TOKYO J. MATH.
VOL. 8, NO. 1, 1985

Formula for the Casimir Operator in Iwasawa Coördinates

Floyd L. WILLIAMS

University of Massachusetts at Amherst and Sophia University

(Communicated by Y. Kawada)

Introduction

Let G be a connected non-compact semisimple Lie group with an Iwasawa decomposition $G=KAN$, where the Lie algebra of K is $\mathfrak{k}$ in the Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of G, the Lie algebra $\mathfrak{a}$ of A is a maximal abelian subspace of $\mathfrak{p}$, and the Lie algebra $\mathfrak{n}$ of N is a sum of root spaces corresponding to a choice of positive *restricted* roots. Let Ω be the Casimir operator of G. We give a formula for Ω in terms of first order differential operators $\delta_z, \delta_H, \delta_x$ defined for left-invariant vector fields $(z, H, x)\in\mathfrak{k}\times\mathfrak{a}\times\mathfrak{n}$. Theorem 1.10 is the main result. Such a formula was first proposed in [1]. The arguments given there, however, are incomplete. Corrections are made in the present paper. In particular our formula reduces to Takahashi's formula [3] when $G=SL(2, \boldsymbol{R})$.

§1. Statement of the result.

Let $\mathfrak{g}$ be a non-compact real semisimple Lie algebra with a Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$, and Cartan involution θ. The Killing form B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. The formula

$$\langle x, y\rangle=-B(x, \theta y) \qquad x, y\in\mathfrak{g} \tag{1.1}$$

defines a real positive definite inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. Let $\mathfrak{a}\subset\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$. For $\alpha\in\mathfrak{a}^*$ (the real dual space of $\mathfrak{a}$) let $\mathfrak{g}_\alpha=\{x\in\mathfrak{g}\mid[H, x]=\alpha(H)x$ for every H in $\mathfrak{a}\}$. α is a *restricted* root (relative to $\mathfrak{a}$) if both α and the root space $\mathfrak{g}_\alpha$ are non-zero. Let $\Sigma\subset\mathfrak{a}^*$ denote the set of restricted roots, let $\Sigma^+\subset\Sigma$ denote a choice of a system of positive restricted roots, let $\mathfrak{n}$ denote the sum of the positive root spaces, and let (for $\alpha\in\Sigma$)

Received November 17, 1983

$$m_\alpha = \dim \mathfrak{g}_\alpha \qquad \rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha \,. \tag{1.2}$$

Then $\mathfrak{g}$ has an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ and a root space decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n} + \theta\mathfrak{n}$, where $\mathfrak{m}$ is the *centralizer* of $\mathfrak{a}$ in $\mathfrak{k}$. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. We regard elements of $\mathfrak{g}$ as left-invariant vector fields on G. Let $K \subset G$ be the Lie subgroup with Lie algebra $\mathfrak{k}$. Then G has an Iwasawa decomposition $G = KAN$ where $A = \exp \mathfrak{a}$, $N = \exp \mathfrak{n}$. The subgroups K, A, N are closed; K is compact when and only when the center of G is finite. Given $(z, H, x) \in \mathfrak{k} \times \mathfrak{a} \times \mathfrak{n}$ we define differential operators δ_z, δ_H, δ_x: $C^\infty(G) \to C^\infty(G)$ on G by

$$\begin{aligned} (\delta_z f)(kan) &= \frac{d}{dt} f(k(\exp tz)an)\big|_{t=0} \\ (\delta_H f)(kan) &= \frac{d}{dt} f(k(\exp tH)an)\big|_{t=0} \\ (\delta_x f)(kan) &= \frac{d}{dt} f(ka(\exp tx)n)\big|_{t=0} \end{aligned} \tag{1.3}$$

where $f \in C^\infty(G)$ and $kan \in KAN$ is the Iwasawa decomposition of an element of G. For $y \in \mathfrak{g}$ arbitrary (a left-invariant operator) we also define $\tilde{y}$ by

$$(\tilde{y} f)(b) = \frac{d}{dt} f((\exp ty)b)\big|_{t=0} \tag{1.4}$$

for $f \in C^\infty(G)$, $b \in G$. For $a = \exp H \in A$, $H \in \mathfrak{a}$, we write $H = \log a$ and we set

$$e^\alpha(kan) = e^{\alpha(\log a)} \tag{1.5}$$

for $\alpha \in \Sigma$, $kan \in KAN$. Thus $e^\alpha \in C^\infty(G)$. Let Ω be the Casimir operator of G. If $\{E_i\}$, $\{F_j\}$ are orthonormal bases of $\mathfrak{k}$, $\mathfrak{p}$, relative to $\langle\ ,\ \rangle$ given in (1.1), then $\Omega = -\sum_i E_i^2 + \sum_j F_j^2 \in U\mathfrak{g}$, the universal enveloping algebra of $\mathfrak{g}$. Let $\tilde{\Omega} = -\sum_i \tilde{E}_i^2 + \sum_j \tilde{F}_j^2$; see (1.4). Then one knows that

$$\Omega = \tilde{\Omega} \quad \text{on the domain} \quad C^\infty(G) \,. \tag{1.6}$$

(1.6) can be proved using the Cartan decomposition $G = K \exp(\mathfrak{p})$ of G. Now choose orthonormal bases $\{H_i\}_{i=1}^r$ of $\mathfrak{a}$, $\{u_i\}_{i=1}^s$ of $\mathfrak{m}$, and $\{x_i\}_{i=1}^t$ of $\mathfrak{n}$. Set

$$z_i = \frac{x_i + \theta x_i}{\sqrt{2}} \in \mathfrak{k} \,, \qquad y_i = \frac{x_i - \theta x_i}{\sqrt{2}} \in \mathfrak{p} \tag{1.7}$$

for $1 \leqq i \leqq t$. Let $\mathfrak{m}^\perp$ denote the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{k}$.

PROPOSITION 1.8. *$\{z_i\}_{i=1}^t$ is an orthonormal basis of $\mathfrak{m}^\perp$. Hence $\{u_i\}_{i=1}^s \cup \{z_i\}_{i=1}^t$ is an orthonormal basis of $\mathfrak{k}$. Also $\{H_i\}_{i=1}^r \cup \{y_i\}_{i=1}^t$ is an orthonormal basis of $\mathfrak{p}$. Hence $\Omega = -\sum_{i=1}^s \tilde{u}_i^2 - \sum_{i=1}^t \tilde{z}_i^2 + \sum_{i=1}^r \tilde{H}_i^2 + \sum_{i=1}^t \tilde{y}_i^2$ by* (1.6). *We make the choice of the orthonormal basis $\{x_i\}_{i=1}^t$ of $\mathfrak{n}$ as follows. Write $\Sigma^+ = \{\alpha_1, \cdots, \alpha_q\}$, and for $1 \leqq j \leqq q$ let $\{x_{i(j)}\}_{i=1}^{m_{\alpha_j}}$ be an orthonormal basis of $\mathfrak{g}_{\alpha_j}$. Then*

$$\{x_i\}_{i=1}^t = \bigcup_{1 \leqq j \leqq q} \{x_{i(j)}\}_{i=1}^{m_{\alpha_j}} \tag{1.9}$$

is an orthonormal basis of $\mathfrak{n}$; $\sum_{j=1}^q m_{\alpha_j} = t$ and $2\rho = \sum_{j=1}^q m_{\alpha_j}\alpha_j$; see (1.2).

Using the expression for Ω in Proposition 1.8 we will derive the following main

THEOREM 1.10. *As above let $r = \dim \mathfrak{a}$ (= the rank of G), $s = \dim \mathfrak{m}$, $t = \dim \mathfrak{n}$, and $q = |\Sigma^+|$, the cardinality of Σ^+. Relative to the inner product defined in* (1.1) *choose orthonormal bases $\{H_i\}_{i=1}^r$ of $\mathfrak{a}$, $\{u_i\}_{i=1}^s$ of $\mathfrak{m}$, and $\{x_i\}_{i=1}^t = \bigcup_{1 \leqq j \leqq q} \{x_{i(j)}\}_{i=1}^{m_{\alpha_j}}$ of $\mathfrak{n}$ (as in* (1.9)). *Let $z_i = (x_i + \theta x_i)/\sqrt{2}$ as in* (1.7). *Then*

$$\begin{aligned}\Omega = &\sum_{i=1}^r (\delta_{H_i}^2 + 2\rho(H_i)\delta_{H_i}) + \sum_{j=1}^q \sum_{i=1}^{m_{\alpha_j}} (2e^{-2\alpha_j}\delta_{x_{i(j)}}^2 - 2\sqrt{2}\, e^{-\alpha_j}\delta_{z_{i(j)}}\delta_{x_{i(j)}}) \\ &+ \sum_{i=1}^t \delta_{z_i}^2 - \sum_{i=1}^t \tilde{z}_i^2 - \sum_{i=1}^s \tilde{u}_i^2 \quad \text{on} \quad C^\infty(G)\,;\end{aligned} \tag{1.11}$$

see (1.2), (1.3), (1.4), and (1.5). *Moreover we have*

$$\left(\sum_{i=1}^t \delta_{z_i}^2 - \sum_{i=1}^t \tilde{z}_i^2 - \sum_{i=1}^s \tilde{u}_i^2\right) f = \left(-\sum_{i=1}^s u_i^2\right) f \quad \text{on } KA \text{ for } f \in C^\infty(G)\,.$$

Because of the latter equation and the definitions (1.3), (1.4), we immediately obtain

COROLLARY 1.12. *If $f \in C^\infty(G)$ satisfies $f(bn) = f(b)$ for $(b, n) \in G \times N$ (i.e. $f \in C^\infty(G/N)$) then*

$$\Omega f = \left[\sum_{i=1}^r (\delta_{H_i}^2 + 2\rho(H_i)\delta_{H_i}) - \sum_{i=1}^s u_i^2\right] f\,.$$

Also since δ_z, δ_x commute we have for $f \in C^\infty(G)$ satisfying $f(kb) = f(b)$ for $(k, b) \in K \times G$ (i.e. $f \in C^\infty(K\backslash G)$)

$$\Omega f = \left[\sum_{i=1}^r (\delta_{H_i}^2 + 2\rho(H_i)\delta_{H_i}) + 2\sum_{j=1}^q \sum_{i=1}^{m_{\alpha_j}} e^{-2\alpha_j}\delta_{x_{i(j)}}^2\right] f\,.$$

§ 2. Proof of Theorem 1.10.

Set $w_i=y_i$, $1\leqq i\leqq t$, $w_i=H_{i-t}$, $t+1\leqq i\leqq r+t$, $k_{ij}(kan)=\langle \mathrm{Ad}(k)w_j, w_i\rangle$; $k_{ij}\in C^\infty(G)$. By Proposition 1.8 $\{w_i\}_{i=1}^{r+t}$ is an orthonormal basis of $\mathfrak{p}$ and $\Omega\overset{(a.)}{=}-\sum_{i=1}^{s}\tilde{u}_i^2-\sum_{i=1}^{t}\tilde{z}_i^2+\sum_{i=1}^{r+t}\tilde{w}_i^2$. The main point is to compute $\sum_{j=1}^{r+t}\tilde{w}_j^2$.

LEMMA 2.1. *$\tilde{w}_j=\sum_{i=1}^{r}k_{j,i+t}\delta_{H_i}+\sum_{\mu=1}^{q}\sum_{\nu=1}^{m_{\alpha_\mu}}k_{j,\nu(\mu)}\{-\delta_{z_{\nu(\mu)}}+\sqrt{2}\,e^{-\alpha_\mu}\delta_{x_{\nu(\mu)}}\}$; here $k_{j,\nu(\mu)}=\langle\mathrm{Ad}(\cdot)y_{\nu(\mu)}, w_j\rangle$, $1\leqq j\leqq r+t$.*

PROOF. For $(a, b, x)\in G\times G\times\mathfrak{g}$ define $L_{axb}\colon C^\infty(G)\xrightarrow{\text{linear}}C$ by

$$L_{axb}f=\frac{d}{dt}f(a(\exp tx)b)\,|_{t=0} \tag{2.2}$$

for $f\in C^\infty(G)$. For $kan\in G=KAN$ arbitrary $(\tilde{w}_jf)(kan)=(d/dt)f((\exp tw_j)kan)|_{t=0}$ (by (1.4)) $=(d/dt)f(k\exp t\,\mathrm{Ad}(k^{-1})w_jan)|_{t=0}=L_{k\mathrm{Ad}(k^{-1})w_jan}f\overset{(i)}{=}\sum_{i=1}^{r+t}k_{ij}(k^{-1})L_{kw_ian}f$. For $1\leqq i\leqq t$, $w_i=y_i=\sqrt{2}\,x_i-z_i$ (by (1.7)). $x_i=x_{\nu(\mu)}\in g_{\alpha_\mu}$, $1\leqq\nu\leqq m_{\alpha_\mu}$, for some ν, μ (by (1.9)) so that $L_{kw_ian}\overset{(ii)}{=}\sqrt{2}\,L_{kx_{\nu(\mu)}an}-L_{kz_ian}$ where for $a=\exp H$, $H\in\mathfrak{a}$, $L_{kx_{\nu(\mu)}an}f=(d/dt)f(k\exp tx_{\nu(\mu)}an)|_{t=0}=(d/dt)f(ka\exp t\,\mathrm{Ad}\,(a^{-1})x_{\nu(\mu)}n)|_{t=0}=(d/dt)f(ka\exp te^{-\alpha_\mu(H)}x_{\nu(\mu)}n)|_{t=0}=L_{kae^{-\alpha_\mu(H)}x_{\nu(\mu)}n}f=e^{-\alpha_\mu(H)}(\delta_{x_{\nu(\mu)}}f)(kan)$. Also by definition $L_{kz_ian}f=(\delta_{z_i}f)(kan)$. Thus (ii) becomes $L_{kw_ian}f\overset{(iii)}{=}\sqrt{2}\,e^{-\alpha_\mu(H)}(\delta_{x_{\nu(\mu)}}f)(kan)-(\delta_{z_i}f)(kan)$. On the other hand for $t+1\leqq i\leqq r+t$, $w_i=H_{i-t}$ so that $\sum_{i=t+1}^{t+r}k_{ij}(k^{-1})L_{kH_{i-t}an}f\overset{(iv)}{=}\sum_{i=1}^{r}k_{i+t,j}(k^{-1})(\delta_{H_i}f)(kan)$. Since $k_{ij}(k^{-1})=k_{ji}(k)$, Lemma 2.1 follows from (i), (iii), (iv), and (1.5).

LEMMA 2.2. $\sum_{j=1}^{r+t}\sum_{\mu=1}^{q}\sum_{\nu=1}^{m_{\alpha_\mu}}k_{j,\nu(\mu)}\delta_{z_{\nu(\mu)}}\sum_{i=1}^{r}k_{k,i+t}\,\delta_{H_i}=-\sum_{i=1}^{r}2\rho(H_i)\delta_{H_i}$.

PROOF. For $(z, H, f)\in\mathfrak{k}\times\mathfrak{a}\times C^\infty(G)$, $kan\in G=KAN$ $(\delta_zk_{ij}\delta_Hf)(kan)=(d/dt)k_{ij}(k\exp tz)|_{t=0}(\delta_Hf)(kan)+k_{ij}(kan)(\delta_z\delta_Hf)(kan)$. Now $k_{ij}(k\exp tz)=\langle\mathrm{Ad}\,(\exp tz)w_j,\ \mathrm{Ad}\,(k^{-1})w_i\rangle\Longrightarrow(d/dt)k_{ij}(k\exp tz)|_{t=0}=\langle[z, w_j],\ \mathrm{Ad}\,(k^{-1})w_i\rangle=\sum_{l=1}^{r+t}k_{il}(kan)\ \langle[z, w_j], w_l\rangle$; i.e.

$$\delta_zk_{ij}\delta_H=\sum_{l=1}^{r+t}\langle[z, w_j], w_l\rangle k_{il}\delta_H+k_{ij}\delta_z\delta_H \tag{2.3}$$

for $(z, H)\in\mathfrak{k}\times\mathfrak{a}$. Hence for $1\leqq j\leqq r+t$

$$\begin{aligned}\sum_{\mu=1}^{q}\sum_{\nu=1}^{m_{\alpha_\mu}}k_{j,\nu(\mu)}\delta_{z_{\nu(\mu)}}\sum_{i=1}^{r}k_{j,i+t}\delta_{H_i}&=\sum_{\mu=1}^{q}\sum_{\nu=1}^{m_{\alpha_\mu}}\sum_{i=1}^{r}\sum_{l=1}^{r+t}\langle[z_{\nu(\mu)}, w_{i+t}], w_l\rangle k_{j,\nu(\mu)}k_{jl}\delta_{H_i}\\&\quad+\sum_{\mu=1}^{q}\sum_{\nu=1}^{m_{\alpha_\mu}}\sum_{i=1}^{r}k_{j,\nu(\mu)}k_{j,i+t}\delta_{z_{\nu(\mu)}}\delta_{H_i}\,.\end{aligned} \tag{2.4}$$

Now

$$\sum_{j=1}^{r+t} k_{ji}k_{jk}=\delta_{ik}\ , \tag{2.5}$$

and for $1\leqq i\leqq r$, $\delta_{\nu(\mu),i+t}=0$, $w_{i+t}=H_i$. Also from (1.7), $[z_i, y_i]\overset{(v)}{=}[\theta x_i, x_i]$ and since $\{x_{\nu(\mu)}\}_{\nu=1}^{m_{\alpha_\mu}}$ is an orthonormal basis of $\mathfrak{g}_{\alpha_\mu}$ we get $\langle [z_{\nu(\mu)}, H_i], y_{\nu(\mu)}\rangle = -B(H_i, [z_{\nu(\mu)}, y_{\nu(\mu)}]) = -B(H_i, [\theta x_{\nu(\mu)}, x_{\nu(\mu)}])$ (by (v)) $= B(\theta x_{\nu(\mu)}, [H_i, x_{\nu(\mu)}]) = -\alpha_\mu(H_i)\langle x_{\nu(\mu)}, x_{\nu(\mu)}\rangle = -\alpha_\mu(H_i)$. Thus summing over j in (2.4) gives $\sum_{j=1}^{r+t}\sum_{\mu=1}^{q}\sum_{\nu=1}^{m_{\alpha_\mu}} k_{j,\nu(\mu)}\delta_{z_{\nu(\mu)}}\sum_{i=1}^{r} k_{j,i+t}\delta_{H_i}=\sum_{\mu=1}^{q}\sum_{\nu=1}^{m_{\alpha_\mu}}\sum_{i=1}^{r}\langle [z_{\nu(\mu)}, H_i], y_{\nu(\mu)}\rangle\delta_{H_i}= -\sum_{\mu=1}^{q}(\sum_{\mu=1}^{q} m_{\alpha_\mu}\alpha_\mu)(H_i)\delta_{H_i}=-\sum_{i=1}^{r}2\rho(H_i)\delta_{H_i}$ (by (1.9)), which proves Lemma 2.2.

LEMMA 2.6. $\sum_{j=1}^{r+t}[\sum_{\mu=1}^{q}\sum_{\nu=1}^{m_{\alpha_\mu}} k_{j,\nu(\mu)}(-\delta_{z_{\nu(\mu)}}+\sqrt{2}\,e^{-\alpha_\mu}\delta_{x_{\nu(\mu)}})]^2=\sum_{\mu=1}^{q}\sum_{\nu=1}^{m_{\alpha_\mu}}\{\delta^2_{z_{\nu(\mu)}}+2e^{-2\alpha_\mu}\delta^2_{x_{\nu(\mu)}}-2\sqrt{2}\,e^{-\alpha_\mu}\delta_{z_{\nu(\mu)}}\delta_{x_{\nu(\mu)}}\}$.

PROOF. Similar to (2.3) we have $k_{j,\nu(\mu)}\delta_{z_{\nu(\mu)}}k_{j,\lambda(\gamma)}\delta_{z_{\lambda(\gamma)}}=k_{j,\nu(\mu)}\sum_{l=1}^{r+t}\langle [z_{\nu(\mu)}, w_{\lambda(\gamma)}], w_l\rangle k_{jl}\delta_{z_{\lambda(\gamma)}}+k_{j,\nu(\mu)}k_{j,\lambda(\gamma)}\delta_{z_{\nu(\mu)}}\delta_{z_{\lambda(\gamma)}}$, and $-k_{j,\nu(\mu)}\delta_{z_{\nu(\mu)}}k_{j,\lambda(\gamma)}\sqrt{2}\,e^{-\alpha_\gamma}\delta_{x_{\lambda(\gamma)}}= -\sqrt{2}\,e^{-\alpha_\gamma}k_{j,\nu(\mu)}\sum_{l=1}^{r+t}\langle [z_{\nu(\mu)}, w_{\lambda(\gamma)}], w_l\rangle k_{jl}\delta_{x_{\lambda(\gamma)}}-\sqrt{2}\,e^{-\alpha_\gamma}k_{j,\nu(\mu)}k_{j,\gamma(\gamma)}\delta_{z_{\nu(\mu)}}\delta_{x_{\lambda(\gamma)}}$. Then using (2.5) again, the identity $\langle [z_{\nu(\mu)}, w_{\lambda(\gamma)}], w_{\nu(\mu)}\rangle=\langle [z_{\nu(\mu)}, y_{\lambda(\gamma)}], y_{\nu(\mu)}\rangle= -B([z_{\nu(\mu)}, y_{\nu(\mu)}], y_{\lambda(\gamma)})=-B([\theta x_{\nu(\mu)}, x_{\nu(\mu)}], y_{\lambda(\gamma)})$ (by (v) above) $=0$ (since $[\theta x, x]\in\mathfrak{a}$ for $x\in g_\alpha$, $\alpha\in\Sigma$), and the fact that δ_z, δ_x commute for $(z, x)\in\mathfrak{k}\times\mathfrak{n}$, the statement of Lemma 2.6 follows.

We are now in a position to compute $\sum_{j=1}^{r+t}\tilde{w}_j^2$. By Lemma 2.1 $\sum_{j=1}^{r+t}\tilde{w}_j^2=\sum_{j=1}^{r+t}[(\sum_{i=1}^{r} k_{j,i+t}\delta_{H_i})^2+\sum_{i=1}^{r} k_{j,i+t}\delta_{H_i}\sum_{\mu=1}^{q}\sum_{\nu=1}^{m_{\alpha_\mu}} k_{j,\nu(\mu)}(-\delta_{z_{\nu(\mu)}}+\sqrt{2}\,e^{-\alpha_\mu}\delta_{x_{\nu(\mu)}})+\sum_{\mu=1}^{q}\sum_{\nu=1}^{m_{\alpha_\mu}}k_{j,\nu(\mu)}(-\delta_{z_{\nu(\mu)}}+\sqrt{2}\,e^{-\alpha_\mu}\delta_{x_{\nu(\mu)}})\sum_{i=1}^{r}k_{j,i+t}\delta_{H_i}+\{\sum_{\mu=1}^{q}\sum_{\nu=1}^{m_{\alpha_\mu}}k_{j,\nu(\mu)}(-\delta_{z_{\nu(\mu)}}+\sqrt{2}\,e^{-\alpha_\mu}\delta_{x_{\nu(\mu)}})\}^2]$. Now $\sum_{j=1}^{r+t}(\sum_{i=1}^{r} k_{j,i+t}\delta_{H_i})^2=\sum_{l,i=1}^{r}\sum_{j=1}^{r+t}k_{j,i+t}k_{j,l+t}\delta_{H_i}\delta_{H_l}$ (since $\delta_{H_i}k_{j,l+t}\delta_{H_l}=k_{j,l+t}\delta_{H_i}\delta_{H_l}$) $=\sum_{i=1}^{r}\delta^2_{H_i}$ (by (2.5)). Also the sum over j of the second term in the above bracket is zero by (2.5) since $\delta_{i+t,\nu(\mu)}=0$ for $1\leqq i\leqq r$. Similarly $\sum_{j=1}^{r+t}\sum_{\mu=1}^{q}\sum_{\nu=1}^{m_{\alpha_\mu}} k_{j,\nu(\mu)}\sqrt{2}\,e^{-\alpha_\mu}\delta_{x_{\nu(\mu)}}\sum_{i=1}^{r} k_{j,i+t}\delta_{H_i}=0$ by (2.5). Lemma 2.2 and Lemma 2.6 therefore imply

PROPOSITION 2.7. $\sum_{j=1}^{r+t}\tilde{w}_j^2=\sum_{i=1}^{r}(\delta^2_{H_i}+2\rho(H_i)\delta_{H_i})+\sum_{\mu=1}^{q}\sum_{\nu=1}^{m_{\alpha_\mu}}\{\delta^2_{z_{\nu(\mu)}}+2e^{-2\alpha_\mu}\delta^2_{x_{\nu(\mu)}}-2\sqrt{2}\,e^{-\alpha_\mu}\delta_{z_{\nu(\mu)}}\delta_{x_{\nu(\mu)}}\}$.

Since $\sum_{\mu=1}^{q}\sum_{\nu=1}^{m_{\alpha_\mu}}\delta^2_{z_{\nu(\mu)}}=\sum_{i=1}^{t}\delta^2_{z_i}$ Proposition 2.7 and equation (a.) preceding Lemma 2.1 clearly imply formula (1.11) of Theorem 1.10. Given $f\in C^\infty(G)$, $ka\in KA$ (1.3) implies for $z\in\mathfrak{k}$, $(\delta_z f)(ka)=(d/dt)f(ka\exp t\,\mathrm{Ad}(a^{-1})z)|_{t=0}=(\mathrm{Ad}(a^{-1})zf)(ka)$ so that

$$\Big(\Big(\sum_{i=1}^{t}\delta^2_{z_i}\Big)f\Big)(ka)=\sum_{i=1}^{t}(\mathrm{Ad}(a^{-1})z_i\mathrm{Ad}(a^{-1})z_if)(ka)\ . \tag{2.8}$$

On the other hand we have

LEMMA 2.9. *Let* $\{E_i\}$ *be an orthonormal basis of* $\mathfrak{k}$. *Then*

$$\Big(\Big(\sum_i \tilde{E}_i^2\Big)f\Big)(k\exp y)=\sum_i (\mathrm{Ad}(\exp -y)E_i)^2 f)(k\exp y) \qquad \textit{for} \quad (k,y)\in K\times\mathfrak{p}\,.$$

Lemma 2.9 follows essentially from the fact that $\mathrm{Ad}(k^{-1})$ is an orthogonal transformation of $\mathfrak{p}$. Taking $\{E_i\}=\{u_i\}\cup\{z_i\}$ (see Proposition 2.8) and $y\in\mathfrak{a}$ in Lemma 2.9 we get for $a=\exp y\in A$

$$\text{(2.10)}\qquad \Big(\Big(\sum_{i=1}^{s}\tilde{u}_i^2+\sum_{i=1}^{t}\tilde{z}_i^2\Big)f\Big)(ka)=\Big(\Big(\sum_{i=1}^{s}(\mathrm{Ad}(a^{-1})u_i)^2+\sum_{i=1}^{t}(\mathrm{Ad}(a^{-1})z_i)^2\Big)f\Big)(ka)$$
$$=\Big(\Big(\sum_{i=1}^{s}u_i^2+\sum_{i=1}^{t}(\mathrm{Ad}(a^{-1})z_i)^2\Big)f\Big)(ka)$$

since $u_i\in\mathfrak{m}$, the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Thus (2.8) and (2.10) imply $(\sum_{i=1}^{t}\delta_{z_i}^2)f=(\sum_{i=1}^{s}(\tilde{u}_i^2-u_i^2)+\sum_{i=1}^{t}\tilde{z}_i^2)f$ on KA, which completes the proof of Theorem 1.10.

In the very special case when $G=SL(2,\boldsymbol{R})$, Theorem 1.10 reduces to a result of Takahashi. Namely we have the following data. If $\boldsymbol{Z}$ is the ring of integers, an Iwasawa decomposition of G is given by the diffeomorphism $\Phi\colon \boldsymbol{R}/4\pi\boldsymbol{Z}\times\boldsymbol{R}\times\boldsymbol{R}\to G$ defined by

$$\text{(2.11)}\qquad \Phi(\theta,t,\xi)=k_\theta a_t n_\xi \overset{\text{def.}}{=} \begin{bmatrix}\cos\theta/2 & \sin\theta/2\\ -\sin\theta/2 & \cos\theta/2\end{bmatrix}\begin{bmatrix}e^{t/2} & 0\\ 0 & e^{-t/2}\end{bmatrix}\begin{bmatrix}1 & \xi\\ 0 & 1\end{bmatrix}$$
$$\in KAN \overset{\text{def.}}{=} SO(2)\exp\mathfrak{a}\exp\mathfrak{n}\,, \qquad \text{where}$$

$$\text{(2.12)}\qquad \mathfrak{a}=\left\{\begin{bmatrix}r & 0\\ 0 & -r\end{bmatrix}\Bigg|\, r\in\boldsymbol{R}\right\}, \qquad \mathfrak{n}=\mathfrak{g}_\alpha=\left\{\begin{bmatrix}0 & r\\ 0 & 0\end{bmatrix}\Bigg|\, r\in\boldsymbol{R}\right\},$$
$$\alpha\begin{bmatrix}r & 0\\ 0 & -r\end{bmatrix}=2r\,, \qquad \Sigma^+=\{\alpha\};\ \mathfrak{m}=0\,.$$

The vectors $H_1=\begin{bmatrix}1/2\sqrt{2} & 0\\ 0 & -1/2\sqrt{2}\end{bmatrix}$, $x_1=\begin{bmatrix}0 & 1/2\\ 0 & 0\end{bmatrix}$ are an orthonormal basis of $\mathfrak{a}$, $\mathfrak{n}$ respectively. Here $B(x,y)=4\,\mathrm{trace}\,xy$ and $\theta x=-x^t$ for $x,y\in\mathfrak{g}=\mathfrak{sl}(2,\boldsymbol{R})$. Hence, by (1.7), $Z_1=\begin{bmatrix}0 & 1/2\sqrt{2}\\ -1/2\sqrt{2} & 0\end{bmatrix}$ and since K is *abelian* δ_{z_1} and $\tilde{Z}_1$ (in (1.3), (1.4)) coincide. Theorem 1.10 therefore gives

$$\text{(2.13)}\qquad \Omega=\delta_{H_1}^2+\frac{1}{\sqrt{2}}\delta_{H_1}+2e^{-2\alpha}\delta_{X_1}^2-2\sqrt{2}\,e^{-\alpha}\delta_{Z_1}\delta_{X_1}\,.$$

For $s,t\in\boldsymbol{R}$, $\exp sH_1a_t=\exp\begin{bmatrix}(s/\sqrt{2}+t)/2 & 0\\ 0 & -(s/\sqrt{2}+t)/2\end{bmatrix}=a_{s/\sqrt{2}+t}$. For $\psi\in C^\infty(\boldsymbol{R}/4\pi\boldsymbol{Z}\times\boldsymbol{R}\times\boldsymbol{R})$ it follows that $(\delta_{H_1}(\psi\cdot\Phi^{-1}))\cdot\Phi(\theta,t,\xi)=\delta_{H_1}(\Psi\cdot\Phi^{-1})$

$(k_\theta a_t h_\xi) = (d/ds)\psi\cdot\Phi^{-1}(k_\theta(\exp sH_1)a_t n_\xi)|_{s=0} = (d/ds)\psi(\theta, s/\sqrt{2}+t, \xi)|_{s=0} = (1/\sqrt{2})(\partial\psi/\partial t)(\theta, t, \xi)$. Similarly $k_\theta \exp sZ_1 = \exp\begin{bmatrix} 0 & \theta/2 \\ -\theta/2 & 0 \end{bmatrix}\exp\begin{bmatrix} 0 & s/2\sqrt{2} \\ -s/2\sqrt{2} & 0 \end{bmatrix} = \exp\begin{bmatrix} 0 & (\theta+s/\sqrt{2})/2 \\ (\theta+s/\sqrt{2})/-2 & 0 \end{bmatrix} = k_{\theta+s/\sqrt{2}} \Rightarrow (\delta_{Z_1}(\psi\cdot\Phi^{-1}))\cdot\Phi = (1/\sqrt{2})(\partial\psi/\partial\theta)$, and $(\delta_{X_1}(\psi\cdot\Phi^{-1}))\cdot\Phi = (1/2)(\partial\psi/\partial\xi)$ since $\exp sX_1 n_\xi = n_{s/2+\xi}$. Using $e^\alpha(k_\theta a_t n_\xi) = e^t$ (cf. (1.5)) we therefore obtain from (2.13)

COROLLARY 2.14 (R. Takahashi, Lemma 3 of [3]). *For $G=SL(2, \boldsymbol{R})$, the expression of Ω in terms of the Iwasawa coordinates (θ, t, ξ) in* (3.11) *is given by* $(\Omega(\psi\cdot\Phi^{-1}))\cdot\Phi = (1/2)(\partial^2\psi/\partial t^2) + (1/2)(\partial\psi/\partial t) + (1/2)e^{-2t}(\partial^2\psi/\partial\xi^2) - e^{-t}(\partial^2\psi/\partial\theta\partial\xi)$ *for* $\psi \in C^\infty(\boldsymbol{R}/4\pi\boldsymbol{Z}\times\boldsymbol{R}\times\boldsymbol{R})$.

The reader should note that the Casimir operator defined in [3] is not the same as our Ω, but rather coincides with -2Ω.

References

[1] M. ANDERSON, A simple expression for the Casimir operator on a Lie group, Proc. Amer. Math. Soc., **77** (1979), 415–420.

[2] S. HELGASON, Differential Geometry and Symmetric Spaces, Academic Press, N.Y., 1962.

[3] R. TAKAHASHI, Sur les fonctions sphériques et la formule de Plancherel dans le groupe hyperbolique, Japan. J. Math., **31** (1961), 55–90.

Present Address:
DEPARTMENT OF MATHEMATICS AND STATISTICS,
UNIVERSITY OF MASSACHUSETTS
Amherst, Massachusetts 01003
U.S.A.