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On the Length of Modules over Artinian Local Rings

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Introduction

Let R be an Artinian local ring with the maximal ideal $\mathfrak{m}$. Then it is classical and well known that R is Gorenstein if and only if $l(\mathfrak{a})+l(0:\mathfrak{a})=l(R)$ for each ideal $\mathfrak{a}$ of R, where $0:\mathfrak{a}$ denotes the annihilator of $\mathfrak{a}$ and $l(M)$ denotes the length for an Artinian R-module M (cf. [1]). However in general we can say nothing about which is greater $l(\mathfrak{a})+l(0:\mathfrak{a})$ or $l(R)$ and so in this note we shall tackle with the question, introducing a certain invariant $t(M)$ for R-modules M (see Definition 1.1). In [2] the author defined the value $t(\mathfrak{a})=l(R/0:\mathfrak{a})/l(\mathfrak{a})$ for a non-zero ideal $\mathfrak{a}$ of R (for convenience we set $t(0)=1$) and gave the upper bound, the lower bound and other several properties of this value. Furthermore, he defined $t(M)$ for an Artinian R-module M and mentioned that similar results can be obtained, passing to idealization, for $t(M)$. In the present note, we will treat this value $t(M)$ directly.

At first we will prove inequalities $1/r(M)\leqq t(M)\leqq r(M)$ for any non-zero Artinian R-module M, where $r(M)$ denotes the dimension of the socle of M as a vector space over the residue field of R, that is $r(M)=\dim_{R/\mathfrak{m}}(0:\mathfrak{m})_M=l(0:\mathfrak{m})_M$. We will also consider the value $T(M)=\mathrm{Sup}_N\, t(N)$, where N runs over all R-submodules of M. Then from the above inequalities we obviously have $1\leqq T(M)\leqq r(M)$. And it will be proved that $T(M)=r(M)$ if and only if $r(M)=1$ if and only if $t(N)=1$ for each R-submodule N of M, which gives the classical result above mentioned when $M=R$. Next we will give an example which shows the above inequalities are the best possible in some sense. In the rest of the note we will study about when $T(R)=1$ or for what $\mathfrak{a}$, $t(\mathfrak{a})\leqq 1$.

Throughout this note let R denote an Artinian local ring with the maximal ideal $\mathfrak{m}$. An R-module will always means a finitely generated R-module.

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§1. Definitions and main theorems.

DEFINITIONS 1.1

For an R-module M, we define

$$t(M)=t_R(M)=1 \quad \text{if } M=0\,,$$
$$=l(R/0:M)/l(M) \quad \text{if } M\neq 0$$

and $T(M)=T_R(M)=\mathrm{Sup}_N\, t(N)$, where N runs over all R-submodules of M.

We denote the socle of M by $Soc(M)=(0:\mathfrak{m})_M$ and set $r(M)=\dim_{R/\mathfrak{m}} Soc(M)$. Let $\mu(M)$ denote the number of elements in a minimal system of generators for M.

We begin with the following

PROPOSITION 1.2.

Let M be an R-module.

(1) *If* $\mathfrak{m}M=0$, *then* $t(M)\leqq 1$

(2) *If M is cyclic, then* $t(M)=1$.

PROOF. (1) This is obvious.

(2) Recall $R/0: M\cong M$, as M is cyclic.

COROLLARY 1.3.

If $\mathfrak{m}^2=0$, *then* $T(R)=1$.

Now we will prove

THEOREM 1.4.

Let M be a non-zero R-module. Then we have

$$1/r(M)\leqq t(M)\leqq r(M)$$

and hence

$$1\leqq T(M)\leqq r(M)\,.$$

PROOF. First, we have an exact sequence

$$(1) \qquad 0\longrightarrow R/0{:}\, M\longrightarrow \mathrm{Hom}_R(M, M)\,.$$

Let $E=E(R/\mathfrak{m})$ be the injective envelope of the residue field $R/\mathfrak{m}$. Then we have an exact sequence

$$(2) \qquad 0\longrightarrow M\longrightarrow E^{r(M)}$$

by [3], where $E^{r(M)}$ denotes the direct sum of $r(M)$ copies of E. From (2) also we have an exact sequence

$$(3) \qquad 0 \longrightarrow \mathrm{Hom}_R(M, M) \longrightarrow \mathrm{Hom}_R(M, E)^{r(M)} .$$

Hence by (1) and (3) we have $l(R/0: M) \leqq r(M) l(\mathrm{Hom}_R(M, E))$. Since $l(\mathrm{Hom}_R(M, E)) = l(M)$ by [3], we get $t(M) \leqq r(M)$.

On the other hand, we also have the following exact sequence from (2)

$$0 \longrightarrow \mathrm{Hom}_R(R/0: M, M) \longrightarrow \mathrm{Hom}_R(R/0: M, E)^{r(M)} .$$

Note that $\mathrm{Hom}_R(R/0: M, M) \cong M$ and we get $l(M) \leqq r(M) l(R/0: M)$. Thus we have $1/r(M) \leqq t(M)$.

REMARK. See [2] for another elementary proof with no help of Matlis duality.

THEOREM 1.5.
Let M be an R-module. Then the following conditions are equivalent:
(1) $T(M) = r(M)$
(2) $r(M) = 1$
(3) $t(N) = 1$ *for each submodule N of M.*

PROOF. (1)$\Rightarrow$(2): Since the set of values $t(N)$, where N runs over all submodules of M, is finite, there exists a submodule N of M such that $t(N) = r(M)$. Similarly as in the proof of Theorem 1.4, we have

$$R/0: N \hookrightarrow \mathrm{Hom}(N, N) \hookrightarrow \mathrm{Hom}(N, M) \hookrightarrow \mathrm{Hom}(N, E)^{r(M)} .$$

Compairing the length, $t(N) = r(M)$ implies $R/0: N$ is isomorphic to $\mathrm{Hom}(N, E)^{r(M)}$. Since R is local, $r(M)$ must be 1.

(2)$\Rightarrow$(1) and (3): For each non-zero submodule N of M, we have $r(N) = 1$. Hence by Theorem 1.4, $t(N) = 1$ and so $T(M) = 1 = r(M)$.

(3)$\Rightarrow$(2): By the assumption, $t((0: \mathfrak{m})_M) = 1/l((0: \mathfrak{m})_M) = 1$, hence $r(M) = 1$.

If we take $M = R$, we get the classical result. Namely

COROLLARY 1.6.
R is Gorenstein, that is $r(R) = 1$, if and only if $l(R) = l(\mathfrak{a}) + l(0: \mathfrak{a})$ for each ideal $\mathfrak{a}$ of R.

Now we will show that the inequalities in Theorem 1.4 are the best possible in some sense.

PROPOSITION 1.7.
For any integer $r \geqq 2$ and any small real number $e > 0$, there exists

an Artinian local ring R of $r(R)=r$ and $r-e<T(R)<r$.

PROOF. Let K be a field, $X_i^{(k)}$, Y_i $(i=1, \cdots, n, k=1, \cdots, r)$ be indeterminates and

$$R=K[X_i^{(k)}, Y_i \mid 1\leqq i\leqq n, 1\leqq k\leqq r]/I=K[x_i^{(k)}, y_i]$$

where $I=(X_i^{(k)} \mid 1\leqq i\leqq n, 1\leqq k\leqq r)^2+(Y_1, \cdots, Y_n)^2+(X_i^{(k)}Y_j \mid 1\leqq i\neq j\leqq n, 1\leqq k\leqq r)+(X_i^{(k)}Y_i-X_j^{(k)}Y_j \mid 1\leqq i, j\leqq n, 1\leqq k\leqq r)$ and $x_i^{(k)}$, y_i are the images of $X_i^{(k)}$, Y_i, respectively. Then R is an Artinian local ring with the maximal ideal $\mathfrak{m}=(x_i^{(k)}, y_i \mid 1\leqq i\leqq n, 1\leqq k\leqq r)$. We have $\mathfrak{m}^2=(x_1^{(1)}y_1, \cdots, x_1^{(r)}y_1)$ and $\mathfrak{m}^3=0$. Hence $l(R)=rn+n+r+1$. And since $(0:\mathfrak{m})=\mathfrak{m}^2$, $r(R)=l(0:\mathfrak{m})=r$. Now let $\mathfrak{a}=(y_1, \cdots, y_n)$. Then we have $(0:\mathfrak{a})=\mathfrak{a}$ and $\mathfrak{m}\mathfrak{a}=\mathfrak{m}^2$, so $l(\mathfrak{a})=l(0:\mathfrak{a})=n+r$. Therefore $t(\mathfrak{a})=(rn+1)/(r+n)=r-\{(r^2-1)/(r+n)\}$. Thus we obtain a required example taking n large enough.

§2. When $t(\mathfrak{a})\leqq 1$?

In this section we will study when $T(R)=1$ or for what $\mathfrak{a}$, $t(\mathfrak{a})\leqq 1$. Of course if R is Gorenstein, $T(R)=1$ (cf. Cor. 1.6), but the converse is not true in general, see Example 2.2. Hence the research on rings R with $T(R)=1$ may have some interest.

First of all we note

PROPOSITION 2.1.

For an R-module M, if there exists an $m\in M$ such that $(0:M)=(0:m)$, then $t(M)\leqq 1$.

PROOF. $t(M)=l(R/0:M)/l(M)=l(R/0:m)/l(M)=l(Rm)/l(M)\leqq 1$.

EXAMPLE 2.2.

Let K be a field, $X_1, \cdots, X_n$ be indeterminates and $R=K[X_1, \cdots, X_n]/(X_1, \cdots, X_n)^m=K[x_1, \cdots, x_n]$. Then, for any $f\in R$ we have $(0:f)=(x_1, \cdots, x_n)^{m-k}$ for some k. Hence for any ideal $\mathfrak{a}=(f_1, \cdots, f_s)$ of R, $(0:\mathfrak{a})=\bigcap_{i=1}^s(0:f_i)=(0:f_t)$ for some t. So we have $T(R)=1$ by Prop. 2.1 and obviously R is not Gorenstein for $n, m\geqq 2$.

Now let K and $X_1, \cdots, X_n$ be as above, $\mathfrak{M}=(X_1, \cdots, X_n)$ be a maximal ideal of the polynomial ring $K[X_1, \cdots, X_n]$ and $\mathfrak{Q}$ be an $\mathfrak{M}$- primary ideal generated by monomials of X_i's. And let $R=K[X_1, \cdots, X_n]/\mathfrak{Q}=K[x_1, \cdots, x_n]$. Can we say $T(R)=1$? Unfortunately we cannot claim so: even for graded ideals $\mathfrak{a}$ of R, we may have the case $t(\mathfrak{a})>1$ (cf. Example 2.11). But for ideals $\mathfrak{a}$ of R generated by monomials of x_i's, we always have $t(\mathfrak{a})\leqq 1$. Namely

PROPOSITION 2.3.
Let R and $\mathfrak{a}$ be as above. Then we have $t(\mathfrak{a})\leqq 1$.

PROOF. Let $S=K[X_1,\cdots,X_n]/(X_1^{a_1},\cdots,X_n^{a_n})=K[x_1,\cdots,x_n]$ and we can put $R=S/B$ and $\mathfrak{a}=A/B$, where $A=(M_1,\cdots,M_r)$ is an ideal of S generated by monomials M_i's of x_i's and $B=(N_1\cdots,N_s)$ is an ideal of S generated by monomials N_j's of x_i's in A. Let $V=\{(l)=(l_1,\cdots,l_n)\mid l_i\in \boldsymbol{N}\cup\{0\},\ 0\leqq l_i\leqq a_i\}$ and $M_i=x_1^{l_{i1}},\cdots,x_n^{l_{in}}=x^{(l_i)}$, $(l_i)=(l_{i1},\cdots,l_{in})\in V$. Then $l(A)=\#\{(l)\in V\mid x^{(l)}\in A\}=\#[\cup_{i=1}^r\{(l)\in V\mid (l_i)\leqq (l)\}]$, where $(b)\leqq(c)$ means $b_k\leqq c_k$ for $k=1,\cdots,n$. Let $N_j=x^{(m_j)}$ and we have $x^{(l)}M_i\in(N_1,\cdots,N_s)\Leftrightarrow x^{(l)}M_i\in(N_j)$ for some $j\Leftrightarrow(l)+(l_i)\geqq(m_j)$ for some j. Thus we have $l(S/B\colon A)=l(S)-l(B\colon A)=l(S)-l(\cap_{i=1}^r(B\colon M_i))=\#(V)-\#[\cap_{i=1}^r\{(l)\in V\mid (l)+(l_i)\geqq(m_j)$ for some $j\}]=\#[\cup_{i=1}^r\{(l)\in V\mid (l)+(l_i)\ngeqq(m_j)$ for each $j\}]$. On the other hand $l(\mathfrak{a})=l(A)-l(B)=\#[\cup_{i=1}^r\{(l)\in V\mid (l)\geqq(l_i)$ for some i and $(l)\ngeqq(m_j)$ for each $j\}]$. Since $t(\mathfrak{a})=l(R/0\colon\mathfrak{a})/l(\mathfrak{a})=l(S/B\colon A)/l(A/B)$, to prove $t(\mathfrak{a})\leqq 1$ we have only to prove the following

LEMMA 2.4.
For $V_i=\{(l)\in V\mid (l)+(l_i)\ngeqq(m_j)$ for each $j\}\subset V$ we have $\#(V_1\cup\cdots\cup V_r)\leqq\#\{V_1(n_1)\cup\cdots\cup V_r(n_r)\}$, for each n_k such that $(l)+(n_k)\in V$, where $V_i(n_i)=\{(l)+(n_i)\in V\mid (l)\in V_i\}$.

PROOF. We may assume $r=2$ without loss of generality. Looking

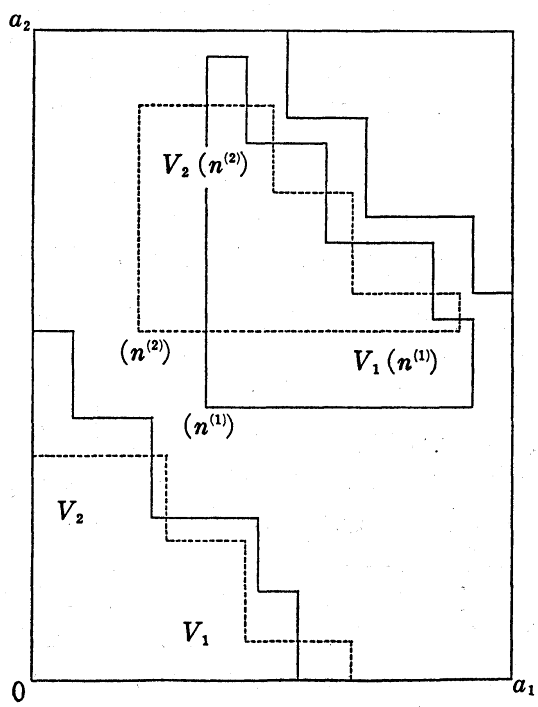

FIGURE 1

at the section by a plane parallel to the X_1X_2-plane, we may also assume $n=2$ then the assertion is now clear (see the above Figure 1).

T. H. Gulliksen [4] has proved that $l(R)\leqq l(M)$ for any faithful R-module M if $r(R)$ is not greater than 3. His result is restated in Theorem 2.7 below. Here we will give another (but essentially the same as his) proof of this fact for the sake of completeness.

First we prove

Lemma 2.5.
Let $0\to M'\to M\to M''\to 0$ be an exact sequence of R-modules such that $t(M')\leqq 1$ and $t(M'')\leqq 1$. If $t_{R/0:M}(0\colon M'/0\colon M)\geqq 1$ or $t_{R/0:M}(0\colon M''/0\colon M)\geqq 1$, then we have $t(M)\leqq 1$.

Proof. Since $t_R(L)=t_{R/\mathfrak{a}}(L)$ if L is an $R/\mathfrak{a}$-module, we may assume that M is faithful. Then, since $(0\colon M')(0\colon M'')=0$, we have $(0\colon M')\subseteqq 0\colon(0\colon M'')$ and $(0\colon M'')\subseteqq 0\colon(0\colon M')$. Hence the assumption that $t(0\colon M')$ or $t(0\colon M'')\geqq 1$ implies that $l(0\colon M')+l(0\colon M'')\leqq l(R)$. Thus $l(M)-l(R)=l(M')+l(M'')-l(R)\geqq(l(M')+l(0\colon M')-l(R))+(l(M'')+l(0\colon M'')-l(R)\geqq 0$, since $t(M')$ and $t(M'')\leqq 1$. Therefore we have $t(M)\leqq 1$.

Corollary 2.6.
Let M be an R-module of $\mu(M)=2$ and $\{m_1, m_2\}$ be a set of minimal generators for M. If $t_{R/0:M}(0\colon m_i/0\colon M)\geqq 1$ for $i=1$ or 2, then $t(M)\leqq 1$.

Proof. $M'=Rm_i$ and $M''=M/Rm_i$ are both cyclic and hence $t(M')$ and $t(M'')=1$ by Prop. 1.2. Thus the assertion follows directly from the above lemma.

Theorem 2.7 (Gulliksen)
Let M be an R-module. If $r(R/0\colon M)\leqq 3$, then we have $t(M)\leqq 1$.

Proof. First, we may assume that M is faithful. Now, if the assertion is not true, we can take a counter example M of minimal length. Then for each non zero submodule M' of M, we have $(0\colon M')\neq 0$ and $(0\colon M/M')\neq 0$. Let $\{m_1, \cdots, m_d\}$ be a set of minimal generators for M and M_i the submodule of M generated by all m_j's except m_i. Then $(0\colon M_i+\mathfrak{m}M)\neq 0$ for each i and $\bigoplus_{i=1}^{d}(0\colon M_i+\mathfrak{m}M)\subseteqq(0\colon \mathfrak{m}M)=(0\colon \mathfrak{m})$. Therefore we have $d=\mu(M)\leqq r(R)\leqq 3$. When $d=1$, obviously $t(M)=1$ by Prop. 1.2. If $d=2$, since $r(R)\leqq 3$, we have $\dim(0\colon M_1+\mathfrak{m}M)$ or $\dim(0\colon M_2+\mathfrak{m}M)=1$, that is $r(0\colon M_1)$ or $r(0\colon M_2)=1$. Thus $t(0\colon M_1)$ or $t(0\colon M_2)=1$ by Theorem 1.4. Hence $t(M)\leqq 1$ by Cor. 2.6. Therefore we must have $d=3$ and in this case $\bigoplus_{i=1}^{3}(0\colon M_i+\mathfrak{m}M)=(0\colon \mathfrak{m}M)=(0\colon \mathfrak{m})$ and we have $(0\colon \mathfrak{m})M=$

$\sum_{i=1}^{3}(0: M_i+\mathfrak{m}M)M=\sum_{i=1}^{3}(0: M_i+\mathfrak{m}M)m_i=(\sum_{i=1}^{3}(0: M_i+\mathfrak{m}M))m=(0: \mathfrak{m})m$, where $m=m_1+m_2+m_3$. Changing the set of generators for M, we may take $m=m_1$. So we get an exact sequence

$$0 \longrightarrow (0: Rm_1+\mathfrak{m}M) \longrightarrow (0: \mathfrak{m}M) \xrightarrow{m_1} (0: \mathfrak{m})M \longrightarrow 0 .$$

Since $(0: M_2+\mathfrak{m}M)\oplus(0: M_3+\mathfrak{m}M)\subseteqq(0: Rm_1+\mathfrak{m}M)$, $\dim(0: Rm_1+\mathfrak{m}M)\geqq 2$ and hence $\dim(0: \mathfrak{m})M=1$. If we have $Soc(M)=(0: \mathfrak{m})M$, then $t(M)=1$ by Theorem 1.4, which is a contradiction and the proof will be completed. So we have only to prove this (this is Lemma 1 of [4]). If $Soc(M)=(0: \mathfrak{m})M\oplus N$ for some $N\neq 0$ then there exists an $r\neq 0\in R$ such that $0\neq rM\subseteqq N\subseteqq Soc(M)$, since $(N: M)\neq 0$. Hence $r\mathfrak{m}M=0$ and so $r\mathfrak{m}=0$. Therefore $0\neq rM\subseteqq(0: \mathfrak{m})M\cap N=0$, which is a contradiction.

Now, we have two cases where $T(R)=1$ that follow from the above theorem.

PROPOSITION 2.8.
If $l(R)\leqq 6$, *then* $T(R)=1$.

PROOF. If $l(0: \mathfrak{m})=1$, then $T(R)=1$ by Theorem 1.4. So we may assume $l(0: \mathfrak{m})\geqq 2$ and hence $l(0: \mathfrak{a})\geqq 2$ for each ideal $\mathfrak{a}$ of R. And if $\mathfrak{m}\mathfrak{a}=0$, $t(\mathfrak{a})\leqq 1$ by Prop. 1.2, so we may have $\mathfrak{m}\mathfrak{a}\neq 0$ and $l(0: \mathfrak{m}\mathfrak{a})\leqq 5$. Therefore $r(R/0: \mathfrak{a})=l(0: \mathfrak{m}\mathfrak{a}/0: \mathfrak{a})\leqq 3$. Thus $t(\mathfrak{a})\leqq 1$ by the above theorem.

PROPOSITION 2.9.
Let $\mathrm{emb}\cdot\dim(R)=\mu(\mathfrak{m})\leqq 3$. *In this case, if* $\mathfrak{m}^2M=0$, *then* $t(M)\leqq 1$. *Therefore if* $\mathfrak{m}^3=0$, *then* $T(R)=1$.

PROOF. We may assume $\mathfrak{m}M\neq 0$ by Prop. 1.2. Then $\mathfrak{m}^2M=0$ implies $(0: \mathfrak{m}M)=\mathfrak{m}$ and $(0: M)\supseteqq\mathfrak{m}^2$. So $(0: \mathfrak{m}M)/(0: M)$ is a homomorphic image of $\mathfrak{m}/\mathfrak{m}^2$. Therefore $r(R/0: M)=l(0: \mathfrak{m}M/0: M)\leqq l(\mathfrak{m}/\mathfrak{m}^2)=\mu(\mathfrak{m})\leqq 3$, which implies $t(M)\leqq 1$ by Theorem 2.7.

In case of $\mu(\mathfrak{m})=4$ we have

PROPOSITION 2.10.
If $\mu(\mathfrak{m})=4$ *and* $\mathfrak{m}^3=0$, *then* $T(R)\leqq 5/4$.

PROOF. Let $\mathfrak{a}$ be an ideal of R such that $t(\mathfrak{a})=T(R)>1$. Then since $\mathfrak{a}\mathfrak{m}^2=0$, we have an exact sequence

$$0 \longrightarrow (0: \mathfrak{a})/\mathfrak{m}^2 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow 0: \mathfrak{a}\mathfrak{m}/0: \mathfrak{a} \longrightarrow 0 .$$

If $\mathfrak{m}^2\subset 0: \mathfrak{a}$, then $r(R/0: \mathfrak{a})=l(0: \mathfrak{a}\mathfrak{m}/0: \mathfrak{a})=l(\mathfrak{m}/\mathfrak{m}^2)-l(0: \mathfrak{a}/\mathfrak{m}^2)\leqq 3$ and $t(\mathfrak{a})\leqq 1$

by Theorem 2.7. So we have $(0:\mathfrak{a})=\mathfrak{m}^2$ and $t(\mathfrak{a})=l(R/\mathfrak{m}^2)/l(\mathfrak{a})=(1+\mu(\mathfrak{m}))/l(\mathfrak{a})=5/l(\mathfrak{a})$. On the other hand, if $\mathfrak{a}\mathfrak{m}\subset\mathfrak{a}\cap(0:\mathfrak{m})$, then there exists an $a\in\mathfrak{a}\cap(0:\mathfrak{m})$ and $a\notin\mathfrak{a}\mathfrak{m}$. Then we may take a as a member of minimal generators for $\mathfrak{a}$ and $\mathfrak{a}=(a)+\mathfrak{a}'$. Then $0:\mathfrak{a}=0:\mathfrak{a}'$ and hence $t(\mathfrak{a})<t(\mathfrak{a}')$. Therefore we have $\mathfrak{a}\mathfrak{m}=\mathfrak{a}\cap(0:\mathfrak{m})=Soc(\mathfrak{a})$. If $l(\mathfrak{a}\mathfrak{m})=1$, $t(\mathfrak{a})=1$ by Theorem 1.4 and hence $l(\mathfrak{a}\mathfrak{m})\geqq 2$. Thus $l(\mathfrak{a})=\mu(\mathfrak{a})+l(\mathfrak{a}\mathfrak{m})\geqq 4$ and $t(\mathfrak{a})\leqq 5/4$.

So if R is an example with the smallest $l(R)$ such that $T(R)>1$, it is hoped that $l(R)=7$. Indeed, we have such an example with $l(R)=7$, $\mu(\mathfrak{m})=4$ and $\mathfrak{m}^3=0$:

EXAMPLE 2.11 (S. Endo)
Let K be a field, X, Y, Z, W be indeterminates and $R=K[X, Y, Z, W]/(X^2, Y^2, Z^2, W^2, XZ, XW, YZ, YW)=K[x, y, z, w]$. Then $\mathfrak{m}=(x, y, z, w)$, $\mathfrak{m}^2=(xy, zw)$, $\mathfrak{m}^3=0$ and $l(R)=7$. Let $\mathfrak{a}=(x+z, y+w)$. Then $\mathfrak{a}\mathfrak{m}=\mathfrak{m}^2=(0:\mathfrak{a})$. So $l(\mathfrak{a})=4$ and $l(0:\mathfrak{a})=2$, hence $t(\mathfrak{a})=5/4$. Of course $T(R)=t(\mathfrak{a})=5/4$ by the above proposition.

§ 3. Problems.

(1) Let $R[t]=R[T]/(T^2)$, where T is an indeterminate.

PROBLEM 3.1.
Find examples of rings R such that $T(R)<T(R[t])$. The author also tried to do in vain. Let us give here a few remarks which may illustrate this problem.

PROPOSITION 3.2.
If $\mathfrak{a}\subseteqq\mathfrak{b}$ *are ideals of* R, *then* $\mathfrak{a}+\mathfrak{b}t$ *is an ideal of* $R[t]$ *and we have* $\mathrm{Min}(t(\mathfrak{a}), t(\mathfrak{b}))\leqq t_{R[t]}(\mathfrak{a}+\mathfrak{b}t)\leqq\mathrm{Max}(t(\mathfrak{a}), t(\mathfrak{b}))$ *and hence* $T(R)\leqq T(R[t])$.

PROOF. It is easily seen that $t_{R[t]}(\mathfrak{a}+\mathfrak{b}t)=(l(R/0:\mathfrak{a})+l(R/0:\mathfrak{b}))/(l(\mathfrak{a})+l(\mathfrak{b}))$, since $0:(\mathfrak{a}+\mathfrak{b}t)_{R[t]}=(0:\mathfrak{b})_R+(0:\mathfrak{a})_R t$-thus the assertion follows.

More generally we have

PROPOSITION 3.3.
Let $R\to S$ *be a local homomorphism of two Artinian local rings such that* S *is a flat* R*-module and* M *be an* R*-module. Then we have* $t_R(M)=t_S(M\otimes_R S)$ *and* $T(R)\leqq T(S)$.

LEMMA 3.4.
Let M *be an* R*-module and* N *be an* S*-module. Then we have* $l_S(M\otimes_R N)\leqq l_R(M)l_S(N/\mathfrak{m}N)$ *where* $\mathfrak{m}$ *is the maximal ideal of* R, *and the equality holds if* N *is* R*-flat.*

PPOOF. We have an exact sequence of S-modules

$$(0 \longrightarrow)\mathfrak{m}M \otimes_R N \longrightarrow M \otimes_R N \longrightarrow (M/\mathfrak{m}M) \otimes_R N \longrightarrow 0$$

where $(0\rightarrow)$ is the case that N is R-flat. Hence we have $l_S(M \otimes_R N) \leqq l_S(\mathfrak{m}M \otimes_R N) + \mu(M) l_S(N/\mathfrak{m}N)$. Hence we get the assertion.

PROOF OF PROP. 3.3. Since S is R-flat, we have an exact sequence $0 \rightarrow (0: M) \otimes_R S \rightarrow S \rightarrow \mathrm{Hom}_R(M, M) \otimes_R S$ and also we have $\mathrm{Hom}_R(M, M) \otimes_R S = \mathrm{Hom}_S(M \otimes_R S, M \otimes_R S)$. Therefore we have $(0: M \otimes_R S)_S \cong (0: M) \otimes_R S$ and $l_S(0: M \otimes_R S) = l_R(0: M) l_S(S/\mathfrak{m}S)$ by the above lemma. Hence $t_S(M \otimes_R S) = (l_S(S) - l_S(0: M \otimes_R S))/l_S(M \otimes_R S) = (l(R) - l(0: M))/l(M) = t_R(M)$.

In case $\mathfrak{m}^2 = 0$, $T(R) = 1$ by Cor. 1.3 and easily we also have $T(R[t]) = 1$. For the ring R given in Example 2.11, this is the case $T(R) = T(R[t])$ too:

PROPOSITION 3.5.

Let $R = K[x, y, z, w]$ be the ring given in Example 2.11. Then we have $T(R) = T(R[t])$.

PROOF. We denote the maximal ideal of R and S by $\mathfrak{m} = (x, y, z, w)$ and $\mathfrak{M} = (x, y, z, w, t)$, respectively. Let $\mathfrak{A}$ be an ideal of S and $\{a_1 + b_1 t, \cdots, a_d + b_d t\}$ be a set of minimal generators for $\mathfrak{A}$. We have to show that $t(\mathfrak{A}) \leqq T(R) = 5/4$. If $d = 1$ or $r(\mathfrak{A}) = 1$, we have nothing to say by Prop. 1.2 and Theorem 1.4. So we may have $d \geqq 2$ and $r(\mathfrak{A}) = 2$, since $r(S) = 2$. In this case $Soc(S) = (0: \mathfrak{M}) \subseteqq \mathfrak{A}$ and $l_S(S/0: \mathfrak{A}) \leqq 12$, since $l(S) = 14$. Hence we may have $l(\mathfrak{A}) \leqq 9$. If $a_i \notin \mathfrak{m}$, $a_i + b_i t$ is a unit of S and hence we may have $a_i \in \mathfrak{m}$ for each i.

Case 1. There exists an $b_i \notin \mathfrak{m}$, say $b_1 \notin \mathfrak{m}$.

In this case we may take $\mathfrak{A} = (a_1 + t, a_2, \cdots, a_d)$. If $a_1 = 0$, then $\mathfrak{A} = (a_2, \cdots, a_d) + St$ and $t(\mathfrak{A}) \leqq T(R)$ by Prop. 3.2. So we may have $a_1 \neq 0$. Let $a_1 \in \mathfrak{m}^2$. If every $a_i \in \mathfrak{m}^2$, then $\mathfrak{M}\mathfrak{A} = \mathfrak{m}t$, and $l(\mathfrak{A}) = d + 6$. On the other hand $\mathfrak{M}\mathfrak{A} \subseteqq (0: \mathfrak{A})$ and $l(0: \mathfrak{A}) \geqq 6$. Therefore $t(\mathfrak{A}) \leqq 8/(d+6) \leqq 1$. If there exists an $a_i \notin \mathfrak{m}^2$, say $a_2 \notin \mathfrak{m}^2$, then we may take $\mathfrak{A} = (zw + t, x, a_3, \cdots, a_d)$. Since $(xt, yt, zt, wt, xy) \subseteqq \mathfrak{M}\mathfrak{A}$, $l(\mathfrak{A}) \geqq d + 7$. Thus $d = 2$, and $(xt, zt, wt) \subseteqq (0: \mathfrak{A})$ and $l(0: \mathfrak{A}) \geqq 5$. Hence $t(\mathfrak{A}) \leqq 1$. In case of $a_1 \notin \mathfrak{m}^2$, we can take $\mathfrak{A} = (x + t, a_2, \cdots, a_d)$. If there exists an $a_i \notin \mathfrak{m}^2$, say $a_2 \notin \mathfrak{m}^2$, then we may assume that $a_2 = y$ or z without loss of generality. In both cases we have $l(\mathfrak{A}) \geqq d + 7$, since (xt, yt, zt, wt, xy) or $(xt, y(x+t), zt, wt) \subseteqq \mathfrak{M}\mathfrak{A}$. Hence $d = 2$ and $l(0: \mathfrak{A}) \geqq 4$, since (zt, wt) or $(xt, zt) \subseteqq (0: \mathfrak{A})$. Therefore $t(\mathfrak{A}) \leqq 10/9 < 5/4$. Thus we may have every $a_i \in \mathfrak{m}^2$. In this case $(xt, y(x+t), zt, wt) \subseteqq \mathfrak{M}\mathfrak{A}$ and $(xt, zt, wt) \subseteqq (0: \mathfrak{A})$ and we have $l(\mathfrak{A}) \geqq d + 6 \geqq 8$ and $l(0: \mathfrak{A}) \geqq 5$. Hence $t(\mathfrak{A}) \leqq 9/8 < 5/4$.

Case 2. Every $b_i \in \mathfrak{m}$.

In this case $\mathfrak{m}^2=(xy, zw)\subseteqq(0:\mathfrak{A})$ and $l(0:\mathfrak{A})\geqq 4$. Hence we may have $l(\mathfrak{A})\leqq 7$. Now let $a_i\notin\mathfrak{m}^2$ for $i\leqq e$ and $a_j\in\mathfrak{m}^2$ for $e<j\leqq d$. If $e\geqq 3$, $l(\mathfrak{A})\geqq 8$. Hence we have $e\leqq 2$. When $e=2$, we may take $a_1=x$ and $a_2=y$ or z without loss of generality. And also we may take $b_1, b_2\in(z, w)$ in first case or $b_1, b_2\in(y, w)$ in second case. In first case $(xt, yt, xy)\subseteqq\mathfrak{M}\mathfrak{A}$ and $l(\mathfrak{A})\geqq d+5$. Hence we have $d=2$ and $l(0:\mathfrak{A})\geqq 6$, since $(zt, wt, xy, zw)\subseteqq(0:\mathfrak{A})$. Thus $t(\mathfrak{A})\leqq 8/7<5/4$. In another case $(xt, zt, xy, zw)\subseteqq\mathfrak{M}\mathfrak{A}$ and $l(\mathfrak{A})\geqq d+6\geqq 8$, that is the case already covered. When $e=1$, we may take $a_1=x$, $a_i\in(zw)$ and $b_j\in(y, z, w)$. Then we have $l(\mathfrak{A})\geqq 6$, since $(xt, xy)\subseteqq\mathfrak{M}\mathfrak{A}$, and $l(0:\mathfrak{A})\geqq 7$, since $(xt, zt, wt, xy, zw)\subseteqq(0:\mathfrak{A})$. Therefore we have $t(\mathfrak{A})\leqq 7/6<5/4$. When $e=0$, $(xy, zw, xt, yt, zt, wt)+(0:(b_1, \cdots, b_d))\subseteqq(0:\mathfrak{A})$. Thus if $d\geqq 3$, we have $t(\mathfrak{A})\leqq 6/5<5/4$. If $d=2$, then we may have (z, w) or $(y, w)\subseteqq(0:(b_1, b_2))$ and $l(S/0:\mathfrak{A})\leqq 4$. Thus $t(\mathfrak{A})\leqq 1$.

(2) From now on, $(R, \mathfrak{m})$ denotes a Noetherian local fing of dimension d with the maximal ideal $\mathfrak{m}$. We also define the invariant $T(R)$ of R by $T(R)=\operatorname{Sup} T(R/\mathfrak{q})$, where $\mathfrak{q}$ runs over all parameter ideals of R.

PROBLEM 3.6.

Explore this invariant $T(R)$. For example

(a) Is $T(R)=\operatorname{Sup}_n T(R[T]/(T^n))$?, or more generally is $T(R)=T(R[\![T]\!])$?

(b) Is $T(R)$ always finite?

(c) Characterize the local ring R of $T(R)=1$.

When $d=\dim R\leqq 3$, Goto-Suzuki [5] have shown that the type of R, that is $\operatorname{Sup} r(R/\mathfrak{q})$ where $\mathfrak{q}$ runs over all parameter ideals, is finite and hence $T(R)$ is finite if $d\leqq 3$.

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