TOKYO J. MATH.
VOL. 8, NO. 1, 1985

Extended Alexander Matrices of 3-manifolds I

Shinji FUKUHARA and Jinko KANNO

Tsuda College

Introduction

In this paper we study some new invariant for Heegaard splittings of 3-manifolds. Stabilizing them we obtain an invariant of 3-manifolds. There are some similarities between our invariant and Reidemeister-Franz torsion ([10], [7], [9]), Fox-Brody's invariant ([6], [3]) and Turaev's invariant [12]. But the formulation is quite different and it is rather easy to calculate our invariant (an EA-matrix) from a given Heegaard diagram of a 3-manifold.

In one aspect our invariant can be thought as an extension of Birman's [2]. She assigned some matrix invariant to a Heegaard splitting of a 3-manifold M. It is an integer matrix while ours is a matrix over a group ring $ZH_1(M)$. Our invariant also can be thought as an extension of the Alexander matrix of the finitely presented group $\pi_1(M)$ but it has more informations than $\pi_1(M)$. For instance it distinguishes lens spaces up to homeomorphism.

In the forthcoming paper [8] the first author will show the further development of this paper. He will give the necessary condition for a homology lens space obtained from S^3 by surgery on a knot to be a genuine lens space in terms of EA-matrices.

§ 1. Preliminaries.

We work in the PL category. Every submanifold is assumed to be locally flat and homeomorphism means PL homeomorphism. Throughout the paper, a 3-manifold means a closed connected orientable 3-manifold. $\bar{A}$ denotes the closure of A while $\operatorname{int} A$ denotes the interior of A.

We use some fundamental facts concerning with Heegaard splittings of 3-manifolds and free differential calculus of Fox [5]. Let T_g be a 3-dimensional orientable handle body of genus g and $F_g=\partial T_g$. We fix a

Received December 15, 1983
Revised September 27, 1984

2-disk D_0 embedded in F_g and a point $p_0 \in \partial D_0$ and call them a preferred disk and a preferred base point. Further we fix a reflection map $r_0: D_0 \to D_0$ which has as a fixed point set an arc l_0 such that $p_0 \in l_0$. We call r_0 and l_0 a preferred reflection and a preferred arc. Let $\hat{F}_g$ stand for $F_g - \text{int}\, D_0$.

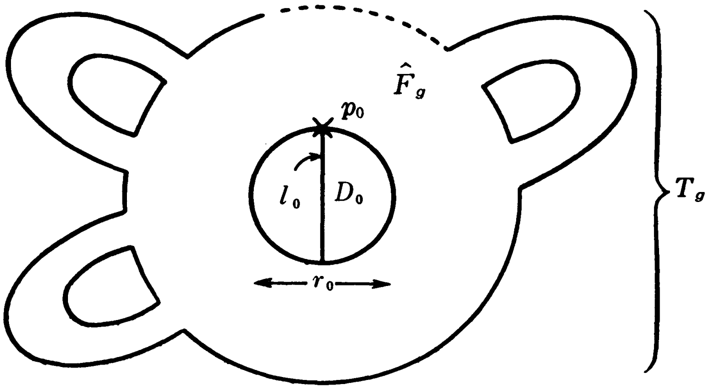


FIGURE 1

DEFINITION 1. Suppose that M is a 3-manifold. We call a pair of embeddings (i, j) a Heegaard splitting (or a H-splitting) of genus g of the 3-manifold M if the following are satisfied:

(i) $i, j: T_g \to M$ are embeddings,

(ii) $i(T_g) \cup j(T_g) = M$,

(iii) $i(T_g) \cap j(T_g) = i(F_g) = j(F_g)$,

(iv) $k = (j | F_g)^{-1} \cdot (i | F_g)$ is an orientation preserving homeomorphism such that $k | D_0 = \text{id}$.

REMARK. The above definition of a Heegaard splitting is slightly different from ordinary one. But it is obvious that every 3-manifold admits a H-splitting in our sense.

DEFINITION 2. H-splittings (i, j) of M and (i', j') of N are called equivalent if there is a homeomorphism $f: M \to N$ which satisfies the following:

(i) $f(i(T_g)) = i'(T_g)$ and $f(j(T_g)) = j'(T_g)$,

(ii) $i'^{-1} \cdot f \cdot i | D_0 = \text{id}_{D_0}$ or r_0 where r_0 is the preferred reflection.

The condition (ii) is not so strict because, by Disk Theorem, an equivalence in usual sense can easily be deformed to an equivalence in our sense.

Now we recall free differential calculus. For a group G, ZG always denotes a group ring of G over an integer ring Z. Let $F = F(x_1, \cdots, x_n)$ be a free group of rank n generated by $x_1, \cdots, x_n$. Then a free derivative is defined as follows (see Fox [5] or Birman [1] for more detail).

DEFINITION 3. A map $\partial/\partial x_j$: $ZF \to ZF$ is called a free derivative with respect to x_j if the following are satisfied:

(i) $\partial x_i/\partial x_j = \delta_{ij}$,

(ii) $\partial(u+v)/\partial x_j = \partial u/\partial x_j + \partial v/\partial x_j$ for any $u, v \in ZF$

and

(iii) $\partial uv/\partial x_j = (\partial u/\partial x_j)v^0 + u(\partial v/\partial x_j)$ for any $u, v \in ZF$

where $v^0 = \sum_{g \in F} n_g \in Z$ for $v = \sum_{g \in F} n_g g$.

By Fox [5] it is known that the free derivative exists and is unique. The following lemma is also proved by Fox ([1], [5]).

LEMMA 1 (Chain rule). *Suppose that u be a word $u(y_1, \cdots, y_n)$ of $F(y_1, \cdots, y_n)$ and v_i ($i=1, \cdots, n$) be words of $F(x_1, \cdots, x_m)$. Let w be a word of $F(x_1, \cdots, x_m)$ defined by the identity*

$$w(x_1, \cdots, x_m) = u(v_1(x_1, \cdots, x_m), \cdots, v_n(x_1, \cdots, x_m)) .$$

Then it follows that $\partial w/\partial x_j = \sum_{i=1}^{n} (\partial u/\partial y_i)_{y_k = v_k(x_1, \cdots, x_m)} (\partial v_i/\partial x_j)$.

§ 2. Extended Alexander matrix.

In what follows we abuse a notation l for an element of a fundamental group represented by a loop l. For a map f: $X \to Y$, f_* denotes a homomorphism $\pi_1(X) \to \pi_1(Y)$ or $H_1(X) \to H_1(Y)$ induced from f. We also abuse f_* for a homomorphism of group rings $Z\pi_1(X) \to Z\pi_1(Y)$ or $ZH_1(X) \to ZH_1(Y)$.

We denote by α_0 an abelianization homomorphism $\pi_1 \to H_1$ or $Z\pi_1 \to ZH_1$. For an element x of a group ring and a homomorphism ϕ we some-

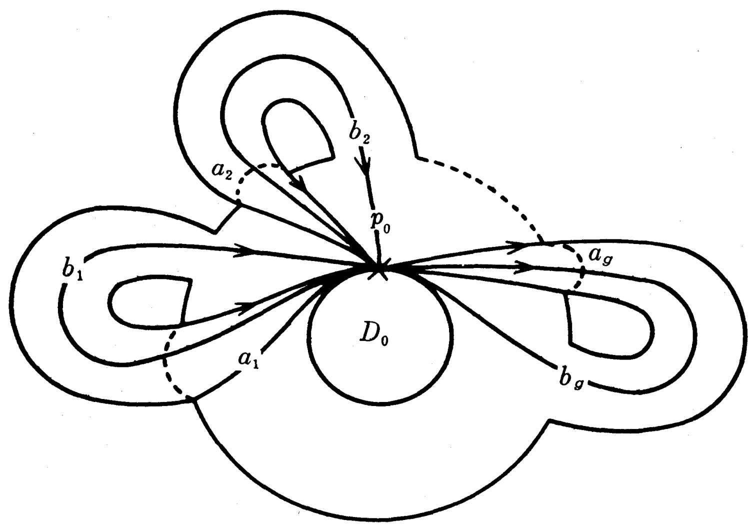


FIGURE 2

times use a notation x^{ϕ} for $\phi(x)$, an image of x by ϕ. Also, for a matrix $A=(a_{ij})$ over a group ring, A^{ϕ} stands for (a_{ij}^{ϕ}).

Let a_i, b_i $(i=1, \cdots, g)$ be simple loops on $\hat{F}_g$ described in Figure 2. Note that they have p_0 as the base point.

DEFINITION 4. A system of loops $\{a_i', b_i'\}_{i=1,\cdots,g}$ on $\hat{F}_g$ is called a meridian-longitude system (or briefly a *m-l* system) of T_g if there is a homeomorphism $f: \hat{F}_g \to \hat{F}_g$ which satisfies:

(i) f extends to $\bar{f}: T_g \to T_g$ such that $\bar{f}|D_0=\mathrm{id}$ or $\bar{f}|D_0=r_0$,

(ii) for loops a_i, b_i $(i=1, \cdots, g)$ as in Figure 2, $f(a_i)=a_i'$ and $f(b_i)=b_i'$.

In particular a *m-l* system $\{a_i, b_i\}$ as in Figure 2 is called a standard *m-l* system of T_g.

Now we define the extended Alexander matrix. Suppose that (i, j) is a H-splitting of a 3-manifold M. Let $h_0=(j|\hat{F}_g)^{-1}\cdot(i|\hat{F}_g): \hat{F}_g \to \hat{F}_g$ and let $h: \hat{F}_g \to T_g$ be the composition of h_0 and an inclusion map $\hat{F}_g \hookrightarrow T_g$.

DEFINITION 5. Let $\{a_i, b_i\}_{i=1,\cdots,g}$ be a *m-l* system and $\{x_1, \cdots, x_g\}$ be a free basis of $\pi_1(T_g)$. Then $h(a_i)$, $h(b_i)$ can be thought as words of $\{x_1, \cdots, x_g\}$. Thus we obtain the following matrix over $ZH_1(M)$:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \left(\dfrac{\partial h(a_i)}{\partial x_j}\right) \\ \left(\dfrac{\partial h(b_i)}{\partial x_j}\right) \end{pmatrix}^{\alpha}$$

where $\alpha=j_*\cdot\alpha_0: Z\pi_1(T_g) \xrightarrow{\alpha_0} ZH_1(T_g) \xrightarrow{j_*} ZH_1(M)$.

We call the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ an extended Alexander matrix (or briefly an *EA*-matrix) of the H-splitting (i, j) with respect to the *m-l* system $\{a_i, b_i\}$ and the free basis $\{x_1, \cdots, x_g\}$.

Now let us see how an *EA*-matrix changes when one chooses other *m-l* system and free basis. Let $\{a_i, b_i\}$, $\{a_i', b_i'\}$ and f be as in Definition 4. Then *EA*-matrices

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \left(\dfrac{\partial h(a_i)}{\partial x_j}\right) \\ \left(\dfrac{\partial h(b_i)}{\partial x_j}\right) \end{pmatrix}^{\alpha} \quad \text{and} \quad \begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} \left(\dfrac{\partial h(a_i')}{\partial x_j}\right) \\ \left(\dfrac{\partial h(b_i')}{\partial x_j}\right) \end{pmatrix}^{\alpha}$$

are related as follows.

LEMMA 2.

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} \left(\dfrac{\partial f(a_i)}{\partial a_j}\right) & \left(\dfrac{\partial f(a_i)}{\partial b_j}\right) \\ \left(\dfrac{\partial f(b_i)}{\partial a_j}\right) & \left(\dfrac{\partial f(b_i)}{\partial b_j}\right) \end{pmatrix}^{h_*\alpha} \begin{pmatrix} A \\ B \end{pmatrix}.$$

PROOF. Since $f(a_i)=a_i'$ are represented by words of a_k, b_k $(k=1, \cdots, g)$, set $a_i'=w_i(a_1, \cdots, a_g, b_1, \cdots, b_g)$. Also set $h(a_i)=u_i(x_1, \cdots, x_g)$ and $h(b_i)=v_i(x_1, \cdots, x_g)$. Then $h(a_i')=w_i(u_1(x_1, \cdots, x_g), \cdots, u_g(x_1, \cdots, x_g), v_1(x_1, \cdots, x_g), \cdots, v_g(x_1, \cdots, x_g))$. Hence by Lemma 1

$$\begin{aligned} \frac{\partial h(a_i')}{\partial x_j} &= \sum_{k=1}^{g} \left(\frac{\partial w_i}{\partial u_k}\right)_{\substack{u_m=u_m(x_1,\cdots,x_g) \\ v_n=v_n(x_1,\cdots,x_g)}} \frac{\partial u_k}{\partial x_j} + \sum_{k=1}^{g} \left(\frac{\partial w_i}{\partial v_k}\right)_{\substack{u_m=u_m(x_1,\cdots,x_g) \\ v_n=v_n(x_1,\cdots,x_g)}} \frac{\partial v_k}{\partial x_j} \\ &= \sum_{k=1}^{g} \left(\frac{\partial f(a_i)}{\partial a_k}\right)^{h_*} \frac{\partial h(a_k)}{\partial x_j} + \sum_{k=1}^{g} \left(\frac{\partial f(a_i)}{\partial b_k}\right)^{h_*} \frac{\partial h(b_k)}{\partial x_j} . \end{aligned}$$

Similarly we obtain

$$\frac{\partial h(b_i')}{\partial x_j} = \sum_{k=1}^{g} \left(\frac{\partial f(b_i)}{\partial a_k}\right)^{h_*} \frac{\partial h(a_k)}{\partial x_j} + \sum_{k=1}^{g} \left(\frac{\partial f(b_i)}{\partial b_k}\right)^{h_*} \frac{\partial h(b_k)}{\partial x_j} .$$

Mapping these identities to $ZH_1(M)$ by α, we obtain Lemma 2.

Next we see how an EA-matrix changes when a free basis of $\pi_1(T_g)$ is replaced.

LEMMA 3. *Suppose that* $\begin{pmatrix} A \\ B \end{pmatrix}$ *and* $\begin{pmatrix} A' \\ B' \end{pmatrix}$ *are EA-matrices with respect to free bases* $\{x_1, \cdots, x_g\}$ *and* $\{x_1', \cdots, x_g'\}$ *of* $\pi_1(T_g)$ *(and with respect to the common m-l system). Then there is a* $g \times g$ *matrix* G *over* $ZH_1(M)$ *such that* $\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} G$ *and* $\det G \in \pm H_1(M)$.

PROOF. Let $\{a_i, b_i\}$ be a m-l system. Then $h(a_i)$ and $h(b_i)$ can be represented by word u_i and v_i of $\{x_1, \cdots, x_g\}$. Further x_j can be represented by words w_j of $\{x_1', \cdots, x_g'\}$. That is $h(a_i)=u_i(w_1(x_1', \cdots, x_g'), \cdots, w_g(x_1', \cdots, x_g'))$ and $h(b_i)=v_i(w_1(x_1', \cdots, x_g'), \cdots, w_g(x_1', \cdots, x_g'))$.

Then by Lemma 1 we obtain

$$\frac{\partial h(a_i)}{\partial x_j'} = \sum_{k=1}^{g} \left(\frac{\partial u_i}{\partial x_k}\right)_{x_n=w_n(x_1',\cdots,x_g')} \frac{\partial w_k}{\partial x_j'} = \sum_{k=1}^{g} \frac{\partial h(a_i)}{\partial x_k} \frac{\partial w_k}{\partial x_j'}$$

and

$$\frac{\partial h(b_i)}{\partial x_j'} = \sum_{k=1}^{g} \left(\frac{\partial v_i}{\partial x_k}\right)_{x_n=w_n(x_1',\cdots,x_g')} \frac{\partial w_k}{\partial x_j'} = \sum_{k=1}^{g} \frac{\partial h(b_i)}{\partial x_k} \frac{\partial w_k}{\partial x_j'} .$$

Mapping these identities to $ZH_1(M)$ by α and setting $G=(\partial w_i/\partial x'_j)^{\alpha}$ we obtain $\binom{A'}{B'}=\binom{A}{B}G$. Since $(\partial w_i/\partial x'_j)^{\alpha_0}$ is invertible, $\det(\partial w_i/\partial x'_j)^{\alpha_0}\in \pm H_1(T_g)$. Hence $\det G\in \pm H_1(M)$. This completes the proof.

§ 3. Equivalence classes.

In this section we study the relation between equivalence classes of H-splittings and EA-matrices.

For a group ring ZG let $\overline{}: ZG\to ZG$ be an involution defined by $\overline{\sum_{g\in G} n_g g}=\sum_{g\in G} n_g g^{-1}$. Suppose that $A=(a_{ij})$ is a matrix over ZG. Then let $\bar{A}$ denote $(\bar{a}_{ij})$ and $*A$ denote ${}^t\bar{A}$.

DEFINITION 6. For $g\times g$ matrices A, B, A' and B' over ZG, $\binom{A}{B}$ and $\binom{A'}{B'}$ are called equivalent if there are $g\times g$ matrices U, W and G_0 over ZG such that $\det U$, $\det G_0\in \pm G$ and the identity

$$\begin{pmatrix} U & 0 \\ W & {}^*U^{-1}\end{pmatrix}\begin{pmatrix} A \\ B\end{pmatrix}G_0=\begin{pmatrix} A' \\ B'\end{pmatrix} \quad \text{or} \quad \begin{pmatrix} U & 0 \\ W & -{}^*U^{-1}\end{pmatrix}\begin{pmatrix} A \\ B\end{pmatrix}G_0=\begin{pmatrix} A' \\ B'\end{pmatrix}$$

holds. We deonte $\binom{A}{B}\sim\binom{A'}{B'}$ when they are equivalent.

One of our main results is the following.

THEOREM 1. *Let $f: M\to N$ be an equivalence from a H-splitting (i, j) of M to a* H*-splitting (i', j') of N. Suppose that $\binom{A}{B}$ and $\binom{A'}{B'}$ are EA-matrices of (i, j) and (i', j') respectively. Then $\binom{A}{B}^{f_*}\sim\binom{A'}{B'}$.*

The following is the key lemma to prove Theorem 1.

LEMMA 4. *Let $f: \hat{F}_g\to\hat{F}_g$ be a homeomorphism which extends to a homeomorphism $\bar{f}: T_g\to T_g$ such that $\bar{f}|D_0=\mathrm{id}$ or $\bar{f}|D_0=r_0$. Let $\beta: Z\pi_1(\hat{F}_g)\to ZH_1(T_g)$ be a composition of an abelianization $\alpha_0: Z\pi_1(\hat{F}_g)\to ZH_1(\hat{F}_g)$ and a homomorphism $ZH_1(\hat{F}_g)\to ZH_1(T_g)$ induced from the inclusion map $\hat{F}_g\hookrightarrow T_g$.*

Consider a m-l system $\{a_i, b_i\}_{i=1,\cdots,g}$ and set

$$\begin{pmatrix} \left(\dfrac{\partial f(a_i)}{\partial a_j}\right) & \left(\dfrac{\partial f(a_i)}{\partial b_j}\right) \\ \left(\dfrac{\partial f(b_i)}{\partial a_j}\right) & \left(\dfrac{\partial f(b_i)}{\partial b_j}\right)\end{pmatrix}^{\beta}=\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22}\end{pmatrix}.$$

Then the following hold:

(i) $U_{12}=0$,

(ii) $U_{22}={}^*U_{11}^{-1}$ *or* $-{}^*U_{11}^{-1}$ *according as* f *is orientation preserving or not.*

PROOF. First we will prove (i). Since f extends to a homeomorphism $\bar{f}: T_g \to T_g$, $f(a_i)$ is a product of conjugates of $a_1, \cdots, a_g$ and their inverses. Let $f(a_i)=\prod_{k=1}^{n} g_k a_{i_k}^{\varepsilon_k} g_k^{-1}$ $(\varepsilon_k=\pm 1)$. Since $(\partial g_k a_{i_k}^{\varepsilon_k} g_k^{-1}/\partial b_j)^\beta=(1-g_k a_{i_k}^{\varepsilon_k} g_k^{-1})^\beta \times (\partial g_k/\partial b_j)^\beta=0$, we obtain

$$(1)\qquad \left(\frac{\partial f(a_i)}{\partial b_j}\right)^\beta=0 .$$

This means (i).

Next we will prove (ii). Throughout the proof G stands for $H_1(T_g)$. Let $\pi: \tilde{F}_g \to \hat{F}_g$ be a covering space associated with $\beta: \pi_1(\hat{F}_g) \xrightarrow{\alpha_0} H_1(\hat{F}_g) \to H_1(T_g)=G$ (where $H_1(\hat{F}_g) \to H_1(T_g)$ denotes the homomorphism induced from the inclusion $\hat{F}_g \hookrightarrow T_g$). Let $\partial\tilde{F}_g$ denote $\pi^{-1}(\partial\hat{F}_g)$.

We choose a preferred base point $\tilde{p}_0 \in \pi^{-1}(p_0)$ and consider liftings $\tilde{a}_i, \tilde{b}_i$ $(i=1, \cdots, g)$ of a_i, b_i which have $\tilde{p}_0$ as starting points. Note that $\tilde{a}_i$ become loops again while $\tilde{b}_i$ become paths which start from $\tilde{p}_0$ and end at points in $\partial\tilde{F}_g$. We abuse the symbols $\tilde{a}_i, \tilde{b}_i$ as the elements of $H_1(\tilde{F}_g, \partial\tilde{F}_g)$ which are represented by $\tilde{a}_i, \tilde{b}_i$. $\tilde{a}_i$ are also regarded as elements of $H_1(\tilde{F}_g)$. Note that $H_1(\tilde{F}_g, \partial\tilde{F}_g)$ and $H_1(\tilde{F}_g)$ can be thought as left ZG-module naturally.

Now let us consider the intersection pairing

$$\langle\ ,\ \rangle: H_1(\tilde{F}_g)\otimes H_1(\tilde{F}_g, \partial\tilde{F}_g) \longrightarrow ZG$$

which is defined by $\langle x, y\rangle=\sum_{g\in G} g(gx, y)$ where

$$(\ ,\ ): H_1(\tilde{F}_g)\otimes H_1(\tilde{F}_g, \partial\tilde{F}_g) \longrightarrow Z$$

denotes the ordinary intersection pairing. Then it follows immediately that the pairing $\langle\ ,\ \rangle$ has the properties:

$$(2)\qquad \langle x+x', y\rangle=\langle x, y\rangle+\langle x', y\rangle , \qquad \langle x, y+y'\rangle=\langle x, y\rangle+\langle x, y'\rangle ,$$
$$\langle gx, y\rangle=g^{-1}\langle x, y\rangle \quad \text{and} \quad \langle x, gy\rangle=g\langle x, y\rangle .$$

It is also obvious that

$$(3)\qquad \langle\tilde{a}_i, \tilde{b}_j\rangle=\delta_{ij}\in ZG \quad \text{and} \quad \langle\tilde{a}_i, \tilde{a}_j\rangle=0 .$$

Now we recall the formula of free differential calculus. Let w be a loop on $\hat{F}_g$ with the base point p_0 then the lifting $\tilde{w}$ of w with starting point $\tilde{p}_0$ can be thought as an element of $H_1(\tilde{F}_g, \partial\tilde{F}_g)$. On the other hand w is represented by a word of $\{a_1, \cdots, a_g, b_1, \cdots, b_g\}$ and we can consider $\partial w/\partial a_i$ and $\partial w/\partial b_i$. It is known that among these there is a following relation (Fox [5]):

$$\tilde{w}=\sum_{i=1}^{g}\left(\frac{\partial w}{\partial a_i}\right)^{\beta}\tilde{a}_i+\sum_{i=1}^{g}\left(\frac{\partial w}{\partial b_i}\right)^{\beta}\tilde{b}_i . \tag{4}$$

Let $\tilde{f}: \tilde{F}_g \to \tilde{F}_g$ be a lifting of f such that $\tilde{f}(\tilde{p}_0)=\tilde{p}_0$. Since $\tilde{f}$ is a homeomorphism and $\tilde{f}$ commutes with covering transformations it follows that

$$(g\tilde{a}_i, \tilde{b}_j)=\pm(\tilde{f}(g\tilde{a}_i), \tilde{f}(\tilde{b}_j))=\pm(g\tilde{f}(\tilde{a}_i), \tilde{f}(\tilde{b}_j)) \quad \text{for any} \quad g\in G ,$$

where the sign depends on whether f preserves orientation or not. From this we obtain

$$\langle\tilde{f}(\tilde{a}_i), \tilde{f}(\tilde{b}_j)\rangle=\pm\langle\tilde{a}_i, \tilde{b}_j\rangle=\pm\delta_{ij} . \tag{5}$$

From (4) and (1) it follows that

$$\tilde{f}(\tilde{a}_i)=\widetilde{f(a_i)}=\sum_{j=1}^{g}\left(\frac{\partial f(a_i)}{\partial a_j}\right)^{\beta}\tilde{a}_j+\sum_{j=1}^{g}\left(\frac{\partial f(a_i)}{\partial b_j}\right)^{\beta}\tilde{b}_j=\sum_{j=1}^{g}\left(\frac{\partial f(a_i)}{\partial a_j}\right)^{\beta}\tilde{a}_j$$

and

$$\tilde{f}(\tilde{b}_i)=\widetilde{f(b_i)}=\sum_{j=1}^{g}\left(\frac{\partial f(b_i)}{\partial a_j}\right)^{\beta}\tilde{a}_j+\sum_{j=1}^{g}\left(\frac{\partial f(b_i)}{\partial b_j}\right)^{\beta}\tilde{b}_j .$$

Hence from (5) and the identities above we have

$$\pm\delta_{ij}=\left\langle\sum_{k=1}^{g}\left(\frac{\partial f(a_i)}{\partial a_k}\right)^{\beta}\tilde{a}_k, \sum_{k=1}^{g}\left(\frac{\partial f(b_j)}{\partial a_k}\right)^{\beta}\tilde{a}_k+\sum_{k=1}^{g}\left(\frac{\partial f(b_j)}{\partial b_k}\right)^{\beta}\tilde{b}_k\right\rangle .$$

Further from (2), (3) and above we have

$$\pm\delta_{ij}=\sum_{k=1}^{g}\overline{\left(\frac{\partial f(a_i)}{\partial a_k}\right)^{\beta}}\left(\frac{\partial f(b_j)}{\partial b_k}\right)^{\beta} .$$

This means $\bar{U}_{11}{}^{t}U_{22}=\pm E_g$ completing the proof.

PROOF OF THEOREM 1. Suppose that $\binom{A}{B}$ and $\binom{A'}{B'}$ are obtained from m-l systems $\{a_i, b_i\}$, $\{a_i', b_i'\}$ and free bases $\{x_1, \cdots, x_g\}$, $\{x_1', \cdots, x_g'\}$. Let $f_0=(i'|\hat{F}_g)^{-1}\cdot(f|\hat{F}_g)\cdot(i|\hat{F}_g)$: $\hat{F}_g\to\hat{F}_g$ and set $\bar{a}_i=f_0(a_i)$ and $\bar{b}_i=f_0(b_i)$ $(i=1, \cdots, g)$. Further set $\bar{x}_i=(j'^{-1}\cdot f\cdot j)_*(x_i)$. Let $\binom{A''}{B''}$ be an EA-matrix of a H-splitting (i', j') with respect to $\{\bar{a}_i, \bar{b}_i\}$ and $\{\bar{x}_1, \cdots, \bar{x}_g\}$.

First we will show that $\begin{pmatrix} A \\ B \end{pmatrix}^{f_*} = \begin{pmatrix} A'' \\ B'' \end{pmatrix}$. To see this let h and h' be maps $h, h'\colon \hat{F}_g \to T_g$ which correspond to h in Definition 5 with respect to the H-splittings (i, j) and (i', j'). Let $h(a_i)$, $h(b_i)$ be represented as

$$h(a_i) = u_i(x_1, \cdots, x_g) \quad \text{and} \quad h(b_i) = v_i(x_1, \cdots, x_g) .$$

Then corresponding to them $h'(\bar{a}_i)$ and $h'(\bar{b}_i)$ are represented as

$$h'(\bar{a}_i) = u_i(\bar{x}_1, \cdots, \bar{x}_g) \quad \text{and} \quad h'(\bar{b}_i) = v_i(\bar{x}_1, \cdots, \bar{x}_g) .$$

Thus noticing that $f_* \cdot \alpha(x_i) = \alpha(\bar{x}_i)$ we obtain

$$\left(\left(\frac{\partial h(a_i)}{\partial x_j}\right)^{\alpha}\right)^{f_*} = \left(\frac{\partial h'(\bar{a}_i)}{\partial \bar{x}_j}\right)^{\alpha}$$

and

$$\left(\left(\frac{\partial h(b_i)}{\partial x_j}\right)^{\alpha}\right)^{f_*} = \left(\frac{\partial h'(\bar{b}_i)}{\partial \bar{x}_j}\right)^{\alpha} .$$

This means $\begin{pmatrix} A \\ B \end{pmatrix}^{f_*} = \begin{pmatrix} A'' \\ B'' \end{pmatrix}$.

Next let us consider an EA-matrix $\begin{pmatrix} A''' \\ B''' \end{pmatrix}$ of (i', j') with respect to $\{\bar{a}_i, \bar{b}_i\}$ and $\{x'_1, \cdots, x'_g\}$. Then by Lemma 3 there is a matrix G such that $\det G \in \pm H_1(M)$ and $\begin{pmatrix} A'' \\ B'' \end{pmatrix} = \begin{pmatrix} A''' \\ B''' \end{pmatrix} G$.

Further let us compare $\begin{pmatrix} A''' \\ B''' \end{pmatrix}$ with $\begin{pmatrix} A' \\ B' \end{pmatrix}$. Note that they are EA-matrices with respect to m-l systems $\{\bar{a}_i, \bar{b}_i\}$ and $\{a'_i, b'_i\}$ and the common free basis $\{x'_1, \cdots, x'_g\}$. Between these m-l systems there is a homeomorphism, say f', such as f in Definition 4. Then by Lemma 2 it follows that, by setting

$$F = \begin{pmatrix} \left(\dfrac{\partial f'(a_i)}{\partial a_j}\right) & \left(\dfrac{\partial f'(a_i)}{\partial b_j}\right) \\ \left(\dfrac{\partial f'(b_i)}{\partial a_j}\right) & \left(\dfrac{\partial f'(b_i)}{\partial b_j}\right) \end{pmatrix}, \qquad \begin{pmatrix} A''' \\ B''' \end{pmatrix} = F^{h'_* \alpha} \begin{pmatrix} A' \\ B' \end{pmatrix} .$$

Since by Lemma 4 F^{β} has a form $\begin{pmatrix} U & 0 \\ W & \pm{}^* U^{-1} \end{pmatrix}$ where $\det U \in \pm H_1(T_g)$, $F^{\beta i'_*}$ has also such a form (Note that i'_* is a homomorphism $ZH_1(T_g) \to ZH_1(M)$ induced from the map $i'\colon T_g \to M$). But, by the definition of α, β, h'_* and i'_*, $i'_* \cdot \beta = \alpha \cdot h'_*$ holds. Thus $F^{h'_* \alpha}$ has the form $\begin{pmatrix} U & 0 \\ W & \pm{}^* U^{-1} \end{pmatrix}$.

Hence we have proved that there is a matrix of the form $\begin{pmatrix} U & 0 \\ W & \pm{}^* U^{-1} \end{pmatrix}$

such that $\det U \in \pm H_1(M)$ and $\begin{pmatrix} A''' \\ B''' \end{pmatrix} = \begin{pmatrix} U & 0 \\ W & \pm{}^*U^{-1} \end{pmatrix} \begin{pmatrix} A' \\ B' \end{pmatrix}$. As the conclusion, we have $\begin{pmatrix} A \\ B \end{pmatrix}^{f_*} \sim \begin{pmatrix} A' \\ B' \end{pmatrix}$ as required.

The similar argument is available to prove the following theorem.

THEOREM 2. *Let* $\begin{pmatrix} A \\ B \end{pmatrix}$ *and* $\begin{pmatrix} A' \\ B' \end{pmatrix}$ *be EA-matrices of a* H-*splitting of* M *with respect to possibly different m-l systems and free bases of* $\pi_1(T_g)$. *Then it follows that* $\begin{pmatrix} A \\ B \end{pmatrix} \sim \begin{pmatrix} A' \\ B' \end{pmatrix}$.

§ 4. Connected sum and EA-matrices.

In this section we will study EA-matrices of a H-splitting obtained by connected sum. We understand the connected sum of H-splittings as follows.

Let (i, j) and (i', j') be H-splittings of M and N of genus m and n respectively. Let D_+, D_- be "half disks" in T_m such that $D_+ \cup D_- = D_0$ and $D_+ \cap D_- = l_0$ as described in Figure 3. Similarly consider "half disks" D'_+, D'_- in a preferred disk D'_0 of T_n. Let B and B' be 3-balls in T_m and T_n such that $B \cap \partial T_m = D_-$ and $B' \cap \partial T_n = D'_+$. Set $T = \overline{(T_m - B)} \cup \overline{(T_n - B')}$ where $\overline{T_m - B}$ and $\overline{T_n - B'}$ are attached along $\overline{\partial B - D_-}$ and $\overline{\partial B' - D'_+}$ such that l_0 and l'_0 are identified naturally.

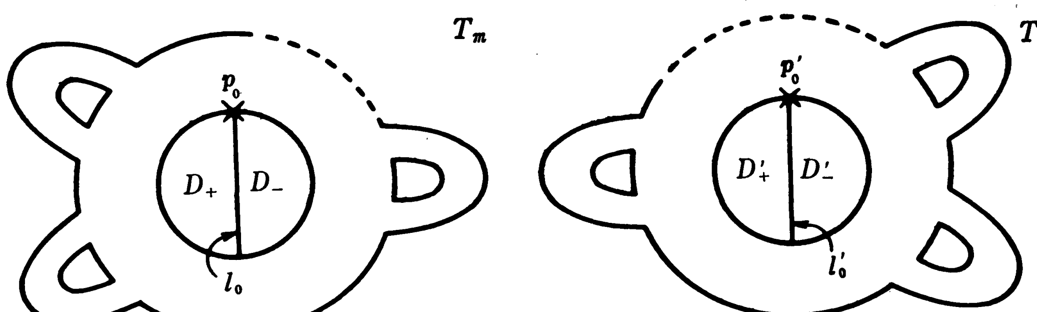


FIGURE 3

Let $\{a_i, b_i\}$, $\{a'_i, b'_i\}$ and $\{a''_i, b''_i\}$ be standard m-l systems of T_m, T_n and T_{m+n} respectively. Then $\{a_i, b_i\} \cup \{a'_i, b'_i\}$ is thought as a system of loops on T and there is a natural homeomorphism $t\colon T_{m+n} \to T$ such that $t(a''_i) = a_i$, $t(b''_i) = b_i$ $(i = 1, \cdots, m)$ and $t(a''_i) = a'_{i-m}$, $t(b''_i) = b'_{i-m}$ $(i = m+1, \cdots, m+n)$.

Further we can assume that, for a preferred disk $D''_0 \subset T_{m+n}$, $t(D''_0) = (D_+ \cup D'_-) \subset T$ holds. Then using t we can easily construct a genus $m+n$ H-splitting on $M \# N$, which we denote by $(i, j) \# (i', j')$.

Let free bases $\{x_1, \cdots, x_m\}$, $\{x'_1, \cdots, x'_n\}$ and $\{x''_1, \cdots, x''_{m+n}\}$ be defined so that x_i, x'_i, x''_i are images of b_i, b'_i, b''_i by inclusions.

Suppose that h, h' and h'' are maps which correspond to h as in Definition 5 with respect to (i, j), (i', j') and $(i, j)\#(i', j')$.

Let $h(a_i)=u_i(x_1, \cdots, x_m)$, $h(b_i)=v_i(x_1, \cdots, x_m)$ and $h'(a_i')=u_i'(x_1', \cdots, x_n')$, $h'(b_i')=v_i'(x_1', \cdots, x_n')$ be representations by words in $\pi_1(T_m)$ and $\pi_1(T_n)$. Then by the construction of connected sum of H-splittings h'' can be represented as

$$\begin{aligned} &h''(a_i'')=u_i(x_1'', \cdots, x_m'') \quad (i=1, \cdots, m)\,, \\ &h''(b_i'')=v_i(x_1'', \cdots, x_m'') \quad (i=1, \cdots, m) \qquad \text{and} \\ &h''(a_i'')=u_i'(x_{m+1}'', \cdots, x_{m+n}'') \quad (i=m+1, \cdots, m+n)\,, \\ &h''(b_i'')=v_i'(x_{m+1}'', \cdots, x_{m+n}'') \quad (i=m+1, \cdots, m+n)\,. \end{aligned}$$

Under these notations it follows immediately that:

LEMMA 5. *Let* $\begin{pmatrix} A \\ B \end{pmatrix}$ *and* $\begin{pmatrix} A' \\ B' \end{pmatrix}$ *be EA-matrices of* (i, j) *and* (i', j') *with respect to m-l systems* $\{a_i, b_i\}$, $\{a_i', b_i'\}$ *and free bases* $\{x_1, \cdots, x_m\}$, $\{x_1', \cdots, x_n'\}$. *Then the EA-matrix* $\begin{pmatrix} A'' \\ B'' \end{pmatrix}$ *of* $(i, j)\#(i', j')$ *with respect to* $\{a_i'', b_i''\}$ *and* $\{x_1'', \cdots, x_{m+n}''\}$ *is represented as*

$$\begin{pmatrix} A'' \\ B'' \end{pmatrix} = \begin{pmatrix} A^f \oplus A'^g \\ B^f \oplus B'^g \end{pmatrix}$$

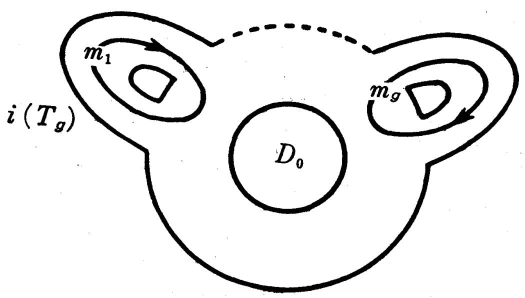
where $f\colon ZH_1(M) \to ZH_1(M\#N)$ *and* $g\colon ZH_1(N) \to ZH_1(M\#N)$ *are canonical homomorphisms.*

The proof is straightforward and we omit it.

§ 5. Stabilization.

First we consider the standard H-splitting of genus g of a 3-sphere S^3 as in Figure 4. We denote this H-splitting by (i_g, j_g).

Let $\{a_i, b_i\}$ be the standard m-l system and $\{x_1, \cdots, x_g\}$ the free basis



(where $m_1, \cdots, m_g$ denote boundaries of meridian disks of $j(T_g)$)

FIGURE 4

of $\pi_1(T_g)$ such that each x_i are images of b_i by the inclusion map. Then we obtain $h(a_i)=x_i$ and $h(b_i)=1$. Thus by easy computation we obtain:

LEMMA 6. *The EA-matrix of* (i_g, j_g) *with respect to* $\{a_i, b_i\}$ *and* $\{x_1, \cdots, x_g\}$ *is* $\begin{pmatrix} E_g \\ O_g \end{pmatrix}$ *where* E_g *is the* $g\times g$ *unit matrix and* O_g *is the* $g\times g$ *zero matrix.*

Now we will compare EA-matrices of H-splittings of two manifolds M and N which are homeomorphic each other. Let (i, j) and (i', j') be H-splittings of M and N. The following is proved by Reidemeister [10], Singer [11] and Craggs [4].

THEOREM (R-S-C). *For some* $m, n \in N$ $(i, j)\#(i_m, j_m)$ *and* $(i', j')\#(i_n, j_n)$ *are equivalent as* H-*splittings.*

Combining Theorem 1, Lemma 5, Lemma 6 and Theorem (R-S-C) we obtain the following:

THEOREM 3. *Suppose that there is a homeomorphism* $f: M\to N$. *Let* $\begin{pmatrix} A \\ B \end{pmatrix}$ *and* $\begin{pmatrix} A' \\ B' \end{pmatrix}$ *be EA-matrices of* H-*splittings of* M *and* N. *Then there are* $m, n \in N$ *such that*

$$\begin{pmatrix} A\oplus E_m \\ B\oplus O_m \end{pmatrix}^{f_*} \sim \begin{pmatrix} A'\oplus E_n \\ B'\oplus O_n \end{pmatrix}.$$

This theorem means that the stable equivalence class of EA-matrices is an invariant of a 3-manifold.

§6. Examples.

Here we present some simple examples of EA-matrices. First we consider a lens space $L(p, q)$ and its standard H-splitting. In Figure 5 m_1 denotes a boundary of a meridian disk of $j(T_g)$. Then m_1 goes q times

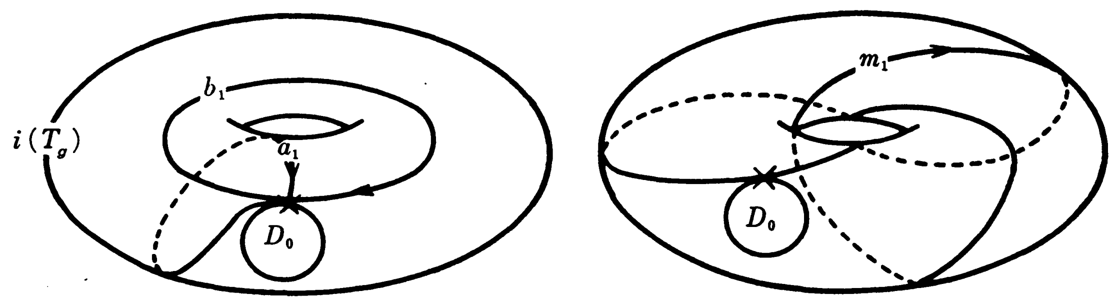


FIGURE 5

around a meridian of $i(T_g)$ while going p times around a longitude of $i(T_g)$. Since the loop a_1 meets m_1 p times and the loop b_1 meets m_1 q times, we obtain $h(a_1)=x_1^p$ and $h(b_1)=x_1^q$ with some basis $\{x_1\}$ of $\pi_1(T_1)$. Let $\alpha(x_1)=t$ then the EA-matrix is presented by $\begin{pmatrix}1+t+\cdots+t^{p-1}\\1+t+\cdots+t^{q-1}\end{pmatrix}$.

REMARK. Comparing stabilized EA-matrices of $L(p, q)$ and $L(p, r)$ we can show famous Reidemeister [10], Franz [7] and Brody's [3] Theorem that they are homeomorphic if and only if $q=\pm r^{\pm1} \pmod p$. But we shall not prove it here because more general version of this theorem will be presented in [8].

Next we consider a 3-manifold obtained from S^3 by 0-framed surgery on a knot k. We denote this manifold by $M(k)$. Then an EA-matrix $\begin{pmatrix}A\\B\end{pmatrix}$ of $M(k)$ has a form

$$A=\begin{pmatrix}0 & 0\\0 & A(k)\end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix}1 & 0\\ * & *\end{pmatrix}$$

where $A(k)$ denotes the Alexander matrix of k. More generally, since $(x_1, \cdots, x_g | h(a_1), \cdots, h(a_g))$ is a presentation of $\pi_1(M)$, A coincides with the Alexander matrix of the finitely presented group $\pi_1(M)$. This is a reason why we call our matrix an EA-matrix.

In [8] the first author will calculate the EA-matrices for 3-manifolds which are obtained from S^3 by Dehn surgery on knots. This and the further consideration will give us necessary conditions that these manifolds are homeomorphic to lens spaces.

References

[1] J. BIRMAN, Braids, link and mapping class groups, Ann. of Math. studies series **82**, Princeton Univ. Press, Princeton, 1974.

[2] J. BIRMAN, On the equivalence of Heegaard splittings of closed, orientable 3-mansfolds, Ann. of Math. studies series **84**, Princeton Univ. Press, Princeton, 1975, 137-164.

[3] E. J. BRODY, The topological classification of the lens spaces, Ann. of Math. (2), **71** (1960), 163-184.

[4] R. CRAGGS, A new proof of the Reidemeister-Singer theorem on stable equivalence of Heegaard splittings of 3-manifolds, Proc. Amer. Math. Soc., **57** (1976), 143-147.

[5] R. H. FOX, Free differential calculus I, Ann. of Math., **57** (1953), 547-560.

[6] R. H. FOX, Free differential calculus V, Ann. of Math., **71** (1960), 408-422.

[7] W. FRANZ, Über die Torsion einer Überdeckung, J. Reine Angew. Math., **173** (1935), 245-254.

[8] S. FUKUHARA, Extended Alexander matrices of 3-manifolds II, to appear.

[9] J. W. MILNOR, Whitehead torsion, Bull. Amer. Math. Soc., **72** (1966), 385-426.

[10] K. REIDEMEISTER, Homotopieringe und Linsenräume, Abh. Math. Sem. Univ. Hamburg., **11** (1935), 102-109.

[11] J. SINGER, Three-dimensional manifolds and their Heegaard diagrams, Trans. Amer. Math. Soc., **35** (1933), 88–111.

[12] V. G. TURAEV, Reidemeister torsion and the Alexander polynomial, Math. USSR-Sb., **30** (1976), 221–237.

Present Address:
DEPARTMENT OF MATHEMATICS
TSUDA COLLEGE
TSUDA-MACHI, KODAIRA, TOKYO 187