^\sigma_x$ for $\sigma=u, s$. Put $U'_x=U_x\cap[W^u_{loc}(x)\setminus[\partial^s_x, x], W^s_{loc}(x)\setminus[x, \partial^u_x]]$. Then, by (b) of Proposition 3, U'_x is dense in U_x. If $y\in U'_x$, then there exist $y_1\in W^u_{loc}(x)\setminus[\partial^s_x, x]$ and $y_2\in W^s_{loc}(x)\setminus[x, \partial^u_x]$ with $y=[y_1, y_2]$. If $z\in W^s_{loc}(y)\cap\partial^s_x\neq\emptyset$, then $y_1=[y, x]=[[y, z], x]=[z, x]\in[\partial^s_x, x]$ which is a contradiction. Hence $W^s_{loc}(y)\cap\partial^s_x=\emptyset$. In the same way, $W^u_{loc}(y)\cap\partial^u_x=\emptyset$. This tells us that $y\in U'_x$ implies $y\in Z^*$. Hence Z^* is dense in Z. Since Z is dense in X, the conclusion is obtained.

For $T_j, T_k\in T$ with $T_j\cap T_k\neq\emptyset$, we define the sets

$$T^1_{j,k}=\{x\in T_j\mid W^u_{loc}(x)\cap T_k\neq\emptyset,\ W^s_{loc}(x)\cap T_k\neq\emptyset\}=T_j\cap T_k\,,$$
$$T^2_{j,k}=\{x\in T_j\mid W^u_{loc}(x)\cap T_k\neq\emptyset,\ W^s_{loc}(x)\cap T_k=\emptyset\}\,,$$

$$T^3_{j,k}=\{x\in T_j \mid W^u_{loc}(x)\cap T_k=\emptyset,\ W^s_{loc}(x)\cap T_k\neq\emptyset\}\ ,$$
$$T^4_{j,k}=\{x\in T_j \mid W^u_{loc}(x)\cap T_k=\emptyset,\ W^s_{loc}(x)\cap T_k=\emptyset\}\ ,$$

Clearly $T_j=\bigcup_{n=1}^4 T^n_{j,k}$ is a disjoint union. It is clear that $T^1_{j,k}$ is a rectangle. If $x, y\in T^2_{j,k}$, then $z\in W^u_{loc}(y)\cap T_k\neq\emptyset$ and $W^s_{loc}(x)\cap T_k=\emptyset$. Since $z=[z, y]=[z, [x, y]]$ and $d(z, [x, y])\leqq 2\beta<\delta_1$, $z\in W^u_{loc}([x, y])\cap T_k\neq\emptyset$. If $z'\in W^s_{loc}([x, y])\cap T_k\neq\emptyset$, then $z'=[[x, y], z']=[x, z']$. Since $d(z', x)\leqq 2\beta<\delta_1$, $z'\in W^s_{loc}(x)\cap T_k\neq\emptyset$ which is a contradiction. Hence $W^s_{loc}([x, y])\cap T_k=\emptyset$ and $[x, y]\in T^2_{j,k}$. This shows that $T^2_{j,k}$ is a rectangle. Similarly, $T^3_{j,k}$ and $T^4_{j,k}$ are rectangles.

LEMMA 11. *For $T_j, T_k\in T$ with $T_j\cap T_k\neq\emptyset$ and any n,*

$$\operatorname{int} T^n_{j,k}=\{x\in T_{j,k} \mid W^s_{loc}(x)\cap(\partial^s T_j\cup\partial^s T_k)=\emptyset\ ,\ W^u_{loc}(x)\cap(\partial^u T_j\cup\partial^u T_k)=\emptyset\}\ .$$

PROOF. Suppose $x\in\operatorname{int} T^n_{j,k}$ and $z\in W^s_{loc}(x)\cap(\partial^s T_j\cup\partial^s T_k)\neq\emptyset$. If $z\in W^s_{loc}(x)\cap\partial^s T_j$, then by the definition of $\partial^s T_j$ there exist $z_1\in\partial W^u(x, T_j)$ and $z_2\in W^s(x, T_j)$ with $z=[z_1, z_2]$. Since $x=[z_2, z_1]$, $x=[[z_1, z_2], [z_2, z_1]]=z_1\in\partial^s T_j$. This contradicts $x\in\operatorname{int} T_j$ by (b) of Lemma 7. If $z\in W^s_{loc}(x)\cap\partial^s T_k$, by the definition of $\partial^s T_k$ there exist $z_1\in\partial W^u(z, T_k)$ and $z_2\in W^s(z, T_k)$ with $z=[z_1, z_2]$. Since $z=[z_1, z_2]$, $z=[[z_1, z_2], [z_2, z_1]]=z_1$ and so $z\in\partial W^u(z, T_k)$. Since $d(x, z)\leqq\rho$, by Lemma 4 we can consider the homeomorphism $[\ , z]\colon D^u_{x,z}\to D^u_{z,x}$. Then $[x, z]=z$. Since $D^u_{x,z}\cap T^n_{j,k}$ is a neighborhood of x in $D_{x,z}$, there exists $v\in D^u_{x,z}\cap T^n_{j,k}$ such that $[v, z]\notin W^u(z, T_k)$. If $w\in W^s_{loc}(v)\cap T_k\neq\emptyset$, then $[v, z]=[[w, v], z]=[w, z]\in W^u(z, T_k)$ which is a contradiction. Hence we have $W^s_{loc}(v)\cap T_k=\emptyset$. This contradicts the fact that x and v belong to the same set $T^n_{j,k}$. Therefore $x\in\operatorname{int} T^n_{j,k}$ implies $W^s_{loc}(x)\cap(\partial^s T_j\cup\partial^s T_k)=\emptyset$. In the similar way, $x\in\operatorname{int} T^n_{j,k}$ implies $W^u_{loc}(x)\cap(\partial^u T_j\cup\partial^u T_k)=\emptyset$.

Next we suppose $x\in\partial^s T^n_{j,k}$ and $W^s_{loc}(x)\cap(\partial^s T_j\cup\partial^s T_k)=\emptyset$. Then $x\in\partial W^u(x, T^n_{j,k})$. If $x\in\partial W^u(x, T_j)$, then $x\in\partial^s T_j$ and so $W^s_{loc}(x)\cap\partial^s T_j\neq\emptyset$ which is a contradiction. Hence $x\in\operatorname{int} W^u(x, T_j)$. Since $d(x, y_k)\leqq\rho$ $(y_k\in T_k)$, by Lemma 4 we can consider the homeomorphism $[\ , y_k]\colon D^u_{x,y_k}\to D^u_{y_k,x}$. Since $W^s_{loc}(x)\cap\partial^s T_k=\emptyset$, $W^s_{loc}(x)\cap\partial W^u(y_k, T_k)=\emptyset$. Hence $[x, y_k]\notin\partial W^u(y_k, T_k)$. Since T_k is a closed set, by Lemma 9 $W^u(y_k, T_k)$ is closed in X. Since $W^u(y_k, T_k)\subset D^u_{y_k,x}$, $\partial W^u(y_k, T_k)$ is the boundary of $W^u(y_k, T_k)$ in $D^u_{y_k,x}$. Hence there exists a neighborhood $U^u_x(\subset W^u(x, T_j))$ of x in D^u_{x,y_k} such that $[U^u_x, y_k]\subset W^u(y_k, T_k)$ or $[U^u_x, y_k]\subset D^u_{y_k,x}\setminus W^u(y_k, T_k)$, i.e., $W^s_{loc}(v)\cap T_k\neq\emptyset$ for all $v\in U^u_x$ or $W^s_{loc}(v)\cap T_k=\emptyset$ for all $v\in U^u_x$. On the other hand, $W^u_{loc}(v)\cap T_k\neq\emptyset$ for all $v\in U^u_x$ or $W^u_{loc}(v)\cap T_k=\emptyset$ for all $v\in U^u_x$ since $U^u_x\subset W^u(x, T_j)$. Hence

$U_x^u \subset T_{j,k}^n$. This contradicts $x \in \partial W^u(x, T_{j,k}^n)$. Therefore, for $x \in T_{j,k}^n$, $W_{loc}^s(x) \cap (\partial^s T_j \cup \partial^s T_k) = \emptyset$ implies $x \notin \partial^s T_{j,k}^n$. Similarly $W_{loc}^u(x) \cap (\partial^u T_j \cup \partial^u T_k) = \emptyset$ implies $x \notin \partial^u T_{j,k}^n$. By using (b) of Lemma 7 we obtain the conclusion.

For $x \in Z^*$, if $T_j \cap T_k \neq \emptyset$ for $T_j \in T(x)$ and $T_k \in T$, then $x \in \operatorname{int} T_{j,k}^n$ for some n (Lemma 11). For $x \in Z^*$ we define

$$R(x) = \cap \{\operatorname{int} T_{j,k}^n | T_j \in T(x),\ T_j \cap T_k \neq \emptyset (T_k \in T) \text{ and } x \in \operatorname{int} T_{j,k}^n\} .$$

Then $R(x)$ is an open rectangle. Assume $y \in R(x)$. Then $T(x) \subset T(y)$. If $T_k \in T(y)$ and $T_j \in T(x)$, then $y \in T_j \cap T_k \neq \emptyset$. Hence $y \in R(x) \subset T_{j,k}^1$ and $T_k \in T(x)$. Therefore we have $T(y) = T(x)$. By the definition of $R(x)$ and Lemma 11, $y \in Z^*$ and $R(y) = R(x)$. Since T is finite, $\{R(x) | x \in Z^*\}$ is finite. Therefore there exist points $x_1, \cdots, x_m \in Z^*$ such that $Z^* = R(x_1) \cup \cdots \cup R(x_m)$ is a disjoint union.

LEMMA 12. *If* $x \in R(x_i) \cap f^{-1}(R(x_j)) \neq \emptyset$, *then*
(a) $f W^s(x, R(x_i)) \subset W^s(f(x), R(x_j))$,
(b) $f W^u(x, R(x_i)) \supset W^u(f(x), R(x_j))$.

PROOF. For $v \in X$, there exist $q \in \Sigma(P)$ with $v = \theta(q)$. Assume $q_0 = P_s$ and $q_1 = P_t$. For $w \in W^s(v, T_s)$, there exists $q' \in \Sigma(P)$ such that $w = \theta(q')$ and $q_0' = P_s$. Hence $w = [v, w] = [\theta(q), \theta(q')] = \theta([q, q'])$, so that $f(w) = f \circ \theta([q, q']) = \theta \circ \sigma([q, q']) \in T_t$. Since $d(v, w) < \beta$, we have $f(w) \in W_{loc}^s(f(v))$. Hence $f(w) \in W^s(f(v), T_t)$; i.e.,

$$(1) \qquad f W^s(v, T_s) \subset W^s(f(v), T_t) .$$

In the similar way $f^{-1} W^u(f(v), T_t) \subset W^u(v, T_s)$; i.e.,

$$(2) \qquad f W^u(v, T_s) \supset W^u(f(v), T_t) .$$

Assume $y \in W^s(x, R(x_i))$. Then $y \in W_{loc}^s(x)$ and $R(y) = R(x) = R(x_i)$. First we show $T(f(x)) = T(f(y))$. If $f(x) \in T_j$ and $f(x) = \theta \circ \sigma(q)$ with $q_1 = P_j$, $q_0 = P_s$, then $x = \theta(q) \in T_s$. By (1), $f(y) \in f W^s(x, T_s) \subset W^s(f(x), T_j)$. Hence $f(y) \in T_j$. Similarly $f(y) \in T_j$ implies $f(x) \in T_j$. Next we prove that $f(x)$ and $f(y)$ belong to the same set $T_{j,k}^n$ when $T_j \in T(f(x)) = T(f(y))$ and $T_j \cap T_k \neq \emptyset (T_k \in T)$. Since $f(y) \in W^s(f(x), T_j)$, $f(x)$ and $f(y)$ belong to $T_{j,k}^1 \cup T_{j,k}^3$ or $T_{j,k}^2 \cup T_{j,k}^4$. Now we assume that $W_{loc}^u(f(y)) \cap T_k = \emptyset$ and $W_{loc}^u(f(x)) \cap T_k \neq \emptyset$. And we derive a contradiction. This assumption is equivalent to

$$(3) \qquad W^u(f(y), T_j) \cap T_k = \emptyset,\ f(z) \in W^u(f(x), T_j) \cap T_k \neq \emptyset .$$

Let $f(x) = \theta(q)$ with $q_1 = P_j$, $q_0 = P_s$. By (2), $f(z) \in W^u(f(x), T_j) \subset f W^u(x, T_s)$

and so $z \in W^u(x, T_s)$. Let $f(z)=\theta\circ\sigma(q')$ where $q'_1=T_k$ and $q'_0=T_t$. Then $z\in T_t$. Hence $T_s \in T(x)=T(y)$ and $z\in T_s\cap T_t\neq\emptyset$. Since $z\in W^u(x, T_s)\cap T_t\neq\emptyset$ and x, y belong to the same set $T^n_{s,t}$, it follows that $W^u_{loc}(y)\cap T_t\neq\emptyset$. This is equivalent to $z'\in W^u(y, T_s)\cap T_t\neq\emptyset$. Then $z''=[z, y]=[z, z']\in W^s(z, T_t)$. By (1), $f(z'')\in W^s(f(z), T_k)$. Since $f(z)$ and $f(y)$ belong to T_j and f is expansive, we have $f(z'')=[f(z), f(y)]\in W^u(f(y), T_j)$. This contradicts (3).

Let us put

$$R(f(x))'=\cap\{T^n_{j,k}\mid T_j\in T(f(x)),\ T_j\cap T_k\neq\emptyset\quad (T_k\in T)\quad\text{and}\quad f(x)\in T^n_{j,k}\}\,.$$

Then $\operatorname{int} R(f(x))'=R(f(x))=R(x_j)$. Since $\operatorname{diam} R(x_i)\leqq\beta$, $fW^s(x, R(x_i))\subset W^s_{loc}(f(x))$. Hence $fW^s(x, R(x_i))\subset W^s(f(x), R(f(x))')$ by the above argument. On the other hand, $fW^s(x, R(x_i))$ is open in $W^u_{loc}(f(x))$ by Lemma 5. Therefore $fW^s(x, R(x_i))\subset \operatorname{int} W^s(f(x), R(f(x))')=W^s(f(x), R(x_j))$ by Lemma 8. Similarly, we have $fW^u(x, R(x_i))\supset W^u(f(x), R(x_j))$.

§ 3. Proof of Theorem.

For the proof we use the above notations. Put $R_i=\operatorname{cl} R(x_i)$ for $1\leqq i\leqq m$. Then it is enough to prove that $\{R_1, \cdots, R_m\}$ is a Markov partition of X. By Lemma 10, $\{R_1, \cdots, R_m\}$ is a covering of X. Clearly R_i are proper rectangles, $\operatorname{diam} R_i\leqq\beta$ and $\operatorname{int} R_i\cap\operatorname{int} R_j=\emptyset$ for $i\neq j$. If $\operatorname{int} R_i\cap f^{-1}\operatorname{int} R_j\neq\emptyset$, then there exists $x\in R(x_i)\cap f^{-1}(R(x_j))$. And then $fW^s(x, R(x_j))\subset W^s(f(x), R(x_j))$ by Lemma 12. Hence $fW^s(x, R_i)\subset W^s(f(x), R_j)$ by Lemma 9. If $y\in R_i\cap f^{-1}(R_j)$, then $fW^s(y, R_i)=f[y, W^s(x, R_i)]=[f(y), fW^s(x, R_i)]\subset[f(y), W^s(f(x), R_j)]=W^s(f(y), R_j)$. Similarly $fW^u(y, R_i)\supset W^u(f(y), R_j)$. The proof of the Theorem is completed.

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References

[1] N. AOKI, Topological stability of solenoidal automorphisms, Nagoya Math. J., **90** (1983), 119–135.

[2] N. AOKI, M. DATEYAMA and M. KOMURO, Solenoidal automorphisms with specification, Monatsh. Math., **93** (1982), 79–110.

[3] R. BOWEN, Markov partitions for Axiom A diffeomorphisms, Amer. J. Math., **92** (1970), 725–747.

[4] R. BOWEN, Markov partitions and minimal sets for Axiom A diffeomorphisms, Amer. J. Math., **92** (1970), 907–918.

[5] R. Bowen, Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms, Lecture Notes in Math., **470**, Springer, 1975.

[6] K. Hiraide, Expansive homeomorphisms of tori with the pseudo orbit tracing property, preprint.

[7] A. Manning, Axiom A diffeomorphisms have rational zeta functions, Bull. London. Math. Soc., **3** (1971), 215–220.

[8] D. Ruelle, Thermodynamic Formalism-The Mathematical Structure of Classical Equilibrium Statistical Mechanics, Encyclopedia of Math. Appl., **5**, Addison-Wesley, 1978.

[9] Ja. G. Sinai, Markov partitions and C-diffeomorphisms, Functional Anal. Appl., **2** (1968), 61–82.

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