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Schur Indices of Some Finite Chevalley Groups of Rank 2, I

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Introduction

Let $\boldsymbol{F}_q$ be a finite field with q elements, of characteristic p. Let us consider the special orthogonal group $SO_5(q)$ of degree 5 over $\boldsymbol{F}_q$, the conformal symplectic group $CSp_4(q)$ of degree 4 over $\boldsymbol{F}_q$ and the Chevalley group $G_2(q)$ of type (G_2) over $\boldsymbol{F}_q$. If $p=2$, then $CSp_4(q)\simeq \boldsymbol{F}_q^*\times Sp_4(q)$, and the irreducible characters of $Sp_4(2^f)$ were described by H. Enomoto [24]. The character table of $CSp_4(q)$, q odd, was obtained by K. Shinoda in [19] (according to him, the table had also been obtained by S. Reid independently). The characters of $G_2(q)$ were calculated by B. Chang and R. Ree [4] when $p\neq 2, 3$ and by Enomoto [7, 8] when $p=2, 3$ ([8] has not been published yet). The complete table of characters of $SO_5(q)$, q odd, seems to have not been obtained yet. However much information about it can be gotten from G. Lusztig's theory [15] on the classification of the irreducible representations of finite classical groups (see §3 below). As to the rationality-properties of the characters of these groups, R. Gow has proved in [10] that all the irreducible characters of $Sp_4(q)$, q even, have the Schur index 1 over the field $\boldsymbol{Q}$ of rational numbers. Therefore, if $p=2$, all the irreducible characters of $CSp_4(q)$ $(\simeq \boldsymbol{F}_q^*\times Sp_4(q))$ and $SO_5(q)$ $(\simeq Sp_4(q))$ have the Schur index 1 over $\boldsymbol{Q}$. In this paper we shall prove the following.

MAIN THEOREM. *Suppose q is odd. Then all the irreducible characters of $SO_5(q)$, $CSp_4(q)$ and $G_2(q)$ have the Schur index 1 over $\boldsymbol{Q}$.*

It can be shown that all the irreducible characters of $G_2(2^f)$ also have the Schur index 1 over $\boldsymbol{Q}$. This case will be treated in the subsequent paper.

Now let G be a simple adjoint algebraic group defined and split over $\boldsymbol{F}_q$, and $G(q)$ be the group of its $\boldsymbol{F}_q$-points. If the rank of G is 1, then $G(q)=PGL_2(q)$, and if the rank is 2, then $G(q)$ is a homomorphic image

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of $GL_3(q)$, $SO_5(q)$, $CSp_4(q)$, $CO_4^{+,0}(q)(\simeq GL_2(q)\times GL_2(q))$ or $G_2(q)$. It is known that all the irreducible characters of $GL_n(q)$ have the Schur index 1 over $\boldsymbol{Q}$ (A. V. Zelevinsky [23]), and the same is true for $CO_4^{+,0}(q)$. Therefore we get

COROLLARY. *Let G be a simple adjoint algebraic group defined and split over* $\boldsymbol{F}_q$, *of rank* $\leqq 2$. *Suppose* $p\neq 2$. *Then all the irreducible characters of* $G(q)$ *have the Schur index* 1 *over* $\boldsymbol{Q}$.

I wish to thank Professor H. Enomoto for sending me his preprint [8] and kindly showing me the character table of a Sylow 2-subgroup of $G_2(2^f)$, which has been very useful. I also thank Professor G. Lusztig for kindly teaching me the result (7.6) of [13]. Finally I thank Professor K. Shinoda for kindly showing me the manuscript for [19].

§ 1. Schur index of $G_2(q)$.

In this section we prove

THEOREM 1. *All the irreducible characters of* $G_2(q)$, q *odd, have the Schur index* 1 *over* $\boldsymbol{Q}$.

First, we state two lemmas.

LEMMA 1 (Schur's Theorem). *Let H be a finite group, K a field of characteristic* 0 *and* ξ *an ordinary character of H realizable in K. Then, for any irreducible character* χ *of H, the Schur index* $m_K(\chi)$ *of* χ *with respect to K divides the inner product* $\langle\chi, \xi\rangle_H$.

LEMMA 2. *Let H be a finite group, r a prime number dividing* $|H|$ *and g an element of H of order r. Assume that there exists an element h of order* $r-1$ *such that* $hgh^{-1}=g^{\nu}$, *where* ν *is an integer such that* ν *mod r has order* $r-1$ *in* $(Z/rZ)^*$. *Then, for any irreducible character* χ *of H,* $\chi(g)$ *is a rational integer and* $m_{\boldsymbol{Q}}(\chi)|\chi(g)$.

For a proof of Lemma 1, see, for instance, W. Feit [9, 11.4]. The first assertion of Lemma 2 is well-known and easy to prove. The second one is implicitely proved in Gow [10, page 105]. However, since it plays an important role in the arguments below, we prove it here. Let $K=\langle g, h\rangle$ be the subgroup of H generated by g and h. By assumption, we have $K=\langle h\rangle\ltimes\langle g\rangle$ (semidirect product). Let λ be any linear character $\neq 1$ of $\langle g\rangle$. Then it is easy to check that λ^K is a rational-valued irreducible character of K. Since $\lambda^K|\langle h\rangle$ is the character of the regular representation of $\langle h\rangle$ (as can be easily seen), by Lemma 1, we have $m_Q(\lambda^K)=1$ and λ^K is realizable in $\boldsymbol{Q}$. Now let χ be an irreducible character of $\langle g\rangle$, and put $m=m_Q(\chi)$. Then we can write: $\chi|\langle g\rangle=\sum_\lambda a_\lambda\cdot\lambda$, where the sum is taken over all linear characters λ of $\langle g\rangle$, and, for each λ, $a_\lambda=\langle\chi, \lambda^H\rangle_H$, which is divisible by m as we have seen above (cf. Lemma 1). From this we get an expression: $\chi(g)/m=\sum_\lambda (a_\lambda/m)\cdot\lambda(g)$, where the right-hand side is an algebraic integer and the left-hand side is a rational number. Hence $\chi(g)/m$ is a rational integer, and $m|\chi(g)$. This completes the proof.

In order to proceed our arguments we need some preparations. First, we quote from Ree [18] and Enomoto [7] the following (some notations are changed). Being R the root system of type (G_2), the positive roots arranged in increasing order (with respect to some ordering in R) are: $a, b, a+b, 2a+b, 3a+b$ and $3a+2b$, where a and b are the simple roots. For $r\in R$, let X_r denote the corresponding root subgroup of $G_2(q)$ and x_r be an isomorphism of the additive group $\boldsymbol{F}_q^+$ of $\boldsymbol{F}_q$ with X_r induced by a homomorphism ϕ_r: $SL_2(q)\to G_2(q)$, i.e., $x_{-r}(t)=\phi_r\begin{bmatrix}1&0\\t&1\end{bmatrix}$, $x_r(t)=\phi_r\begin{bmatrix}1&t\\0&1\end{bmatrix}$, $t\in\boldsymbol{F}_q$. For two elements g, h of a group, we put $[g, h]=g^{-1}h^{-1}gh$. Then (see Ree [18, (3.10)]):

$$
(1)\qquad
\begin{aligned}
&[x_a(t), x_b(u)]=x_{a+b}(-tu)x_{2a+b}(-t^2u)x_{3a+b}(t^3u)x_{3a+2b}(-2t^3u^2)\ ,\\
&[x_a(t), x_{a+b}(u)]=x_{2a+b}(-2tu)x_{3a+b}(3t^2u)x_{3a+2b}(3tu^2)\ ,\\
&[x_a(t), x_{2a+b}(u)]=x_{3a+b}(3tu)\ ,\\
&[x_b(t), x_{3a+b}(u)]=x_{3a+2b}(tu)\ ,\\
&[x_{a+b}(t), x_{2a+b}(u)]=x_{3a+2b}(3tu)\ ,\\
&[x_r(t), x_s(u)]=1,\ \text{for all the other pairs } r, s\in R,\ r, s>0\ .
\end{aligned}
$$

Let $U=\langle X_r|r\in R, r>0\rangle$ (a Sylow p-subgroup of $G_2(q)$), and put $B=N_{G_2(q)}(U)(N_*(\ )$ denotes the normalizer). Then $B=H\ltimes U$, where H is a subgroup of B isomorphic to $\boldsymbol{F}_q^*\times\boldsymbol{F}_q^*$. One can write: $H=\{h(z_1, z_2, z_3)|z_1z_2z_3=1, z_i\in\boldsymbol{F}_q^*\}$, and the multiplication in H is given by

$$h(z_1, z_2, z_3)h(z_1', z_2', z_3')=h(z_1z_1', z_2z_2', z_3z_3')\ .$$

For a positive root r, put $\omega_r=x_r(1)x_{-r}(-1)x_r(1)$. Then the action of H, ω_a and ω_b on U is given by (see Enomoto [7]; for two elements x, h of a finite group, we put $x^h=hxh^{-1}$):

(2)

x	$x^{h(z_1,z_2,z_3)}$	x^{ω_a}	x^{ω_b}
$x_a(t)$	$x_a(tz_2)$	$x_{-a}(-t)$	$x_{a+b}(t)$
$x_b(t)$	$x_b(tz_1z_2^{-1})$	$x_{3a+b}(t)$	$x_{-b}(-t)$
$x_{a+b}(t)$	$x_{a+b}(tz_1)$	$x_{2a+b}(t)$	$x_a(-t)$
$x_{2a+b}(t)$	$x_{2a+b}(tz_3^{-1})$	$x_{a+b}(-t)$	$x_{2a+b}(t)$
$x_{3a+b}(t)$	$x_{3a+b}(tz_2z_3^{-1})$	$x_b(-t)$	$x_{3a+2b}(t)$
$x_{3a+2b}(t)$	$x_{3a+2b}(tz_1z_3^{-1})$	$x_{3a+2b}(t)$	$x_{3a+b}(-t)$

Now every element $u \in U$ can be written uniquely as $u=\prod_{r>0}x_r(c_r)$, where the product is taken over all the positive roots r according as increasing order and the c_r are some elements of $\boldsymbol{F}_q$. We recall that $u \in G_2(q)$ is a "regular unipotent element" if u is conjugate to some $\prod_{r>0} x_r(c_r)$ with $c_a \neq 0$ and $c_b \neq 0$.

PROPOSITION 1. *Let u be unipotent element of $G_2(q)$. If $p=2$ or 3, we assume that u is non-regular. Then, for any irreducible character χ of $G_2(q)$, $\chi(u)$ is a rational integer and $m_Q(\chi)|\chi(u)$.*

The assertion on the integralness of the value follows directly from the character tables in [4], [7], [8]. However we give here another proof. Let χ be an arbitrary irreducible character of $G_2(q)$ and put $m=m_Q(\chi)$. First, suppose $p\neq 2, 3$. Suppose u is regular. Let G be the Chevalley group of type (G_2) over an algebraic closure of $\boldsymbol{F}_q$. Then G is a simple adjoint algebraic group defined over $\boldsymbol{F}_q$, and $G_2(q)$ can be identified with the group $G(q)$ of $\boldsymbol{F}_q$-points of G. As $p\neq 2, 3$, p is a good prime for G (see T. A. Springer and R. Steinberg [20, E-12, 4.3]), so that by J. A. Green, G. I. Lehrer and G. Lusztig [11, Theorem 3] we have $\chi(u)=0$, 1 or -1, and by Ohmori [17] $m|\chi(u)$. Suppose therefore u is non-regular $\neq 1$. We show that $u^p=1$ and that there exists an element $t \in N_{G_2(q)}(\langle u\rangle)$, of order $p-1$, such that $tut^{-1}=t^{\nu}$ (cf. Lemma 2). In view of Chang [3, (3.2), (3.9)], we may assume that u is one of the following elements: $x_b(1)$, $x_{a+b}(1)$, $x_b(1)x_{2a+b}(1)$, $x_b(1)x_{2a+b}(\lambda)$, $x_b(1)x_{3a+b}(\mu)$ and $x_b(1)x_{2a+b}(-1)x_{3a+b}(\zeta)$, where λ, μ and ζ are some elements of $\boldsymbol{F}_q$. First, we prove that $u^p=1$. For the first four elements, the assertion is clear from the commutator relations (1). Suppose $u=x_b(1)x_{3a+b}(\mu)$. Then, using (1), we can check by induction on i that

$$u^i=x_b(i)x_{3a+b}(i\mu)x_{3a+2b}\left(-\frac{i(i-1)}{2}\mu\right), \qquad 1\leqq i\leqq p\ ;$$

hence $u^p=1$. Suppose $u=x_b(1)x_{2a+b}(-1)x_{3a+b}(\zeta)$. Then, noting that

$x_{2a+b}(c_{2a+b})$ commutes with $x_b(c_b)$ and $x_{3a+b}(c_{3a+b})$, we have:

$$\begin{aligned} u^i &= x_{2a+b}(-1)^i(x_b(1)x_{3a+b}(\zeta))^i \\ &= x_{2a+b}(-i)x_b(i)x_{3a+b}(i\zeta)x_{3a+2b}\left(-\frac{i(i-1)}{2}\zeta\right) \\ &= x_b(i)x_{2a+b}(-i)x_{3a+b}(i\zeta)x_{3a+2b}\left(-\frac{i(i-1)}{2}\zeta\right), \qquad 1 \leqq i \leqq p ; \end{aligned}$$

hence $u^p=1$. Next we find t. In view of (2), if $u=x_b(1)$, $x_{a+b}(1)$, $x_b(1)x_{2a+b}(1)$ or $x_b(1)x_{2a+b}(\lambda)$, we can take: $t=h(\nu, 1, \nu^{-1})$. If $u=x_b(1)x_{3a+b}(\mu)$ or $x_b(1)x_{2a+b}(-1)x_{3a+b}(\zeta)$, we can take: $t=x_b((1-\nu)/2)h(\nu, 1, \nu^{-1})$. In fact, using (2), we have by induction on i that

$$t^i = x_b\left(\frac{1-\nu^i}{2}\right)h(\nu^i, 1, \nu^{-i}) , \qquad 1 \leqq i \leqq p-1 ,$$

hence $t^{p-1}=1$, and by (1), (2), we have $tut^{-1}=u^\nu$. Therefore, by Lemma 2, we conclude that $\chi(u)$ is a rational integer and $m \mid \chi(u)$.

Secondly, suppose $p=3$. Let u be any non-regular unipotent element $\neq 1$. Then, using the results of Enomoto [6], we easily see that $u^3=1$ and find an element $t \in N_{G_2(q)}(\langle u \rangle)$, of order 2, such that $tut^{-1}=u^2$. Hence the assertion follows from Lemma 2.

Finally, suppose $p=2$. Let u be any non-regular unipotent element $\neq 1$. By Enomoto [6], u is conjugate to one of the elements: $x_3=x_{3a+2b}(1)$, $x_4=x_{2a+b}(1)$, $x_5=x_{a+b}(1)x_{2a+b}(1)$, $x_6=x_{a+b}(1)x_{2a+b}(1)x_{3a+b}(\eta)$ and $x_7=x_b(1)x_{2a+b}(1)x_{3a+b}(\zeta)$, where η and ζ are some elements of $\boldsymbol{F}_q$. If $u=x_3$ or x_4, then $u^2=1$ and the assertion is clear. Suppose $u=x_5$. Then, by (1), we have:

$$u^2 = [x_{a+b}(1), x_{2a+b}(1)] = x_{3a+2b}(1) = x_3 ;$$

hence $u^4=1$. Put $t=x_{a+b}(1)$. Then, by (1), we have $t^2=1$ and $tut^{-1}=u^3$. Hence $D=\langle t, u\rangle$ is isomorphic to the dihedral group of order 8. D has five irreducible characters: four rational-valued linear characters and one character ϕ of degree 2 with $\phi(u)=0$. Hence $\chi(u)=(\chi|D)(u)$ is a rational integer. Moreover we see easily that, for any linear character λ of $\langle u\rangle$, λ^D is realizable in $\boldsymbol{Q}$. Therefore, by a method similar to the proof of Lemma 2, we can prove that m divides $\chi(u)$. Suppose $u=x_6$. Put $t=x_{a+b}(1)$. Then, using (1), we see that $u^2=x_3$, $u^4=1$, $t^2=1$ and $tut^{-1}=u^3$. Hence by the same argument as above, we conclude that $\chi(u)$ is a rational integer divisible by m. Suppose finally $u=x_7$. Then $u^2=x_{3a+2b}(\zeta)$ and $u^4=1$. Put $t=x_b(1)$. Then, by (1), we have $t^2=1$ and $tut^{-1}=u^3$. Hence the assertion follows as in the case when $u=x_5$ or x_6. This completes the proof of

Proposition 1.

We can now prove Theorem 1. Assume that $p\neq 2$. For the notation of the characters of $G_2(q)$, we follow Chang and Ree [4] and Enomoto [7].

LEMMA 3 (Chang and Ree [4] and Enomoto [7]). *If $p\neq 2, 3$ (resp. $p=3$), then except for two characters X_{19} and $\bar{X}_{19}$ (resp. $\theta_{12}(1)$ and $\theta_{12}(-1)$) all the other irreducible characters are real and hence have the Schur indices at most two. X_{19} and $\bar{X}_{19}$ (resp. $\theta_{12}(1)$ and $\theta_{12}(-1)$) are complex conjugate of each other and $\boldsymbol{Q}(X_{19})=\boldsymbol{Q}(\bar{X}_{19})=\boldsymbol{Q}(\zeta_3)$ (resp. $\boldsymbol{Q}(\theta_{12}(1))=\boldsymbol{Q}(\theta_{12}(-1))=\boldsymbol{Q}(\zeta_3)$). Hence $m_{\boldsymbol{Q}}(X_{19})(=m_{\boldsymbol{Q}}(\bar{X}_{19}))$ (resp. $m_{\boldsymbol{Q}}(\theta_{12}(\pm 1))$) divides 6. Here, for an irreducible character ξ of a finite group and a field K of characteristic 0, $K(\xi)$ is the field generated over K by the values of ξ, and ζ_3 is a primitive cubic root of 1.*

If $p=3$, the assertion on the values can be checked immediately from the character table (see Enomoto [7]). Suppose therefore $p\neq 2, 3$, where the assertion is not so immediate from Chang and Ree [4]. In the list of the characters of $G_2(q)$ in page 412 of [4], reading from the top, characters $X_1, X_2, \cdots, X_{12}$ are $\boldsymbol{Q}$-linear combinations of generalized characters $X_\alpha(\pi_\alpha)$, $\alpha=1, 2, a, b, 3, 6$ (see [4, pages 399-402]). Therefore, to see that $X_1, X_2, \cdots, X_{12}$ are real, it suffices to check that the $X_\alpha(\pi_\alpha)$ are real. In view of the table of the values of $X_\alpha(\pi_\alpha)$ in [4, pages 409-10], for doing so, it suffices to check that the functions $\hat{\pi}_\alpha$, $\alpha=1, 2, a, b, 3, 6$, are real. By the definition, for each α, the function $\hat{\pi}_\alpha$ on H_α is defined by

$$\hat{\pi}_\alpha(h)=\frac{1}{|C_{W_\alpha}(h)|}\sum_{w\in W_\alpha} w\pi_\alpha(h)\ , \qquad h\in H_\alpha\ ,$$

(see [4, pape 396]). We note that each W_α contains the element w_2, where $h(z_1, z_2, z_3)^{w_2}=h(z_1^{-1}, z_2^{-1}, z_3^{-1})$ (see [loc. cit.]). Therefore, for each α, we have:

$$\begin{aligned}\hat{\pi}_\alpha(h)&=\frac{1}{2|C_{W_\alpha}(h)|}\Big\{\sum_{w\in W_\alpha} w\pi_\alpha(h)+\sum_{w\in w_2W_\alpha} w\pi_\alpha(h)\Big\}\\&=\frac{1}{2|C_{W_\alpha}(h)|}\Big\{\sum_{w\in W_\alpha} w\pi_\alpha(h)+\sum_{w\in W_\alpha} w_2w\pi_\alpha(h)\Big\}\\&=\frac{1}{2|C_{W_\alpha}(h)|}\Big\{\sum_{w\in W_\alpha} w\pi_\alpha(h)+\sum_{w\in W_\alpha} w\pi_\alpha(h^{-1})\Big\}\ ,\end{aligned}$$

which is clearly real. Next, the characters $X_{13}, X_{14}, \cdots, X_{18}, X_{19}$ and $\bar{X}_{19}$ are $\boldsymbol{Q}$-linear combinations of the $X_\alpha(\pi_\alpha)$ and the four class functions

$Y_i (1 \leq i \leq 4)$ (see [4, page 402]). The functions Y_1 and Y_2 are rational, and Y_3 and Y_4 are complex conjugate of each other and take values in $\boldsymbol{Q}(\zeta_3)$. But, if X is any one of the characters $X_{13}, \cdots, X_{18}$, the coefficient of Y_3 and that of Y_4 coincide with each other, so that X is real. If $X=X_{19}$ or $\bar{X}_{19}$, the coefficient of Y_3 is different from that of Y_4, so that X is not real and $\boldsymbol{Q}(X)=\boldsymbol{Q}(\zeta_3)$. The assertion on the Schur indices follows from the following.

LEMMA 4 (M. Benard and M. M. Schacher [1]). *If χ is an irreducible character of a finite group and K is a field of characteristic* 0, *then $K(\chi)$ contains a primitive $m_K(\chi)$-th root of* 1. *Especially, if χ is real, then $m_Q(\chi) \leq 2$* (The Brauer-Speiser Theorem).

The roots of 1 contained in $\boldsymbol{Q}(\zeta_3)$ are $1, -1, \zeta_3$ and $-\zeta_3$. Therefore, by Lemma 4, $m_Q(\chi)|6$ if $\chi=X_{19}, \bar{X}_{19}, \theta_{12}(1)$ or $\theta_{12}(-1)$, and $m_Q(\chi) \leq 2$ otherwise. This proves Lemma 3.

LEMMA 5 (Chang and Ree [4] and Enomoto [7]). *Let χ be any irreducible character of $G_2(q)$, q odd. Then if $p \neq 3$ (resp. $p=3$), the greatest common divisor of the values of χ at all unipotent elements (resp. all non-regular unipotent elements) of $G_2(q)$ is equal to the p-part of $\chi(1)$. Hence $m_Q(\chi)$ divides a power of p.*

This can be checked directly from [4] and [7] (cf. Ohmori [16, Theorem C]). The assertion on the Schur indices follows from Proposition 1.

By Lemmas 4, 5, except for two characters $\theta_{12}(1)$ and $\theta_{12}(-1)$ in case $p=3$, we find that all the other characters have the Schur index 1 over $\boldsymbol{Q}$. Suppose therefore $p=3$, and let us consider the characters $\theta_{12}(1)$ and $\theta_{12}(-1)$. Let V be the subgroup $\langle x_b(1)\rangle X_a X_{a+b} X_{2a+b} X_{3a+b} X_{3a+2b}$ of U, and let $\varepsilon_\kappa=\varepsilon(\kappa, 1, 1)$, $\kappa=\pm 1$, be two linear characters of V which is defined in [7, page 197]; we see at once that $\boldsymbol{Q}(\varepsilon_\kappa)=\boldsymbol{Q}(\zeta_3)$, $\kappa=\pm 1$, (see [loc. cit.]).

LEMMA 6. *One has $\langle \theta_{12}(\kappa)|V, \varepsilon_\lambda \rangle_V=\delta_{\kappa\lambda}$, $(\kappa, \lambda=\pm 1)$, where $\delta_{\kappa\lambda}$ denotes Kronecker's symbol.*

Assume that Lemma 6 is proved. Then, since ε_κ, $\kappa=\pm 1$, are realizable in $\boldsymbol{Q}(\theta_{12}(\kappa))$, $\kappa=\pm 1$, by Lemma 1, we have $m_Q(\theta_{12}(\kappa))=1$, $\kappa=\pm 1$, and the proof of Theorem 1 is finished.

Let $P=B \cup B\omega_a B$ be the parabolic subgroup of $G_2(q)$ generated by B and ω_a. Then we find from [7, pages 197 and 205] that $\theta_4(\kappa)=(\varepsilon_\kappa)^P$, $\kappa=\pm 1$. Hence it suffices to prove

LEMMA 7. $\langle\theta_{12}(\kappa)|P, \theta_4(\lambda)\rangle_P=\delta_{\kappa\lambda}$, $\kappa, \lambda=\pm 1$ (cf. [7, page 205]).

In order to know the inner products, we have to know the relation between the conjugacy classes of P and those of $G_2(q)$. Since $\theta_4(\kappa)$, $\kappa=\pm 1$, vanish outside of the unipotent elements (see [7, Table III-2, page 221]), in view of [7, Table III-1, page 217], it suffices to check only the classes of P denoted by $A_{65}(t)$ and $A_{66}(t)$ in [loc. cit.]. Set $S=(\boldsymbol{F}_q^*)^2$ and $N=\boldsymbol{F}_q^*-S$. For $t\in\boldsymbol{F}_q^*$, put $u(t)=x_a(1)x_{3a+b}(1)x_{3a+2b}(1-t)$. Then $u(t)$ belongs to the class $A_{65}(t)$ (resp. $A_{66}(t)$) if $t\in S$ (resp. $t\in N$) (see [7, page 217]). We have

LEMMA 8. *If $t=1$, then $u(t)$ belongs to the class A_{32} of $G_2(q)$. If $t\neq 1$ and $1-t\in S$ (resp. $1-t\in N$), then $u(t)$ belongs to the class A_{41} (resp. A_{42}) of $G_2(q)$.*

We find from (2) and (1) that

$$\begin{aligned} u(t)^{\omega_b} &= x_{a+b}(1)x_{3a+2b}(1)x_{3a+b}(t-1) \\ &= x_{a+b}(1)x_{3a+b}(t-1)x_{3a+2b}(1) . \end{aligned}$$

Suppose $t=1$. Then $u(t)^{\omega_b}=x_{a+b}(1)x_{3a+2b}(1)$. Hence, by (2), we have

$$(u(t)^{\omega_b})^{\omega_a}=(x_{a+b}(1)x_{3a+2b}(1))^{\omega_a}=x_{2a+b}(1)x_{3a+2b}(1) ,$$

which belongs to the class A_{32}. Suppose therefore $t\neq 1$. Put $y=x_b(1/(1-t))$. Then, by (1), we have

$$\begin{aligned} (u(t)^{\omega_b})^y &= x_{a+b}(1)x_{3a+b}(t-1)^y x_{3a+2b}(1) \\ &= x_{a+b}(1)x_{3a+2b}\left(\frac{1}{t-1}(1-t)\right)x_{3a+b}(t-1)x_{3a+2b}(1) \\ &= x_{a+b}(1)x_{3a+b}(t-1) . \end{aligned}$$

First, suppose $1-t\in S$. Then $1-t=z^2$ for some $z\in\boldsymbol{F}_q^*$. Put $h=h(1, z^{-1}, z)$. Then, by (2), we have

$$\begin{aligned} ((u(t)^{\omega_b})^y)^h &= (x_{a+b}(1)x_{3a+b}(t-1))^h \\ &= x_{a+b}(1)x_{3a+b}(z^{-2}(t-1)) \\ &= x_{a+b}(1)x_{3a+b}(-1) . \end{aligned}$$

Hence $u(t)$ belongs to the class A_{41}. Next, suppose $1-t\in N$. Let γ be an element of $\boldsymbol{F}_q^*$ of order $q-1$; $\gamma\in N$ and we have $N=\gamma S$. Hence we have $1-t=\gamma w^2$ for some $w\in\boldsymbol{F}_q^*$. Putting $h'=h(1, w^{-1}, w)$, by (2), we have

$$
\begin{aligned}
((u(t)^{w_b})^{v})^{h'} &= (x_{a+b}(1)x_{3a+b}(t-1))^{h'} \\
&= x_{a+b}(1)x_{3a+b}(w^{-2}(t-1)) \\
&= x_{a+b}(1)x_{3a+b}(-\gamma) .
\end{aligned}
$$

Hence $u(t)$ belongs to the class A_{42}. This proves Lemma 5.

Now we define:

$$
\begin{aligned}
i &= |\{t \in S | u(t) \text{ belongs to the class } A_{41}\}| , \\
j &= |\{t \in S | u(t) \text{ belongs to the class } A_{42}\}| , \\
m &= |\{t \in N | u(t) \text{ belongs to the class } A_{41}\}| , \\
n &= |\{t \in N | u(t) \text{ belongs to the class } A_{42}\}| .
\end{aligned}
$$

Since $|S|=|N|=(q-1)/2$, we find from Lemma 5 that $i+m=(q-1)/2-1=(q-3)/2$ and $j+n=(q-1)/2$. Using these equalities, we can now prove Lemma 7. Denote by E (resp. F) a complete set of representatives of the unipotent classes of $G_2(q)$ (resp. P). For two elements g, h of $G_2(q)$, we shall write $g \sim h$ if g is conjugate to h in $G_2(q)$. Then we have:

$$
\begin{aligned}
&\langle \theta_{12}(\kappa)|P, \theta_4(\lambda)\rangle_P \\
&= \frac{1}{|P|} \sum_{g \in P} \theta_{12}(\kappa)(g)\theta_4(\lambda)(g^{-1}) \\
&= \sum_{g \in F} \frac{1}{|Z_P(g)|} \theta_{12}(\kappa)(g)\theta_4(\lambda)(g^{-1}) \\
&= \sum_{g \in E} \theta_{12}(\kappa)(g) \sum_{\substack{h \in F \\ h \sim g}} \frac{1}{|Z_P(h)|} \theta_4(\lambda)(h^{-1}) \\
&= \frac{1}{3} q(q^2-1)^2 \cdot \frac{1}{q^6(q^2-1)(q-1)} \cdot \frac{1}{3} q(q-1)(q^2-1) \\
&\quad - \frac{1}{3} q(q^2-1) \cdot \left\{ \frac{1}{q^6(q^2-1)} \cdot \frac{1}{3} q(q-1)(q^2-1) + \frac{1}{q^5(q-1)} \cdot \left(-\frac{1}{3}q\right)(q^2-1) \right\} \\
&\quad - \frac{1}{3} q(q^2-1) \cdot \left\{ \frac{1}{q^6(q-1)} \cdot \left(-\frac{1}{3}q\right)(q-1) + \frac{1}{q^4(q-1)} \cdot \left(-\frac{1}{3}q\right)(q-1) \right\} \\
&\quad + \frac{1}{3} q \cdot \left\{ \frac{1}{q^6} \cdot \left(-\frac{1}{3}q\right)(q-1) + \frac{1}{q^5} \cdot \frac{1}{3}q + \frac{1}{q^4} \cdot \frac{1}{3}q \right\} \\
&\quad + \frac{1}{3} q(q+1) \cdot \left\{ \frac{1}{2q^4} \cdot \frac{1}{3} q(q+1) + \frac{1}{2q^4} \cdot \left(-\frac{1}{3}q\right)(q-1) + \frac{1}{q^4} \cdot \frac{1}{3} q \right. \\
&\quad \left. + i \cdot \frac{1}{q^4} \cdot \frac{1}{3} q + m \cdot \frac{1}{q^4} \frac{1}{3} q \right\} \\
&\quad - \frac{1}{3} q(q-1) \cdot \left\{ \frac{1}{2q^4} \cdot \left(-\frac{1}{3}q\right)(q-1) + \frac{1}{2q^4} \cdot \left(-\frac{1}{3}q\right)(q-1) \right.
\end{aligned}
$$

$$
\begin{aligned}
&+j\cdot\frac{1}{q^4}\cdot\frac{1}{3}q+n\cdot\frac{1}{q^4}\cdot\frac{1}{3}q\Big\}\\
&+\frac{1}{3}q\cdot\frac{1}{3q^2}\cdot\frac{1}{3}q\\
&+\left(\frac{1}{3}q+q\zeta_3^{\kappa}\right)\cdot\frac{1}{3q^2}\cdot\frac{1}{3}q(1+3\zeta_3^{-\lambda})\\
&+\left(\frac{1}{3}q+q\zeta_3^{-\kappa}\right)\cdot\frac{1}{3q^2}\cdot\frac{1}{3}q(1+3\zeta_3^{\lambda})\\
&=\frac{13}{27}+\frac{1}{27}\{(1+3\zeta_3^{\kappa})(1+3\zeta_3^{-\lambda})+(1+3\zeta_3^{-\kappa})(1+3\zeta_3^{\lambda})\}\\
&=\frac{13}{27}+\frac{1}{27}\{-4+9(\zeta_3^{\kappa-\lambda}+\zeta_3^{\lambda-\kappa})\}\\
&=\delta_{\kappa\lambda}\,,\quad(\kappa,\lambda=\pm1).
\end{aligned}
$$

This completes the proof.

§ 2. Schur index of $CSp_4(q)$.

Let k be an algebraic closure of $\boldsymbol{F}_q$. Let $G=CSp_4$ be the group of all matrices $g\in GL_4(k)$ such that ${}^t gJg=\lambda_g J$ for some $\lambda_g\in k^*$, where $J=\begin{bmatrix}0 & s\\ -s & 0\end{bmatrix}$, $s=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$. Then G is a connected, reductive algebraic group which is defined and split over $\boldsymbol{F}_q$ and has the connected centre k^*1_4. In this section we shall prove

THEOREM 2. *All the irreducible characters of $G(q)=CSp_4(q)$, q odd, have the Schur index* 1 *over* $\boldsymbol{Q}$.

Let M be a connected, reductive algebraic group defined over $\boldsymbol{F}_q$. Assume that the centre of M is connected and p is a good prime for M. Let Γ_M be the Gel'fand-Graev character of $M(q)$ (see, for instance, Green, Lehrer and Lusztig [11]). Then, for an irreducible character χ of $M(q)$, we shall say that χ is regular (resp. semisimple) if $\langle\chi,\Gamma_M\rangle_{M(q)}\neq 0$ (resp. $p\nmid\chi(1)$). By Ohmori [17], if M splits over $\boldsymbol{F}_q$, any regular or semisimple character has the Schur index 1 over $\boldsymbol{Q}$.

Now assume that $p\neq 2$. Then, since G is of type (C_2), p is a good prime for G, and any regular or semisimple character of $G(q)$ has the Schur index 1 over $\boldsymbol{Q}$. Sinoda has shown in [19] that an irreducible character χ of $G(q)$ is regular if and only if $\chi(1)$ is, as a polynomial in q, of degree four. Thus the remaining characters are: $\tau_2(\lambda)$, $\lambda\in\boldsymbol{F}_q^*=\mathrm{Hom}(\boldsymbol{F}_q^*,\boldsymbol{C}^*)$, and $\theta_i(\lambda)$, $\lambda\in\boldsymbol{F}_q$, $1\leqq i\leqq 4$. We have $\tau_2(\lambda)=\theta_0(\lambda)\cdot\tau_2$ and $\theta_i(\lambda)=$

$\theta_0(\lambda)\cdot\theta_i(1)$, $1\leqq i\leqq 4$, where the $\theta_0(\lambda)$ are the linear characters of $G(q)$ (see [19, pages 1399 and 1416]). Hence it suffices to show that τ_2 and the $\theta_i(1)$, $1\leqq i\leqq 4$, have the Schur index 1 over $\boldsymbol{Q}$.

PROPOSITION 2. *Let χ be any irreducible character of $G(q)$. Then, for any unipotent element u of $G(q)$, $\chi(u)$ is a rational integer and $m_Q(\chi)|\chi(u)$.*

When u is regular, the assertion can be proved by the method similar to the proof of Proposition 1, §1. Therefore, we may assume that u is non-regular. Then u is conjugate to one of the following:

$$u_1=\begin{bmatrix}1&&&1\\&1&&\\&&1&\\&&&1\end{bmatrix},\quad u_2=\begin{bmatrix}1&1&&\\&1&1&\\&&1&\\&&&1\end{bmatrix},\quad u_3=\begin{bmatrix}1&&&-\gamma_1\\&1&&\\&&1&\\&&&1\end{bmatrix},$$

where γ_1 is a fixed non-square in $\boldsymbol{F}_q^*$ (see [19, page 1376]). Put

$$t=\begin{bmatrix}1&&&\\&1&&\\&&\nu&\\&&&\nu\end{bmatrix}.$$

Then one checks easily that $u^p=t^{p-1}=1$ and $t^{-1}ut=u^\nu$. Hence, by Lemma 2, $\chi(u)$ is a rational integer and $m_Q(\chi)|\chi(u)$.

LEMMA 9 (Shinoda [19]). *For any irreducible character χ of $G(q)$, there is a unipotent element u such that $|\chi(u)|$ is equal to the p-part of $\chi(1)$. Hence $m_Q(\chi)$ divides a power of p.*

First assertion follows from [19]. The second assertion follows from first one and Proposition 2.

Now put $s=\mathrm{diag}(\nu, -\nu, -\nu, \nu)$. Then $s^{p-1}=1$ and s^i belongs to the class A_0 (resp. B_0) if i is even (resp. odd) (see [19, page 1375]). Therefore we have:

$$\begin{aligned}\langle\tau_2, 1_{\langle s\rangle}\rangle_{\langle s\rangle}&=\frac{1}{p-1}\left\{\frac{p-1}{2}q(q^2+1)+\frac{p-1}{2}(q^2+1)\alpha_0(-1)\right\}\\&=\frac{(q^2+1)(q\pm1)}{2}\not\equiv 0(\mathrm{mod}\ p)\quad(\text{note that}\quad \alpha_0(-1)=\pm1);\end{aligned}$$

$$\langle\theta_1(1), 1_{\langle s\rangle}\rangle_{\langle s\rangle}=\frac{1}{p-1}\left\{\frac{p-1}{2}\cdot\frac{q(q+1)^2}{2}+\frac{p-1}{2}\cdot\frac{(q+1)^2}{2}\right\}$$
$$=\frac{(q+1)^3}{4}\not\equiv 0(\mathrm{mod}\, p);$$
$$\langle\theta_2(1), 1_{\langle s\rangle}\rangle_{\langle s\rangle}=\frac{1}{p-1}\left\{\frac{p-1}{2}\cdot\frac{q(q-1)^2}{2}+\frac{p-1}{2}\cdot\frac{-(q-1)^2}{2}\right\}$$
$$=\frac{(q-1)^3}{4}\not\equiv 0(\mathrm{mod}\, p);$$
$$\langle\theta_3(1), 1_{\langle s\rangle}\rangle_{\langle s\rangle}=\frac{1}{p-1}\left\{\frac{p-1}{2}\cdot\frac{q(q^2+1)}{2}+\frac{p-1}{2}\cdot\frac{q^2+2q-1}{2}\right\}$$
$$=\frac{q^3+q^2+3q-1}{4}\not\equiv 0(\mathrm{mod}\, p)\ ;$$
$$\langle\theta_4(1), 1_{\langle s\rangle}\rangle_{\langle s\rangle}=\frac{p-1}{2}\left\{\frac{p-1}{2}\cdot\frac{q(q^2+1)}{2}+\frac{p-1}{2}\cdot\frac{-q^2+2q+1}{2}\right\}$$
$$=\frac{q^3-q^2+3q+1}{4}\not\equiv 0(\mathrm{mod}\, p)\ .$$

Hence, by Lemma 1, if χ is any one of τ_2 and $\theta_i(1)$, $1\leqq i\leqq 4$, $m_Q(\chi)$ is coprime to p. Hence, by Lemma 9, $m_Q(\chi)=1$. This completes the proof of Theorem 2.

REMARK. As is stated in [19, page 1399], the characters $\theta_i(1)$, $0\leqq i\leqq 5$, are the unipotent characters of $G(q)=CSp_4(q)$, and θ_0, $\theta_{i+8}=\theta_i(1)|Sp_4(q)$, $1\leqq i\leqq 5$, are the unipotent characters of $Sp_4(q)$ determined by B. Srinivasan in [21]. We have

$$1_B^{G(q)}=\theta_0(1)+2\theta_1(1)+\theta_3(1)+\theta_4(1)+\theta_5(1)\ .$$

Hence, by a theorem of C. T. Benson and C. W. Curtis [2], the characters $\theta_i(1)$, $0\leqq i\leqq 5$, are realizable in $\boldsymbol{Q}$. The character $\theta_2(1)$ is the cuspidal unipotent character of $G(q)$. Lusztig has shown that $\theta_2(1)$ can be realized in an l-adic cohomology space of an algebraic variety over $\boldsymbol{F}_q$ (see [13]); as a consequence, he proved that $\theta_2(1)$ is realizable in $\boldsymbol{Q}$ (see [13, (7.6)]). Similarly, the characters θ_i, $9\leqq i\leqq 13$, are realizable in $\boldsymbol{Q}$.

On the other hand, Gow showed in [10] that each θ_i is contained in a certain induced module with multiplicity one, and derived from this that if q is an even power of p, the θ_i have the Schur index 1 over $\boldsymbol{Q}$ (see [10, Theorem 7]). In fact, for example, we have $\langle\theta_2(1), \theta_2^0(1, -1)^{G(q)}\rangle_{G(q)}=1$, where $\theta_2^0(1, -1)$ is an irreducible character of B (see [19, page 1381]; we can prove that $\theta_2^0(1, -1)$ is realizable in $\boldsymbol{Q}$).

§ 3. Schur index of $SO_5(q)$.

Let G be $SO_5=\{g \in SL_5(\bar{F}_q) | {}^t gJg=J\}$, where

$$J=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

In this section we shall prove

THEOREM 3. *All the irreducible characters of $SO_5(q)$, q odd, have the Schur index* 1 *over* $\boldsymbol{Q}$.

In the rest of this section we assume that $p \neq 2$. To get an information about the characters of $G(q)=SO_5(q)$, we apply Lusztig's theory [15] to G.

For a connected, reductive algebraic group M defined over F_q, we denote by M^* its "dual group" (see P. Deligne and G. Lusztig [5, 5.21]); M^* is again a connected, reductive group defined over F_q. We have $G^*=Sp_4$. For the future usage, we adopt here the following matrix realization of Sp_4: $Sp_4=\{g \in SL_4(\bar{F}_q) | gA^tg=A\}$, where

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Let $\mathscr{E}_G$ be the set of pairs $((s), \rho)$ of a semisimple class (s) of $G^*(q)=Sp_4(q)$ and a unipotent character ρ of $H(s)(q)=(Z_{G^*}(s))^*(q)$. For an integer m, we define $m_{p'}$, by $m=m_{p'}p^a$, $(m_{p'}, p)=1$. Then, as a special case of [15], we *get*:

THEOREM 4 (Lusztig; also cf. [14]). *There exists a bijection $((s), \rho) \rightarrow R_{s,\rho}$ of $\mathscr{E}_G$ with the set $G(q)^\wedge$ of irreducible characters of $G(q)$ such that*:

(i) $$\deg R_{s,\rho}=\frac{|G(q)|_{p'}}{|H(s)(q)|_{p'}} \cdot \deg \rho \,;$$

(ii) $R_{1,\rho}=\rho$ *for any unipotent character ρ of $G(q)$*;

(iii) *for each $((s), \rho) \in \mathscr{E}_G$, $R_{s,1}$* (resp. $R_{s,St}$) *is semisimple* (*resp. regular*), *where* 1 (*resp. St*) *denotes the principal* (*resp. Steinberg*) *character* *of* $H(s)(q)$, *and conversely, every semisimple (resp. regular) character* χ *of* $G(q)$ *can be expressed as* $\chi=R_{s,1}$ *(resp.* $\chi=R_{s,St}$*) for some semisimple class* (s).

Now we quote from Srinivasan [21] the following list of semisimple classes of $G^*(q)=Sp_4(q)$:

Notation		Number of classes	Order of centralizer
$A_1=(1_4)$		1	$q^4(q^2-1)(q^4-1)$
$A_1'=(-1_4)$		1	$q^4(q^2-1)(q^4-1)$
$B_1(i)$	$i\in R_1$	$(q^2-1)/4$	q^2+1
$B_2(i)$	$i\in R_2$	$(q-1)^2/4$	q^2-1
$B_3(i, j)$	$i, j\in T_1,\ i\neq j$	$(q-3)(q-5)/8$	$(q-1)^2$
$B_4(i, j)$	$i, j\in T_2,\ i\neq j$	$(q-1)(q-3)/8$	$(q+1)^2$
$B_5(i, j)$	$i\in T_2,\ j\in T_1$	$(q-1)(q-3)/4$	q^2-1
$B_6(i)$	$i\in T_2$	$(q-1)/2$	$q(q+1)(q^2-1)$
$B_8(i)$	$i\in T_1$	$(q-3)/2$	$q(q-1)(q^2-1)$
$C_1(i)$	$i\in T_2$	$(q-1)/2$	$q(q+1)(q^2-1)$
$C_1'(i)$	$i\in T_2$	$(q-1)/2$	$q(q+1)(q^2-1)$
$C_3(i)$	$i\in T_1$	$(q-3)/2$	$q(q-1)(q^2-1)$
$C_3'(i)$	$i\in T_1$	$(q-3)/2$	$q(q-1)(q^2-1)$
D_1		1	$q^2(q^2-1)^2$

$$R_1=\{1, 2, \cdots, 1/4(q^2-1)\}\ ,$$

R_2 is a set of $1/2(q-1)^2$ distinct positive integers i such that θ^i, θ^{-i}, θ^{qi}, θ^{-qi} are all distinct, where θ is an element of $F_{q^2}^*$ of order q^2-1,

$$T_1=\{1, 2, \cdots 1/2(q-3)\} \quad \text{and} \quad T_2=\{1, 2, \cdots, 1/2(q-1)\}\ .$$

First, let us consider the class $A_1=(1_4)$. By Theorem 4, (ii), we find that the characters of $G(q)$ associated with A_1 are precisely the unipotent characters of $G(q)$: $\theta_0=1_G$, θ_9, θ_{10}, θ_{11}, θ_{12} and $\theta_{13}=St_G$, where we borrow the notation of the unipotent characters from Srinivasan's list [21] (note that SO_5 is of type (B_2), Sp_4 is of type (C_2) and $(B_2)=(C_2)$).

Secondly, consider the class $A_1'=(z)$, $z=-1_4$. As is stated in [5, page 157], there is a natural isomorphism

$$\mathrm{Hom}(G(q)/\pi(\tilde{G}(q)), \bar{Q}_l^*)\simeq Z(G^*)(q)=\langle z\rangle=(\simeq Z/2Z)\ ,$$

where $\pi\colon \tilde{G}\to G$ is the simply connected covering of the derived group of G, l is a fixed prime number $\neq p$ and Q_l is an l-adic number field. Let θ_z be the character of $G(q)/\pi(\tilde{G}(q))$ corresponding to z. Then, by [15, (7.5.5) and (7.8.3)], regarding θ_z as a character of $G(q)$, we have

$$\theta'_i := R_{A'_1,\theta_i} = \theta_z \cdot \theta_i \quad (i=0, 9, 10, 11, 12, 13) .$$

Thirdly, consider the classes $B_1(i)$, $i \in R_1$. Fixing any $i \in R_1$, let $B_1(i) = (s)$. Then $H(s)$ is a torus, and hence $H(s)(q)$ has pricisely one unipotent character: $1_{H(s)}$. Therefore the character associated with $B_1(i)$ is $R_{B_1(i),1} := R_{s,1}$. As to the classes $B_2(i)$, $B_3(i, j)$, $B_4(i, j)$ and $B_5(i, j)$, the situation is similar. If (s) is any one of the classes $B_6(i)$, $B_8(i)$, $C_1(i)$, $C'_1(i)$, $C_3(i)$ and $C'_3(i)$, then $H(s)$ is a connected, reductive group of semisimple rank 1, so that $H(s)(q)$ has precisely two unipotent characters: $1_{H(s)}$ and $St_{H(s)}$. Therefore the characters associated with (s) are $R_{s,1}$ and $R_{s,St}$.

Finally, consider the class $D_1 = (s)$, $s = \begin{bmatrix} 1_2 & \\ & -1_2 \end{bmatrix}$. We have $Z_{G^*}(s) = SL_2 \times SL_2$, hence $H(s) = PGL_2 \times PGL_2$. Hence $H(s)(q)$ has four unipotent characters: $1 = 1_{PGL_2} \times 1_{PGL_2}$, $\rho_1 = 1_{PGL_2} \times St_{PGL_2}$, $\rho_2 = St_{PGL_2} \times 1_{PGL_2}$ and $S_t = St_{PGL_2} \times St_{PGL_2}$. Therefore the characters associated with D_1 are: $R_{D_1,1}$, R_{D_1,ρ_1}, R_{D_1,ρ_2} and $R_{D_1,St}$. Thus the characters of $G(q) = SO_5(q)$ are as follows:

Character		Degree	Number
$\theta_0 = 1_{SO_5}$		1	1
θ_9		$q(q+1)^2/2$	1
θ_{10}		$q(q-1)^2/2$	1
θ_{11}		$q(q^2+1)/2$	1
θ_{12}		$q(q^2+1)/2$	1
$\theta_{13} = St_{SO_5}$		q^4	1
$\theta'_0 = \theta_z$		1	1
$\theta'_9 = \theta_z \cdot \theta_9$		$q(q+1)^2/2$	1
$\theta'_{10} = \theta_z \cdot \theta_{10}$		$q(q-1)^2/2$	1
$\theta'_{11} = \theta_z \cdot \theta_{11}$		$q(q^2+1)/2$	1
$\theta'_{12} = \theta_z \cdot \theta_{12}$		$q(q^2+1)/2$	1
$\theta'_{13} = \theta_z \cdot \theta_{13}$		q^4	1
$R_{B_1(i),1}$	$i \in R_1$	$(q^2-1)^2$	$(q^2-1)/4$
$R_{B_2(i),1}$	$i \in R_2$	q^4-1	$(q-1)^2/4$
$R_{B_3(i,j),1}$	$i, j \in T_1, i \neq j$	$(q+1)^2(q^2+1)$	$(q-3)(q-5)/8$
$R_{B_4(i,j),1}$	$i, j \in T_2, i \neq j$	$(q-1)^2(q^2+1)$	$(q-1)(q-3)/8$
$R_{B_5(i,j),1}$	$i \in T_2, j \in T_1$	q^4-1	$(q-1)(q-3)/4$
$R_{B_6(i),1}$	$i \in T_2$	$(q-1)(q^2+1)$	$(q-1)/2$
$R_{B_6(i),St}$	$i \in T_2$	$q(q-1)(q^2+1)$	$(q-1)/2$
$R_{B_8(i),1}$	$i \in T_1$	$(q+1)(q^2+1)$	$(q-3)/2$
$R_{B_8(i),St}$	$i \in T_1$	$q(q+1)(q^2+1)$	$(q-3)/2$
$R_{C_1(i),1}$	$i \in T_2$	$(q-1)(q^2+1)$	$(q-1)/2$
$R_{C_1(i),St}$	$i \in T_2$	$q(q-1)(q^2+1)$	$(q-1)/2$
$R_{C'_1(i),1}$	$i \in T_2$	$(q-1)(q^2+1)$	$(q-1)/2$

Character		Degree	Number
$R_{C'_1(i),St}$	$i\in T_2$	$q(q-1)(q^2+1)$	$(q-1)/2$
$R_{C_3(i),1}$	$i\in T_1$	$(q+1)(q^2+1)$	$(q-3)/2$
$R_{C_3(i),St}$	$i\in T_1$	$q(q+1)(q^2+1)$	$(q-3)/2$
$R_{C'_3(i),1}$	$i\in T_1$	$(q+1)(q^2+1)$	$(q-3)/2$
$R_{C'_3(i),St}$	$i\in T_1$	$q(q+1)(q^2+1)$	$(q-3)/2$
$R_{D_1,1}$		q^2+1	1
R_{D_1,ρ_1}		$q(q^2+1)$	1
R_{D_1,ρ_2}		$q(q^2+1)$	1
$R_{D_1,St}$		$q^2(q^2+1)$	1

Now let us prove Theorem 3. As $G=SO_5$ has the trivial centre and $p(\neq 2)$ is a good prime for G, by Ohmori [17], all the semisimple and regular characters have the Schur index 1 over $\boldsymbol{Q}$. Among the unipotent characters θ_i, $i=0, 9, 10, 11, 12, 13$, the θ_i, $i=0, 9, 11, 12, 13$, are in the principal series (i.e. contained in $1_{B(q)}^{G(q)}$, where B is a Borel subgroup of G defined over $\boldsymbol{F}_q$), so that, by the theorem of Benson and Curtis [2], they are realizable in $\boldsymbol{Q}$. θ_{10} is cuspidal unipotent, so that, by Lusztig [13, (7.6)], it is also realizable in $\boldsymbol{Q}$. Next, since $\theta_z^2=1$, θ_z is realizable in $\boldsymbol{Q}$. Hence each $\theta'_i=\theta_z\cdot\theta_i$ is realizable in $\boldsymbol{Q}$. The remaining characters are R_{D_1,ρ_1} and R_{D_1,ρ_2}. Let T be the diagonal maximal torus of G:

$$T=\{\mathrm{diag}(1,\, x,\, y,\, x^{-1},\, y^{-1})|x,\, y\in \boldsymbol{F}_q^*\}\ .$$

Let γ be an element of $\boldsymbol{F}_q^*$ of order $q-1$. Let θ be the character of $T(q)$ defined by

$$\theta(\mathrm{diag}(1,\, \gamma^i,\, \gamma^j,\, \gamma^{-i},\, \gamma^{-j}))=(-1)^i\ .$$

Let R_T^θ be the Deligne-Lusztig character of $G(q)$ associated with the pair (T, θ) (Deligne and Lusztig [5]), we prove

LEMMA 10. *One has the decomposition*:

$$(\#)\qquad R_T^\theta=R_{D_1,1}+R_{D_1,\rho_1}+R_{D_1,\rho_2}+R_{D_1,St}\ .$$

Assume that (#) is proved. A Borel subgroup B of G over $\boldsymbol{F}_q$ can be chosen so that $B\supset T$. Then we have $R_T^\theta=\mathrm{Ind}_{B(q)}^{G(q)}(\tilde\theta)$, where $\tilde\theta=\theta\circ(B(q)\to T(q))$ ($B(q)\to T(q)$ is the natural map) (see [5, Proposition 8.2]). Hence, as $\theta^2=1$, R_T^θ is realizable in $\boldsymbol{Q}$. Therefore, by Lemma 1, we have $m_{\boldsymbol{Q}}(R_{D_1,\rho_i})=1$, $i=1, 2$.

PROOF OF LEMMA 10. Let $W=N_G(T)/T$ be the Weyl group of G (with respect to T). Then $W=\langle\sigma,\, \tau|\sigma^4=\tau^2=1,\, \tau\sigma\tau=\sigma^3\rangle$ ($\simeq$the dihedral group of order 8), where the action of W on T is given by:

$$\sigma\colon \mathrm{diag}(1, x, y, x^{-1}, y^{-1}) \to \mathrm{diag}(1, y^{-1}, x, y, x^{-1})$$
$$\tau\colon \mathrm{diag}(1, x, y, x^{-1}, y^{-1}) \to \mathrm{diag}(1, y, x, y^{-1}, x^{-1}) .$$

W acts on $T(q)^\wedge = \mathrm{Hom}(T(q), \bar{\boldsymbol{Q}}_l^*)$ by $\eta^w(t) = \eta(t^w)$ ($\eta \in T(q)^\wedge$, $t \in T(q)$). For $\eta \in T(q)^\wedge$, we put $W(\eta) = \{w \in W \mid \eta^w = \eta\}$. Let M be a $G(q)$-module which affords $R_T^\theta = \tilde{\theta}^{G(q)}$. Then, by Yokonuma [22, Théorème 5.7], we have an isomorphism $\mathrm{End}_{G(q)}(M) \simeq \bar{\boldsymbol{Q}}_l[W(\theta)]$. It is easy to check that $W(\theta) = \{e, \sigma^2, \tau\sigma, \tau\sigma^3\}$, an abelian group of order 4. Hence R_T^θ is multiplicity-free and the sum of four irreducible characters. Let T' be the diagonal maximal torus of $G^* = Sp_4$:

$$T' = \{\mathrm{diag}(x, x^{-1}, y, y^{-1}) \mid x, y \in \bar{\boldsymbol{F}}_q^*\} .$$

T' is isomorphic to T^*, and we shall identify T' with T^*. Then there is a natural isomorphism $\alpha\colon T(q)^\wedge \xrightarrow{\sim} T^*(q)$ which commutes with the action of W ([5, (5.2.4)]; cf. also W. Kilmoyer [12, Theorem (2.1)]), where we identify W with $N_{G_*}(T^*)/T^*$. As $\theta^2 = 1$, $\theta \neq 1$ and $|W(\theta)| = 4$, up to W-conjugacy, we may assume that $\alpha(\theta) = s = \begin{bmatrix} 1_2 & 0 \\ 0 & -1_2 \end{bmatrix} \in D_1$. In terms of [5, (5.21.5)], the $G(q)$-conjugacy class of (T, θ) corresponds to the $G^*(q)$-conjugacy class of (T^*, s). Hence $R_T^\theta = R_{T^*}^s$ (see [15, 7.5]). In view of the way of the constrution of the bijection $R\colon \mathscr{E}_G \xrightarrow{\sim} G(q)^\wedge$ in [15, pages 160-1], we find that every irreducible component of $R_{T^*}^s$ is of the form $R_{D_1,\rho}$ for some unipotent character ρ of $H(s)(q)$. But the group $H(s)(q) = PGL_2 \times PGL_2$ has exactly four unipotent characters 1, ρ_1, ρ_2 and St. Therefore we must have

$$R_T^\theta = R_{T^*}^s = R_{D_1,1} + R_{D_1,\rho_1} + R_{D_1,\rho_2} + R_{D_1,St} .$$

This completes the proof of Lemma 10.

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