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A Note on Characteristic Functions and Projectively Invariant Metrics on a Bounded Convex Domain

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Introduction

The purpose of this note is to propose two metrics on a bounded convex domain, which are projectively invariant and seem to have similar nature to the Blaschke metric.

To recall the Blaschke metric let us take a bounded convex domain Ω in $\boldsymbol{R}^n$ and consider the differential equation

$$(\#) \qquad \det \partial^2 u/\partial x^i \partial x^j = (-u)^{-n-2} \quad \text{on} \quad \Omega\,, \qquad u=0 \quad \text{on} \quad \partial\Omega\,.$$

Since this equation has the unique negative strictly convex solution as is shown in [13] for dimension 2 and [5] in general, we can define a metric $-(1/u)d^2u$. This metric was first considered by Blaschke [1] and Tzitzèica [16] and can be thought of a possible generalization of the Hilbert metric of the ball. We call this shortly the *Blaschke metric* of the domain Ω. This is known to be complete; [4], [14].

A bounded pseudoconvex domain in $\boldsymbol{C}^n$ on the other hand has in general several biholomorphically invariant metrics. We would like to take two of them. One is the Einstein-Kähler metric, which exist at least under some smoothness condition on the boundary [6], and the other is the Bergman metric. Let us pay our attention to the special case where the domain is a tube over a cone V. In this case let Ω be a nontrivial hyperplane section of the cone. Then these metrics have special forms by the invariance and must have their correspondences on Ω. Namely, on the one hand in Appendix A, we show the Einstein-Kähler metric on the tube domain corresponds to the Blaschke metric on Ω.

On the other hand in §1, corresponding to the Bergman kernel, we will define a kernel function on Ω. In the course of this we need to

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define the characteristic function of the domain Ω, which is nothing but the restriction to Ω of the characteristic function of the cone investigated in [17].

To be more precise, let Ω^* be the dual of Ω defined by $\Omega^*=\mathrm{int}\{\xi\in \boldsymbol{R}^n;\ 1+\langle x, \xi\rangle\geqq 0 \text{ for } x\in\Omega\}$. Then the characteristic function of Ω is by definition

$$\chi_\Omega(x)=\int_{\Omega^*} n!\,(1+\langle x, \xi\rangle)^{-n-1}d\xi$$

and the kernel function of Ω is defined to be

$$k_\Omega(x)=\int_{\Omega^*}(2n+1)!\,(1+\langle x, \xi\rangle)^{-2n-2}\chi_{\Omega^*}(\xi)^{-1}d\xi\ .$$

Using these functions we introduce in §2 two invariant metrics. Set $v=\chi^{-1/n+1}$ and $w=k^{-1/2n+2}$ for the moment. The metrics we are concerned with are defined by

$$\omega=-\frac{1}{v}d^2v \quad\text{and}\quad \kappa=-\frac{1}{w}d^2w\ .$$

After the discussion of elementary properties of these metrics we will prove the completeness in Theorem 1 and give some examples and questions. As a result we can see that both metrics can be thought of a generalization of the Hilbert metric and have similar nature to the Blaschke metric.

The third section is devoted to the investigation of the boundary behavior of both the kernel function and the characteristic function. The result is summarized in Theorems 4 and 5. In order to state Theorem 4 we will assume Ω is a strictly convex bounded domain with smooth boundary. Let ϕ be a defining function of Ω: $\Omega=\{\phi<0\}$. We set

$$J(\phi)=\det\begin{pmatrix}\phi_{ij} & \phi_j\\ \phi_i & 2\phi\end{pmatrix}.$$

Then

THEOREM 4. *The characteristic function has an expansion*

$$\chi_\Omega(x)=d_n\sqrt{J(\phi)}\,(-\phi)^{-(n+1)/2}+\sum_1^{[n/2]}\varepsilon_j(-\phi)^{j-(n+1)/2}+0(A(\phi))\ ,$$

near $\partial\Omega$, where $A(t)$ is a function defined to be 1 *for even n and* $\log t$ *for odd n, and d_n is a constant depending only on n. Moreover ε_j is a* *smooth function on $\bar{\Omega}$ whose boundary value is determined locally by the geometrical data of Ω^*.*

As for the function k_Ω, see Theorem 5.

The problem to look for the precise form of these coefficients ε_j using affiine invariants of the boundary is not treated in general, since it will need the invariant theory of the unimodular group which is not developed well for our use. The thing we can do in §4 is only to get in Theorem 6 the coefficient ε_1 in the special case in terms of the affine scalar curvature of the boundary and the Fubini-Pick invariant. The straightforward calculation for the kernel function seems to be hard and is not discussed here.

In Appendix B we will summarize some facts on the affine geometry of a hypersurface which is necessary to the computation in §4.

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§1. Characteristic function and kernel function of a bounded convex domain.

Let Ω be a bounded convex domain in $\boldsymbol{R}^n(x)$ and V_Ω be the cone over Ω given by $V_\Omega=\{(tx, t); x\in\Omega, t\in\boldsymbol{R}^+\}$. Ω is identified with the set $\{t=1\}$. We denoted by u the unique convex solution of (♯) on Ω. Set $\xi_i=u_i$ and $f=-u+\sum\xi_i x^i$. Since u is strictly convex, f is a function of $\xi=(\xi_i)$ and is called the Legendre transform of u. It is known in [2], [14] that this function f defines a hyperbolic affine hypersphere in the dual cone of V_Ω. Independent of this fact Vinberg [17] defined a metric using the characteristic function on every nondegenerate convex cone. In the special case when Ω is projectively homogeneous we can see that the above transform is an isometric embedding of Ω into the dual cone. So it is natural to transplant Vinberg's construction to our bounded domain. This is one of our aims in this section and in the next section. On the other hand, as we have proved in Appendix A, the Blaschke metric mentioned in Introduction is exactly related to the Einstein-Kähler metric on the tube domain D over V_Ω. This latter metric is generally supposed to behave at the (ideal) boundary like the Bergman metric. Moreover the Bergman kernel function of D can be defined on the real part V_Ω by [12], [11]. So we want to describe this kernel function on the domain Ω. This is the second aim.

Now we are going to give some definitions. In the following we work on a slightly more general domain: a convex domain Ω in $\boldsymbol{R}^n(x)$

containing no straight line.

Let $V=V_{\Omega}$ be the cone over Ω. Let V^* be the dual cone of V. When the closure $\bar{\Omega}$ contains the origin, it is the cone over some domain Ω^*. In a concrete form we set

$$\Omega^*=\operatorname{int}\{\xi\in \boldsymbol{R}^n;\, 1+\langle x, \xi\rangle\geqq 0 \text{ for } x\in\Omega\}\,,$$

and call it the *dual* of Ω. When the domain Ω is a non-degenerate convex cone from the start, then Ω^* is nothing but the dual cone of Ω. If Ω contains the origin and is bounded, then Ω^* is also bounded and contains the origin. In this case, setting

$$H(\xi)=\sup\{\langle x, \xi\rangle;\, x\in\Omega\}\,, \tag{1.1}$$

which is called the support function of the domain Ω, we can see

$$\Omega^*=\operatorname{int}\{H(-\xi)\leqq 1\}\,.$$

EXAMPLES. 1. $\Omega=\{|x|<1,\, x\in\boldsymbol{R}^n\}$ then $\Omega^*=\{|\xi|<1,\, \xi\in\boldsymbol{R}^n\}$.
2. $\Omega=\{|x|<1,\, |y|<1\}\subset\boldsymbol{R}^2$ then $\Omega^*=\{|\xi+\eta|<1,\, |\xi-\eta|<1\}$.
3. $(\Omega_1\times\Omega_2)^*=$ the convex hull of $\Omega_1^*\times\{0\}\cup\{0\}\times\Omega_2^*$.

Let V be a non-degenerate convex cone. Recall that the characteristic function χ_V of V is defined by the integral

$$\chi_V(x)=\int_{V^*} e^{-\langle x,\xi\rangle}d\xi\,, \tag{1.2}$$

and that the Bergman kernel of $V+i\boldsymbol{R}^{n+1}$ is written as $K_V(\operatorname{Re} z)$ up to a constant where K_V is defined by the integral

$$K_V(x)=\int_{V^*} e^{-\langle x,\xi\rangle}\chi_{V^*}(\xi)^{-1}d\xi\,. \tag{1.3}$$

Assume $V=V_{\Omega}$ for a convex domain Ω. Then these functions χ_V and K_V are determined by their restrictions to Ω. To see this, set $V_{\Omega}=\{y=(tx, t);\, x\in\Omega,\, t\in\boldsymbol{R}^+\}$ and $V^*=V_{\Omega^*}=\{\eta=(s\xi, s);\, \xi\in\Omega^*,\, s\in\boldsymbol{R}^+\}$. Then

$$\begin{aligned}\chi_V(y)&=\int_{V^*} e^{-\langle y,\eta\rangle}d\eta \\ &=\int_{\Omega^*} d\xi\int_0^\infty s^n e^{-ts(1+\langle x,\xi\rangle)}ds \\ &=t^{-n-1}\int_{\Omega^*} n!\,(1+\langle x, \xi\rangle)^{-n-1}d\xi\,.\end{aligned} \tag{1.4}$$

So we introduce

DEFINITION 1. The *characteristic function* of a convex domain Ω in $\boldsymbol{R}^n$ containing no straight line is the function χ_Ω defined by

$$\chi_\Omega(x)=\int_{\Omega^*} n!\,(1+\langle x,\xi\rangle)^{-n-1}d\xi \tag{1.5}$$

where $d\xi$ is the usual Lebesgue measure on $\boldsymbol{R}^n(\xi)$. The *kernel function* of the domain Ω is the function k_Ω defined by

$$k_\Omega(x)=\int_{\Omega^*} (2n+1)!\,(1+\langle x,\xi\rangle)^{-2n-2}\chi_{\Omega^*}(\xi)^{-1}d\xi\ . \tag{1.6}$$

When Ω is a cone, we have two characteristic functions defined by (1.2) and (1.5). But both are identical. In fact by (1.5) it is equal to

$$\int_{V^*}(n+1)!\,(1+\langle y,\eta\rangle)^{-n-2}d\eta=\int_{R^+\times\Omega^*}(n+1)!\,(1+st(1+\langle x,\xi\rangle))^{-n-2}s^n ds d\xi$$
$$=t^{-n-1}\int_{\Omega^*} n!\,(1+\langle x,\xi\rangle)^{-n-1}d\xi=\chi_V(y) \qquad \text{by (1.4)}\ .$$

Similarly we can prove $K_V=k_V$ for a cone V.

We list examples. Calculations are lengthy and omitted. Here u denotes the solution of (#) on Ω.

EXAMPLES 4. $\Omega=\{|x|<R\}$; $u=-R^{-1/n+1}(R^2-|x|^2)^{1/2}$, $\chi=n!\,b_nR(R^2-|x|^2)^{-(n+1)/2}$ where b_n is the volume of unit n-ball, $k=b_n'\chi^2$ where b_n' is a constant depending only on n.

5. $\Omega=\{|x|<1,\ |y|<1\}$; $\chi=8(1-x^2)^{-1}(1-y^2)^{-1}$,

$$k=8\{(1+x^2)(1-y^2)^2+(1+y^2)(1-x^2)^2-4(1-x^2)(1-y^2)\}((x^2-y^2)(1-x^2)(1-y^2))^{-2}$$
$$+16(x^2-y^2)^{-3}\log\{(1-y^2)(1-x^2)^{-1}\}\ .$$

When $|x|=|y|$, take limits which can be seen to be finite.

6. $\Omega=$a simplex in $\boldsymbol{R}^n=\{l_1>0,\cdots,l_{n+1}>0\}$ where l_i are linear forms. $u=c(l_1\cdots l_{n+1})^{1/n+1}$, $\chi=c'u^{-n-1}$, $k=c''u^{-2n-2}$ where c, c' and c'' are constants. For calculations see [14] and Proposition 3.

PROPOSITION 1. *Let Ω_1 and Ω_2 be convex domains containing no straight lines and assume $\Omega_1\subset\Omega_2$. Then $\Omega_2^*\subset\Omega_1^*$ and*

$$\chi_{\Omega_1}(x)\geqq\chi_{\Omega_2}(x) \quad \text{and} \quad k_{\Omega_1}(x)\geqq k_{\Omega_2}(x)$$

for $x\in\Omega_1$.

PROOF. We set $l=1+\langle x,\xi\rangle$. Let $x\in\Omega_1\subset\Omega_2$. By definition

$$\chi_{\Omega_1}(x)=\int_{\Omega_1^*} n!\, l^{-n-1}d\xi \geqq \int_{\Omega_2^*} n!\, l^{-n-1}d\xi=\chi_{\Omega_2}(x)\ .$$

Similarly $\chi_{\Omega_1^*}(\xi)\leqq\chi_{\Omega_2^*}(\xi)$ for $\xi\in\Omega_2^*$. Hence

$$\begin{aligned} k_{\Omega_1}(x) &= \int_{\Omega_1^*} (2n+1)!\, l^{-2n-2}\chi_{\Omega_1^*}(\xi)^{-1}d\xi \\ &\geqq \int_{\Omega_2^*} (2n+1)!\, l^{-2n-2}\chi_{\Omega_2^*}(\xi)^{-1}d\xi \\ &= k_{\Omega_2}(x)\ . \end{aligned}$$

COROLLARY 1. *Let Ω be a bounded convex domain. Then both functions χ_Ω and k_Ω tend to infinity at the boundary.*

PROOF. For any boundary point $y\in\partial\Omega$ there exists a simplex which contains Ω and has y at its boundary. Then Example 6 implies the assertion by Proposition 1.

We can get more precise estimates when the domain is strictly convex and smooth:

PROPOSITION 2. *Assume Ω is a strictly convex bounded domain with C^2-boundary. Then there exist constants c_i and r depending on n and the domain such that*

$$c_1 d(x, \partial\Omega)^{-(n+1)/2}\leqq\chi_\Omega(x)\leqq c_2 d(x, \partial\Omega)^{-(n+1)/2}\ ,$$
$$c_3 d(x, \partial\Omega)^{-n-1}\leqq k_\Omega(x)\leqq c_4 d(x, \partial\Omega)^{-n-1}$$

for $d(x, \partial\Omega)<r$. Here $d(x, \partial\Omega)=\min\{d(x, y);\ y\in\partial\Omega\}$.

PROOF. For a boundary point y, we denote by $r(y)$ (resp. $R(y)$) the maximum (resp. minimum) radius of the ball which is contained in Ω (resp. is containing Ω) and is tangent to $\partial\Omega$ at y. We put $r=\min\{r(y);\ y\in\partial\Omega\}$ and $R=\max\{R(y);\ y\in\partial\Omega\}$. Choose a point $x\in\Omega$ such that $d(x, \partial\Omega)<r$. Let y be the point in $\partial\Omega$ with $d(x, y)=d(x, \partial\Omega)$. Pick an inscribing or circumscribing ball B_i or B_e at y. We may assume the radius B_i is r. Then by Proposition 1

$$\chi_{B_i}(x)\geqq\chi_\Omega(x)\geqq\chi_{B_e}(x)\ .$$

Now Example 4 shows

$$\chi_{B_i}(x)\leqq n!\, b_n r^{-(n-1)/2} d(x, \partial\Omega)^{-(n+1)/2}$$

and

$$\chi_{B_0}(x) \geqq n!\, b_n 2^{-(n+1)/2} R^{-(n-1)/2} d(x, \partial\Omega)^{-(n+1)/2} .$$

These inequalities imply proposition for χ_Ω. The proof for k_Ω is similar.

We next see the projective invariance of characteristic functions and kernel functions.

Let $A=\begin{pmatrix} a_i^j & a^j \\ a_i & a \end{pmatrix}$ be a projective transformation acting as $Ax=((a_i^j x^i + a^j)/(a_i x^i + a))$. Assume $a_i x^i + a > 0$ for $x \in \Omega$, i.e. $(a_i/a) \in \Omega^*$, and $a^j \xi_j + a > 0$ for $\xi \in \Omega$, i.e. $(a^j/a) \in \Omega$. The equality

$$(1.7) \qquad (a_i x^i + a)(1 + \langle Ax, \xi \rangle) = (a^j \xi_j + a)(1 + \langle x, {}^tA\xi \rangle) ,$$

implies $(A\Omega)^* = {}^tA^{-1}\Omega^*$. Then we have

PROPOSITION 3.

$$\chi_{A\Omega}(Ax) = (\det A)^{-1}(a_i x^i + a)^{n+1} \chi_\Omega(x) ,$$

$$k_{A\Omega}(Ax) = (\det A)^{-2}(a_i x^i + a)^{2n+2} k_\Omega(x) .$$

PROOF. For $\xi \in (A\Omega)^*$, set $\eta = {}^tA\xi \in \Omega^*$. Then $d\xi = (\det A)^{-1}(a^j\xi_j + a)^{n+1} d\eta$. Substituting this into the integrand of the defining equation of $\chi_{A\Omega}$ we have the first equality. This equality for the domain $(A\Omega)^*$ is

$$\chi_{(A\Omega^*)}(\xi) = \det A (a^j \xi_j + a)^{-n-1} \chi_{\Omega^*}({}^tA\xi) .$$

Using this we have the second equality.

REMARK 1. In Corollary 1 we have assumed the boundedness of the domain. But Proposition 3 implies that this assumption can be replaced by the projective equivalence to a bounded domain.

Vinberg defined in his paper [17] the *-mapping from a non-degenerate convex cone to its dual. For $x \in V_\Omega$, its *-image, which we denote here by x°, is defined as follows:

$$x^\circ = -\operatorname{grad} \log \chi_V(x) .$$

For $x \in \Omega \subset V_\Omega$, we set as its *-image $x^* = \Omega^* \cap$ the half line through x° and the origin. By simple argument we see that

$$(1.8) \qquad x^* = -\operatorname{grad} \chi(x)\{(n+1)\chi(x) + \langle x, \operatorname{grad} \chi(x) \rangle\}^{-1} .$$

By the definition of χ it then holds that

$$(1.9) \qquad x^* = \left(\int_{\Omega^*} \xi l^{-n-2} d\xi\right)\left(\int_{\Omega^*} l^{-n-2} d\xi\right)^{-1} .$$

Namely, x^* is the centre of gravity of Ω^* with respect to the measure $l^{-n-2}d\xi$. Since the mapping $^\circ$ is a diffeomorphism of V_Ω onto V_{Ω^*}, we have

PROPOSITION 4. *The mapping* $*$ *is a diffeomorphism of* Ω *onto* Ω^* *and equivariant under projective transformations*: $(Ax)^*=({}^tA)^{-1}x^*$.

PROOF. The projective equivariance follows from the affine equivariance of the mapping $^\circ$ by [17], p. 352. Note that the straightforward proof using (1.7) and (1.9) is also possible.

Also Vinberg has shown that $x^{\circ\circ}=x$ provided that the cone is affinely homogeneous. From this fact we can see

PROPOSITION 5. $x^{**}=x$ *if* Ω *is projectively homogeneous.*

For the sake of later use we introduce another functions: *p-th characteristic function* $\chi_p=\chi_{\Omega,p}$ by

$$\chi_p(x)=\int_{\Omega^*} p!\,(1+\langle x,\xi\rangle)^{-p-1}d\xi\ , \tag{1.10}$$

for $p=1, 2, \cdots$. The function χ_{n+1} has appeared in the definition of x^*, (1.9). It is easy to get an equality

$$1+\langle x, x^*\rangle=(n+1)\chi(x)\chi_{n+1}(x)^{-1}\ . \tag{1.11}$$

EXAMPLE 7. $\Omega=\{|x|<1, |y|<1\}\subset \boldsymbol{R}^2$. Then

$$\chi_p(x, y)=\begin{cases}\dfrac{4(p-2)!!}{x^2-y^2}\{(1+x)^{1-p}-(1+y)^{1-p}+(1-y)^{1-p}+(1-x)^{1-p}\}\ , & p\geqq 2\ ,\\[2ex] \dfrac{4}{x^2-y^2}\log\dfrac{1-y^2}{1-x^2} & p=1\ .\end{cases}$$

EXAMPLE 8. $\Omega=\{|x|<R\}$ in $\boldsymbol{R}^n$.

$$\chi_p(x)\sim b_{n,p}R^{2p-2n+1}(R^2-|x|^2)^{-p+(n-1)/2}$$

for $p\geqq n/2-1$, where $b_{n,p}=b_n n!!\,(2p-n-1)!!$, b_n=the volume of the unit n-ball.

We can apply the most of the preceding arguments to these functions. Among others we have

PROPOSITION 6. *Let* Ω *be a bounded strictly convex domain with smooth boundary. Then there exist constants* c_1 *and* c_2 *for each* $p\geqq n/2-1$ *such that*

$$c_1 d(x, \partial\Omega)^{-p+(n-1)/2}\leqq \chi_p(x)\leqq c_2 d(x, \partial\Omega)^{-p+(n-1)/2}\ .$$

Using this proposition and (1.11), we have

PROPOSITION 7. *Under the same assumption of Proposition* 6, *the value* $1+\langle x, x^*\rangle$ *tends to zero as* x *tends to the boundary.*

REMARK 2. We should remark here that it might be useful to investigate function

$$\int_{\Omega^*} k!\prod_i \xi_i^{\alpha_i}(1+\langle x, \xi\rangle)^{-k-1}d\xi$$

as Gindikin did in [11] in the case of convex cones.

We give one more proposition which seems to be a composition formula.

PROPOSITION 8. *Let* Ω_1 *and* Ω_2 *be bounded convex domains containing the origin in* R^n *and* R^m *respectively. Let* Ω *denote the join of* Ω_1 *and* Ω_2: $\Omega=\{(tx, t, (1-t)y)\in R^{n+m+1};\ x\in\Omega_1,\ y\in\Omega_2,\ t\in(0,1)\}$. *Then* $\Omega^*=\{((1+\tau)\xi, \tau, \eta);\ \xi\in\Omega_1^*,\ \eta\in\Omega_2^*,\ \tau\in(-1,\infty)\}$ *and*

$$u_\Omega(x, t, y)^{n+m+2}=t^{n+1}(1-t)^{m+1}u_{\Omega_1}(x)^{n+1}u_{\Omega_2}(y)^{m+1}$$
$$\chi_\Omega(x, t, y)=t^{-n-1}(1-t)^{-m-1}\chi_{\Omega_1}(x)\chi_{\Omega_2}(y)$$
$$k_\Omega(x, t, y)=t^{-2n-1}(1-t)^{-2m-1}k_{\Omega_1}(x)k_{\Omega_2}(y)\ .$$

PROOF. The form of Ω^* can be checked easily. Then

$$\chi_\Omega(x, t, y)=\int_{\Omega_1^*\times(0,\infty)\times\Omega_2^*}\frac{(n+m+1)!\ \tau'^n d\xi d\tau' d\eta}{((1-t)(1+\langle y, \eta\rangle)+\tau' t(1+\langle x, \xi\rangle))^{n+m+2}}$$

where $\tau'=1+\tau$. Next integrate the right side first on τ' and use the identity

$$\int_0^\infty \frac{(m+n+1)!\ \tau^n}{(1+a\tau)^{m+n+2}}d\tau=n!\ m!\ a^{-n-1}\ ,$$

then we have the formula for χ_Ω. To show the formula for k_Ω, it is necessary to compute χ_{Ω^*}. By definition

$$\chi_{\Omega^*}(\tau'\xi, \tau, \eta)=\int_{\Omega_1\times(0,1)\times\Omega_2}\frac{(m+n+1)!\ t^n(1-t)^m dxdtdy}{((1+\langle y, \eta\rangle)+t(\tau'(1+\langle x, \xi\rangle))-(1+\langle y, \eta\rangle))}\ .$$

Using the identity

$$\int_0^1 \frac{t^{\alpha-1}(1-t)^{\beta-1}}{(t+c)^{\alpha+\beta}}dt=B(\alpha, \beta)(1+c)^{-\alpha}c^{-\beta}$$

and first integrating on t we come to

$$\chi_{\Omega^*}=\tau'^{-n-1}\chi_{\Omega_1^*}(\xi)\chi_{\Omega_2^*}(\eta)\ .$$

Substituting this identity to the defining integral of k_Ω, we have the formula for k_Ω. As for u_Ω see [14].

§2. Projectively invariant metrics on a bounded convex domain.

In this section we define Riemannian metrics making use of functions χ_Ω and k_Ω and discuss some elementary properties.

Let Ω be a convex domain containing no straight line and χ and k be functions defined in §1.

DEFINITION 2. We will define symmetric two forms ω and κ by

$$\begin{aligned}\omega&=-\chi^{1/n+1}d^2(\chi^{-1/n+1})\ ,\\ \kappa&=-k^{1/2n+2}d^2(k^{-1/2n+2})\ .\end{aligned}\tag{2.1}$$

PROPOSITION 9. *Both ω and κ are positive definite and define Riemannian metrics on Ω.*

PROOF. We have set $l=1+\langle x,\xi\rangle$. By the definition of χ we see for $a\in \boldsymbol{R}^n=T_x\Omega$,

$$\langle \operatorname{grad}\chi, a\rangle=-\int(n+1)!\ \langle\xi,a\rangle l^{-n-2}d\xi\ ,$$

and

$$\langle d^2\chi, a\times a\rangle=\int(n+2)!\ \langle\xi,a\rangle^2 l^{-n-3}d\xi\ .$$

Since $\omega=(1/(n+1))\chi^{-2}\{\chi d^2\chi-(n+2)(n+1)^{-1}\operatorname{grad}\chi\times\operatorname{grad}\chi\}$, we have only to say the part $\{\ \}$ is positive. But this is equal to

$$n!\ (n+2)!\left(\int l^{-n-1}d\xi\int\langle\xi,a\rangle^2 l^{-n-3}d\xi-\left(\int\langle\xi,a\rangle l^{-n-2}d\xi\right)^2\right)$$

which is non-negative by Schwarz inequality. This is in fact positive because l^{-n-1} and $\langle\xi,a\rangle^2 l^{-n-3}$ are not proportional. The case κ is treated in the same way.

EXAMPLE 9. $\Omega=\{|x|<R\}$. Then, consulting Example 4, we have $\omega=\kappa=-(1/u)d^2u=$ the Hilbert metric of Ω.

In order to show the completeness we prepare

PROPOSITION 10. *Let Ω_1 and Ω_2 be convex domains containing no* *straight lines. Assume $\Omega_1 \subset \Omega_2$. Then for $x \in \Omega_1$,*

$$\chi_{\Omega_1}(x)\omega_{\Omega_1}(x) \geqq \chi_{\Omega_2}(x)\omega_{\Omega_2}(x)$$

$$k_{\Omega_1}(x)\kappa_{\Omega_1}(x) \geqq k_{\Omega_2}(x)\kappa_{\Omega_2}(x) \ .$$

PROOF. We will set $f=l^{-(n+1)/2}$ and $g=\langle \xi, a \rangle l^{-(n+3)/2}$. Define, for $x \in \Omega_1$

$$P(t)=\int_{\Omega_1^*}(tf+g)^2 d\xi \ ,$$

$$Q(t)=\int_{\Omega_2^*}(tf+g)^2 d\xi \ .$$

Then obviously $P(t) \geqq Q(t) \geqq 0$. Hence $\min P(t) \geqq \min Q(t)$. But, using notations in proof of the previous proposition, we have

$$\min P(t)=\frac{n+1}{n!\,(n+2)!}\chi_{\Omega_1}(x)\omega_{\Omega_1}(x)(a, a)$$

and a similar equality for $Q(t)$. This implies the proposition. The proof for κ is similar by the use of Proposition 2.

THEOREM 1. *Let Ω be a strictly convex bounded domain with C^2-boundary. Then the metrics ω and κ are complete.*

PROOF. For every boundary point p we choose a circumscribing ball B_p which contains Ω and tangents to $\partial\Omega$ only at p. Since the boundary is of class C^2 and strictly convex, we may assume that the ball B_p is chosen so that its radius is independent of p; say R. Now fix p and set $B=B_p$. Then, by the previous proposition, we first have

$$\chi_\Omega \omega_\Omega \geqq \chi_B \omega_B$$

on Ω. Let l be the inner normal at p. Then Proposition 2 implies that there exist constants c and c_1 independent of p such that

$$\chi_B(q)/\chi_\Omega(q) \geqq c$$

for any point q on the line l and with $d(p, q) < c_1$. Next take an euclidean coordinate system $(x^1, \cdots, x^{n-1}, x)$ at p so that l becomes the x-axis. Then an easy check using the explicit form of ω_B shows that

$$\omega_B=\left(\frac{1}{4x^2}+0\left(\frac{1}{x}\right)\right)dx^2+\sum\left(\frac{1}{2Rx}+0\left(\frac{1}{x}\right)\right)dx^i dx^i$$

on l near p. Here note that the coefficients of $0(1/x)$ depends only on the radius R. Hence letting $y(q)=d(q, \partial\Omega)$ for $q\in\Omega$ and making use of above estimates we have

$$\omega_\Omega \geqq \frac{cdy^2}{8y^2}$$

near the boundary. From this estimate follows the completeness of ω_Ω immediately. The proof for κ_Ω is carried out in the same way.

The next question is to compare these metrics with the Blaschke metric. The answer in a special case is

THEOREM 2. (a) *The metrics ω and κ are projectively invariant.* (b) *Assume Ω is projectively homogeneous. Then $\omega=\kappa=-(1/u)d^2u$.*

PROOF. Let $A=\begin{pmatrix} a_i^j & a^j \\ a_i & a \end{pmatrix}$ be a projective transformation as before. Write $m(x)=a_ix^i+a$. Proposition 3 shows $\chi_\Omega(x)^p=(\det A)^p m(x)\chi_{A\Omega}(Ax)^p$ and $k_\Omega(x)^{p/2}=(\det A)^p m(x)k_{A\Omega}(Ax)^{p/2}$ where $p=-1/n+1$. Assume two functions f and g are related as $f(x)=m(x)g(y)$ where $y=Ax$. Then calculations show $a_i\partial y^k/\partial x^j+a_j\partial y^k/\partial x^i+m\partial^2 y^k/\partial x^i\partial x^j=0$ and, hence, $f_{ij}dx^idx^j=mg_{ij}(y)dy^idy^j$. This implies $-(1/f)d^2f=-(1/g)d^2g$. Therefore we have the projective invariance of ω and κ. Since the solution u_Ω of (#) also satisfies $u_\Omega(x)=(\det A)^p m(x)u_{A\Omega}(Ax)$, which is shown in [13], we have, assuming Ω is projectively homogeneous, the equalities $u(x)=Cte.\,\chi(x)^p=Cte.\,k(x)^{p/2}$. Hence the assertion (b).

EXAMPLE 10. Let Ω be a simplex in $\boldsymbol{R}^n$. The projective transformation group of Ω is abelian and isomorphic to $\boldsymbol{R}^n$. It acts simply transitively. Hence the metric is complete and flat.

In [17] Vinberg considered the metric $-d^2\log\chi_V$ for a non-degenerate convex cone V. When $V=V_\Omega$, there is a relation between this metric and ω_Ω.

Let the coordinate of V_Ω be (y, t) so that $x=y/t\in\Omega$. Then by (1.4) we have

$$\chi_V(y, t)=t^{-n-1}\chi_\Omega(y/t)\ .$$

By a straightforward computation we have

$$-(n+1)d^2\log\chi_V(y, t)=\omega_\Omega(x)+(d\log\chi_V)^2\ , \tag{2.2}$$

(see the computation in Appendix A). Now fix a positive constant c and define a mapping $\iota_c\colon\Omega\to V$ by $\iota_c(x)=$the unique point in V such that $t^{-n-1}\chi_\Omega(x)=c$. Then we have

PROPOSITION 11. *The mapping* ι_c: $(\Omega, (1/(n+1))\omega_\Omega) \to (V, -d^2 \log \chi_V)$ *is isometric. The image of* ι_c *is a level hypersurface of* χ_V.

REMARK 3. The level hypersurface of χ_V is written as $t=\rho(y)$. Let v be the Legendre transform of ρ with respect to y: $v=\rho-y^i\partial\rho/\partial y^i$. Then v can be regarded as a function of $\eta_i=\partial\rho/\partial y^i$. We put $\omega_0=-(1/v)d^2v$. Then it is immediate to see $\omega=\omega_0$ rewriting ω_0 in terms of $x=y/t$.

REMARK 4. Making use of the p-th characteristic functions, we can define metrics analogously and prove Theorem 1. These metrics are shown to be *affinely* invariant, but Theorem 2 does not hold generally.

PROBLEM 1. It is necessary to compute the curvature tensor of ω and κ. With consideration of the boundary behaviour of functions χ and k discussed in §1 and §3, it is plausible to prove that the curvature tensor tends to that of negative constant curvature at the boundary. As for the related result see Appendix B of [15].

§3. Boundary behaviour of the characteristic function and the kernel function.

The aim of this section is to find the asymptotic form of functions χ and k near the boundary in the case that the boundary hypersurface is strictly convex and smooth. The result is Theorems 4 and 5.

Throughout this section, we suppose that Ω is a strictly convex bounded domain with smooth boundary which contains the origin.

Recall the support function $H_\Omega(\xi)=\sup\{\langle x, \xi\rangle; x\in\Omega\}$. Since Ω is strictly convex, there exists for any ξ a unique point $y(\xi)\in\partial\Omega$ with $H_\Omega(\xi)=\langle y(\xi), \xi\rangle$. Let g: $\partial\Omega\to S^{n-1}$ be the euclidean Gauss map defined by $g(y)=$the unit outward normal vector at $y\in\partial\Omega$. Then it is easy to see

$$y(\xi)=g^{-1}(\xi/|\xi|) .$$

Hence $H_\Omega(\xi)$ is smooth on $\boldsymbol{R}^n-\{0\}$. From this fact and the remark that $\Omega^*=\{\xi\in\boldsymbol{R}^n; H_\Omega(-\xi)<1\}$, we have

PROPOSITION 12. *The boundary of* Ω^* *is smooth and equal to*

$$\partial\Omega^*=\{\xi\in\boldsymbol{R}^n; \langle y(-\xi), \xi\rangle=-1\} .$$

We will extend the *-mapping to $\partial\Omega$: for $y\in\partial\Omega$ put $y^*=\lim_{x\to y} x^*$ which is well-defined by Proposition 7. Then

PROPOSITION 13. *The map* $\xi\to y(\xi)$ *is the inverse of the *-mapping* *at the boundary.*

With these preparations we shall proceed to find the asymptotic behaviour of the characteristic function near the boundary. Fix a boundary point $y\in\partial\Omega$. Set $x=ky\in\Omega$ for $0\leqq k\leqq 1$. Assume $d(x, y)=d(x, \partial\Omega)$ for k near 1. Put $l_{\cdot}(\xi)=1+\langle\cdot, \xi\rangle$ as before. Then

$$l_x(\xi)=kl_y(\xi)+(1-k)\ . \tag{3.1}$$

Choose coordinate (x^i) so as $x^1=\cdots=x^{n-1}=0$ at y and $y^n:=x^n(y)>0$. Then $l_y(\xi)=1+y^n\xi_n$. Denote $\eta=y^*\in\partial\Omega^*$. By definition

$$\eta_n=-1/y^n\ . \tag{3.2}$$

Now write $l(\xi)=l_y(\xi)$ and set $\sigma=-\eta_n d\xi_1\cdots d\xi_{n-1}$. Then

$$d\xi=\sigma\wedge dl\ . \tag{3.3}$$

When ξ varies in Ω^*, then l varies from 0 to some positive constant $b=b(y)$ depending on y. With these notations, we have

$$\begin{aligned}\chi_\Omega(x)&=\int_{\Omega^*} n!\ l_x(\xi)^{-n-1}d\xi\\ &=\int_0^b n!\ (kt+(1-k))^{-n-1}dt\int_{\Omega^*\cap\{l=t\}}\sigma\ .\end{aligned}$$

Let us first estimate

$$A(t):=\int_{\Omega^*\cap\{l=t\}}\sigma$$

when t is near zero. Choose $\xi'=(\xi_1, \cdots, \xi_{n-1})$ so as $\xi_i=0$ at η. Then the boundary $\partial\Omega^*$ near η can be written as

$$\xi_n=\eta_n+\sum a_{ij}\xi_i\xi_j+\sum_{2<|\alpha|<N} a_\alpha\xi^\alpha+0(|\xi'|^N)\ .$$

Here N is a certain integer which will be fixed later. The range of indices $i, j, \cdots$ is from 1 to $n-1$ and α is a multiindex with the convention $\xi^\alpha=\xi_1^{i_1}\cdots\xi_{n-1}^{i_{n-1}}$ for $\alpha=(i_1, \cdots, i_{n-1})$. After the change of coordinates by an orthogonal transformation on ξ' we may set

$$\xi_n=\eta_n+\sum \xi_i^2/a_i^2+\sum_{|\alpha|<N} a_\alpha\xi^\alpha+0(|\xi'|^N) \tag{3.4}$$

or equivalently,

$$l=y^n(\sum \xi_i^2/a_i^2+\sum_{|\alpha|<N} a_\alpha\xi^\alpha+0(|\xi'|^N))\ . \tag{3.4$'$}$$

Let (r, θ) be the polar coordinate system of $\xi' : \xi_i = rf_i(\theta)$ and $d\xi' = d\xi_1 \cdots d\xi_{n-1} = r^{n-2} dr dS(\theta)$, where $dS(\theta)$ is the area element of the unit sphere. In this coordinate system the section of $\partial\Omega^*$ by a hyperplane $\{l = y^n \rho^2\}$ can be written as

$$\xi_i = a_i r(\theta) f_i(\theta)$$

where

$$r(\theta) = \rho(1 + \sum \varepsilon_p \rho^p + 0(\rho^{N-2})) . \tag{3.5}$$

Define δ_p by

$$r^{n-1} = \rho^{n-1}(1 + \sum \delta_p \rho^p + 0(\rho^{N-2})) . \tag{3.6}$$

LEMMA 1. *δ_p is a polynomial of f_i with a_α as coefficients. The degree modulo 2 of each monomial is equal to p modulo 2.*

PROOF. It is enough to see this fact for ε_p. We proceed by induction on p. Since

$$\varepsilon_1 = -\frac{1}{2} \sum a_{ijk} a_i a_j a_k f_i f_j f_k ,$$

the Lemma is true for $p=1$. Let $p>1$. By the substitution of (3.5) into (3.4), we can see ε_p is a sum of monomials like

$$\varepsilon_{p_1} \cdots \varepsilon_{p_j} a_\alpha f^\alpha a^\alpha$$

for $1 \leqq p_i < p$, $|\alpha| = k$ and $p_1 + \cdots + p_j = p + 2 - k$. Hence the order modulo 2 of this monomial with respect to f_i is equal to $p_1 + \cdots + p_j + k(\text{mod. } 2) = p(\text{mod. } 2)$.

Let now θ° be the antipodal point of θ. Then $f_i(\theta^\circ) = -f_i(\theta)$ and $dS(\theta^\circ) = dS(\theta)$. This Lemma implies $\int \delta_p dS(\theta) = 0$ for odd p. Hence we have

$$\int_{\Omega^* \cap \{l = y^n \rho^2\}} d\xi' = (\prod a_i / n-1) \rho^{n-1} (\omega_{n-2} + \sum \gamma_j \rho^{2j} + 0(\rho^{N-2})) \tag{3.7}$$

where $\gamma_j = \int \delta_{2j} dS(\theta)$. Set $\gamma = 2^{n-1} / \prod a_i^2$ which is the Gauss curvature of $\partial\Omega^*$ at η and put

$$c = 2^{(n-1)/2} \omega_{n-2} \gamma^{-1/2} / (n-1) = \omega_{n-2} \prod a_i / (n-1) . \tag{3.8}$$

Then rewriting (3.7), we get

$$A(t)=(c/y^n)(t/y^n)^{(n-1)/2}+\sum_{j=1}^{N-1}(\gamma'_j/y^n)(t/y^n)^{j+(n-1)/2} \\ +0(t^{N+(n-1)/2}) \tag{3.9}$$

for another N, where

$$\gamma'_j=(\prod a_i/n-1)\gamma_j \ . \tag{3.10}$$

Let us recall

$$\chi_\Omega(x)=n!\int_0^b (kt+1-k)^{-n-1}A(t)dt \ .$$

Putting $\varepsilon=(1-k)/k$ and $t=\varepsilon s$, we have

$$\int_0^b (kt+1-k)^{-n-1}t^m dt=k^{-n-1}\varepsilon^{m-n}\int_0^{b/\varepsilon}(s+1)^{-n-1}s^m ds \ .$$

Note that this is of order ε^{m-n} as $\varepsilon\to 0$ for $m<n$. Moreover this integral over $[b, \infty)$ is finite independent of ε for $m<n$. Hence, by the formula $B(m+1, n-m)=\int_0^\infty (s+1)^{-n-1}s^m ds$, we have

$$\chi_\Omega(x)=n!\,cB\Big(\frac{n+1}{2}, \frac{n+1}{2}\Big)(y^n)^{-(n+1)/2}k^{-n-1}\varepsilon^{-(n+1)/2} \\ +\sum_{j<(n+1)/2} n!\,\gamma'_j B\Big(j+\frac{n+1}{2}, \frac{n+1}{2}-j\Big)(y^n)^{-j-(n+1)/2}k^{-n-1}\varepsilon^{j-(n+1)/2} \\ +0(A(\varepsilon)) \ , \tag{3.11}$$

where

$$A(\varepsilon)=\begin{cases}1 & \text{for even } n \\ \log\varepsilon & \text{for odd } n \ .\end{cases} \tag{3.12}$$

Now we know $d(x, y)=d(ky, y)=(1-k)y^n$. Therefore we have proved

THEOREM 3. *For x near $\partial\Omega$, choose $y=y(x)$ so that $d(x, y)=d(x, \partial\Omega)$. Then*

$$\chi_\Omega(x)=d_n\gamma(y)^{-1/2}d(x, \partial\Omega)^{-(n+1)/2} \\ +\sum_{j=1}^{[n/2]}\delta_j(y)d(x, \partial\Omega)^{j-(n+1)/2}+0(A(d(x, \partial\Omega))) \ , \tag{3.13}$$

where $d_n=n!\,2^{(n-1)/2}\omega_{n-2}B((n+1)/2, (n+1)/2)/(n-1)$ and $\gamma(y)$ is the Gauss curvature of $\partial\Omega^*$ at y^*. Moreover δ_j's are constants determined locally by the geometrical data of $\partial\Omega^*$ at y^*.

REMARK 5. We have

(3.14) $$\delta_1 = n!\left(B\left(\frac{n+3}{2}, \frac{n-1}{2}\right)\gamma_1' + \frac{n+1}{2}cB\left(\frac{n+1}{2}, \frac{n+1}{2}\right)\right).$$

The constant d_n is equal to $\pi^{(n-1)/2}(n-1)!!$ (n: odd) or $\pi^{n/2}2^{-1/2}(n-1)!!$ (n: even).

We shall next study the first term of the above expression. Let G be the Gauss curvature of the boundary $\partial\Omega$ at y. Then

PROPOSITION 14. $G = \gamma^{-1}$.

To prove this identity, write $\partial\Omega$ near y as

$$x^n = f(x^1, \cdots, x^{n-1}), \qquad y = (0, \cdots, 0, y^n).$$

Then the Gauss map g is defined by

$$g(x) = (f_1, \cdots, f_{n-1}, -1)/a\ ; \quad a = \left(1 + \sum_1^{n-1} f_i^2\right)^{1/2}.$$

Setting $\xi = x^*$, we have $g(x) = \xi/|\xi|$. Hence,

$$\xi_n = -|\xi|/a \quad \text{and} \quad \xi_i = f_i|\xi|/a.$$

This implies $f_i(x) = -\xi_i/\xi_n$ for $1 \leqq i \leqq n-1$. Let u be the Legendre transform of f:

$$u = f - \sum x^i f_i.$$

Then $u = -1/\xi_n$ because $1 + \langle x, \xi\rangle = 0$ by definition. Therefore the *-mapping is given by the formula

(3.15) $$\xi_n = -1/u, \qquad \xi_i = f_i/u.$$

Here note that

(3.16) $$f_i(0) = 0 \quad \text{and} \quad \xi_i(0) = 0.$$

The jacobian of this mapping restricted to $\partial\Omega$ is

$$\partial\xi_j/\partial x^i = f_{ij}/u - f_j u_i/u^2.$$

Hence $(\partial\xi_j/\partial x^i(0)) < 0$ by the concavity of f and (3.16). Next regard ξ_n on $\partial\Omega^*$ as the function of ξ': $\xi_n = v(\xi_1, \cdots, \xi_{n-1})$. Then

$$\partial v/\partial\xi_i = \frac{1}{u^2}\partial u/\partial\xi_i = -\frac{1}{u^2}\sum x^k f_{kj}\partial x^j/\partial\xi_i.$$

From this we have

$$\partial v/\partial \xi_i(0)=0 . \tag{3.17}$$

Taking one more derivation we have

$$\partial^2 v/\partial \xi_i \partial \xi_j(0)=-f^{ij}(0) , \tag{3.18}$$

where (f^{ij}) is the inverse matrix of (f_{ij}). But by (3.16) and (3.17) we have

$$G=\det(-f_{ij}) \quad \text{and} \quad \gamma=\det(\partial^2 v/\partial \xi_i \partial \xi_j) .$$

Then (3.18) implies Proposition 14.

The second step is to rewrite the expansion in terms of a defining function of the boundary. Let ϕ be a strictly convex defining function of $\partial\Omega$ near y: $\partial\Omega=\{\phi=0\}$ and $\Omega=\{\phi<0\}$. With coordinates $(x^1, \cdots, x^n)$ as before, taking derivatives we have

$$\phi_i(y)=0 , \qquad f_{ij}(0)=-\phi_{ij}/\phi_n(y) . \tag{3.19}$$

Hence $G(y)=\det(\phi_{ij}/\phi_n)$. At the same time we have

$$\phi(x)=-d(x, y)\phi_n(y)+0(d(x, y)^2)$$

near y. Hence,

$$\sqrt{G}\, d(x, y)^{-(n+1)/2}=(\phi_n^2 \det \phi_{ij})^{1/2}((-\phi)^{-(n+1)/2}+0(\phi^{-(n-1)/2})) . \tag{3.20}$$

Let us introduce an operator J by

$$J(\phi)=-\det\begin{pmatrix}\phi_{\alpha\beta} & \phi_\beta \\ \phi_\alpha & 2\phi\end{pmatrix} ; \quad 1\leqq\alpha,\ \beta\leqq n . \tag{3.21}$$

Then in the present coordinate

$$J(\phi)(y)=\phi_n^2 \det \phi_{ij}(0) .$$

Therefore, getting above discussions together, we have

THEOREM 4. *The characteristic function χ has the expansion*

$$\chi_\Omega(x)=d_n J(\phi)^{1/2}(-\phi)^{-(n+1)/2}+\sum_{j=1}^{[n/2]} \varepsilon_j(-\phi)^{j-(n+1)/2}+0(A(\phi)) , \tag{3.22}$$

near $\partial\Omega$, where A is the function defined by (3.12).

PROBLEM 2. ε_j's are functions on $\bar{\Omega}$ but not defined uniquely in this setting. The value of ε_j at $\partial\Omega$ also depends on the choice of ϕ. However, in consideration of the projective invariance of χ, it is highly desirable to determine ε_j uniquely using affine or projective invariants of the boundary hypersurface $\partial\Omega$. In the next section we will compute the value δ_1 in (3.13) using the boundary data.

REMARK 6. The differential operator $J(\phi)$ is just a real analogue of the operator $I(v)$ of complex Monge-Ampère type, which is introduced and studied by C. Fefferman in [9]. Here

$$I(v)=\pm\det\begin{pmatrix} v_{i\bar{j}} & v_{\bar{j}} \\ v_i & v \end{pmatrix},$$

where derivations are with respect to complex variables. The important fact which we should mention here is that the equation (♯) in Introduction is transformed to

$$(3.23)\qquad J(\phi)=1 \quad \text{on} \quad \Omega\,, \qquad \phi=0 \quad \text{on} \quad \partial\Omega\,,$$

setting $u=-(-\phi/2)^{1/2}$ and vice versa. The discussion which is analogous to that in [9] can be carried out. We would like to treat the operator $J(\phi)$ in another paper.

Let us next consider the kernel function. Calculation is carried out similarly. Let $y\in\partial\Omega$ and set $\eta=y^*\in\partial\Omega^*$. Choose coordinates $x=(x', x^n)$ and $\xi=(\xi', \xi_n)$ as before. We want to estimate $k_\Omega(x)$ for $x=ky$ as k tends to 1. Assume, for simplicity, $y^n=1$. We have set $l=1+\xi_n y^n=1+\xi_n$. $\partial\Omega^*$ is written as

$$l(\xi)=\sum \xi_i^2/a_i^2+0(|\xi'|^3)$$

near η. Let ξ be one point on $\{l=t\}$. Then $\xi_n=t-1$. For this ξ we can pick up a boundary point $\zeta\in\partial\Omega^*$ so that $d(\xi, \partial\Omega^*)=d(\xi, \zeta)$. If ξ is sufficiently near η, then ζ is unique and satisfies

$$\zeta_i-\xi_i=(t-l(\zeta'))\partial l/\partial\xi_i(\zeta')\,.$$

Hence we have

$$d(\xi, \zeta)=|t-l(\zeta')|(1+0(|\zeta'|^2))$$

and

$$d\xi'=(1+0(|\zeta'|^2))+0(t-l(\zeta'))d\zeta'\,.$$

Let $\gamma(\zeta)$ denote the Gauss curvature of $\partial\Omega^*$ at ζ. Then by Theorem 3, we know

$$\chi_{\Omega^*}(\xi)=d_n\sqrt{\gamma(\zeta)}\, d(\xi,\zeta)^{-(n+1)/2}+0(d(\xi,\zeta)^{-(n-1)/2})\ .$$

Consequently, for sufficiently small t and hence for small values of ξ' and ζ', we have

$$\chi_{\Omega^*}(\xi)^{-1}d\xi'=\frac{1}{d_n\sqrt{\gamma(0)}}|t-l(\zeta')|^{(n+1)/2}(1+0(|\zeta'|^2))+0(|t-l(\zeta')|)d\zeta'\ .$$

Now use the polar coordinate $\zeta_i=a_i r f_i(\theta)$. Then $d\zeta'=\prod a_i r^{n-2}drdS(\theta)$. When ξ varies on $\{l=t\}\cap\Omega^*$, then ζ' varies on $\{l(\zeta')<t\}$ and r moves from zero to $r(\theta)$, where

$$r(\theta)=\sqrt{t}+0(t)\ .$$

Hence we have

$$\int_{\{l=t\}\cap\Omega^*}\frac{d\xi'}{\chi}=\int\prod a_i dS(\theta)\int_0^{r(\theta)}\frac{1}{d_n\sqrt{\gamma(0)}}(t-l(\zeta'))^{(n+1)/2}(1+0(t))r^{n-2}dr\ .$$

Let us change the variable r to $s=r^2$. Then $l(\zeta')=s+0(s^{3/2})$ and $s(\theta)=t+0(t^{3/2})$, and we have

$$\int\frac{d\xi'}{\chi}=\frac{e_n}{\gamma(0)}t^n(1+0(t^{1/2}))\ , \tag{3.24}$$

where

$$e_n=\sqrt{\gamma(0)}\prod a_i\omega_{n-2}B\Big(\frac{n-1}{2},\frac{n+3}{2}\Big)\Big/2d_n\ .$$

But using equalities for d_n and $\gamma(0)$ we know

$$e_n=\frac{n+1}{2(n!)}\ . \tag{3.25}$$

Thus we have, by definition,

$$k_\Omega(x)=\frac{(n+1)(2n+1)!}{2(n!)\gamma(0)}\int_0(kt+1-k)^{-2n-2}(t^n+0(t^{n+1/2}))dt\ .$$

From this we get by the same reasoning as before,

$$k_\Omega(x)=\frac{(n+1)!}{2\gamma(0)}d(x,y)^{-n-1}+0(d(x,y)^{-n-1/2})\ .$$

However we have already rewritten $\gamma(0)^{-1/2}d(x, y)^{-(n+1)/2}$ in terms of the defining function ϕ of the domain, (3.20). We finally have

THEOREM 5. *Let $\Omega=\{\phi<0\}$ be a strictly convex bounded domain with smooth boundary in $\boldsymbol{R}^n$. Then the kernel function has an expansion*

$$k_\Omega(x)=\frac{(n+1)!}{2}J(\phi)(-\phi)^{-n-1}+0(\phi^{-n-1/2})$$

near the boundary.

REMARK 7. More careful arguments show that $k_\Omega(x)$ has an expansion $((n+1)!/2)J(\phi)(-\phi)^{-n-1}+\sum_{1\leq j\leq n}\bar{\varepsilon}_j(-\phi)^{-n-1+j}+0(\log(-\phi))$, where $\bar{\varepsilon}_j|_{\partial\Omega}$ is determined locally.

PROBLEM 3. We will continue Problem 2. Let us recall the function u defined by (#). In [13], Loewner and Nirenberg showed in dimension 2 that, for a smooth and strictly convex bounded domain Ω,

$$c_1d(x, \partial\Omega)\leqq u^2(x)\leqq c_2d(x, \partial\Omega)\ , \quad c_3\leqq|\text{grad}\, u^2|\leqq c_4$$

for some positive constants c_i. These are true for any dimension by the maximum principle for the equation $\det u_{ij}=(-u)^{-n-1}$ (e.g. [5]). Hence it is reasonable, referring the asymptotic expansion of the Bergman kernel due to Fefferman, to expect u^2 can be a defining function of the domain at the certain order of differentiability. So, besides the study of the boundary regularity of the function u, it is desirable to expand k with respect to u.

§4. Explicit value of δ_1.

We compute in this section the value δ_1 which is the coefficients of the second term of the expansion (3.13) of the characteristic function. The computation is simple in principle but is complicated. So we will write here only the sketch.

In order to do calculations, we assume the boundary $\partial\Omega$ is written as

$$x^n=f(x^1, \cdots, x^{n-1})\ , \qquad y=(0, \cdots, 0, 1)$$

where

$$f(x)=1-\frac{1}{2}x^ix^i+\frac{1}{6}a_{ijk}x^ix^jx^k+\frac{1}{24}a_{ijkl}x^ix^jx^kx^l+0(|x|^5)\ . \tag{4.1}$$

We use the convention that repeated indices are summed from 1 to $n-1$. Then by a somewhat lengthy calculation, the dual hypersurface $\partial\Omega^*$ is given by

$$\xi_n=-1+\frac{1}{2}\xi_i\xi_i+\frac{1}{6}b_{ijk}\xi_i\xi_j\xi_k+\frac{1}{24}b_{ijkl}\xi_i\xi_j\xi_k\xi_l+0(|\xi|^5)\ , \tag{4.2}$$

where

$$\begin{aligned} b_{ijk}&=-a_{ijk}\ ,\\ b_{ijkl}&=a_{ijkl}+2(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\\ &\quad+(a_{ijm}a_{klm}+a_{ikm}a_{jlm}+a_{ilm}a_{jkm})\ ; \end{aligned} \tag{4.3}$$

Using this expression we can compute the coefficients ε_p defined in (3.5) as follows. Set

$$b_3=\frac{1}{6}b_{ijk}f_if_jf_k\ ;\qquad b_4=\frac{1}{24}b_{ijkl}f_if_jf_kf_l\ , \tag{4.4}$$

where f_i's are angular coordinates. Then we have

$$\varepsilon_1=-\sqrt{2}\,b_3\ ,\qquad \varepsilon_2=5b_3^2-2b_4\ . \tag{4.5}$$

Repeating the calculation in §3, we have

$$\begin{aligned} A(t)=&\frac{2^{(n-1)/2}}{n-1}\omega_{n-2}t^{(n-1)/2}+2^{(n-1)/2}t^{(n+1)/2}\int\left(\varepsilon_2+\frac{n-2}{2}\varepsilon_1^2\right)dS\\ &+0(t^{(n+3)/2})\ . \end{aligned} \tag{4.6}$$

Here note that

$$\varepsilon_2+\frac{n-2}{2}\varepsilon_1^2=(n+3)b_3^2-2b_4\ .$$

Let b = the volume of the unit $(n-1)$-ball. Then we know the following formulae.

$$\int f_i^2f_j^2dS(\theta)=\begin{cases}\dfrac{b}{n+1} & i\neq j\\[2mm] \dfrac{3b}{n+1} & i=j\end{cases} \tag{4.7}$$

$$(4.7)' \qquad \int f_i^2 f_j^2 f_k^2 dS(\theta) = \begin{cases} \dfrac{b}{(n+3)(n+1)} & i, j, k \neq \\ \dfrac{3b}{(n+3)(n+1)} & i=j\neq k \\ \dfrac{15b}{(n+3)(n+1)} & i=j=k\,. \end{cases}$$

Hence we have

$$\begin{aligned} 36\int b_3^2 dS &= \frac{b}{(n+3)(n+1)}\{(9 \sum_{i,j,k\neq} b_{iij}b_{jkk} + 6 \sum_{i,j,k\neq} b_{ijk}b_{ijk}) \\ &\quad \times 3(6 \sum_{i\neq j} b_{iii}b_{ijj} + 9 \sum_{i\neq j} b_{iij}b_{iij}) + 15 \sum b_{iii}b_{iii}\} \\ &= \frac{b}{(n+3)(n+1)}(9b_{iij}b_{jkk} + 6b_{ijk}b_{ijk}) \end{aligned}$$

and

$$24\int b_4 dS = \frac{3b}{n+1} b_{iikk}\,.$$

Consequently the coefficient of the second term of $A(t)$ is

$$\gamma_1' = \frac{2^{(n-1)/2}b}{12(n+1)}(3b_{iij}b_{jkk} + 2b_{ijk}b_{ijk} - 3b_{iikk})\,.$$

Using identities (4.3), we get

$$(4.8) \qquad \gamma_1' = -\frac{2^{(n-1)/2}b}{12(n+1)}(3a_{iikk} + 4a_{ijk}a_{ijk} + 6(n^2-1))\,.$$

Inserting this identity into the formula (3.14), we get

$$(4.9) \qquad \delta_1 = -\frac{2^{(n-3)/2}}{12}\pi^{(n-1)/2}\Gamma\left(\frac{n-1}{2}\right)(3a_{iikk} + 4a_{ijk}a_{ijk})\,.$$

Let us next see the meaning of the quantity Q defined by

$$(4.10) \qquad Q = 3a_{iikk} + 4a_{ijk}a_{ijk}\,.$$

For this purpose we will need the affine unimodular invariants of the hypersurface. We have given in Appendix B some of definitions and calculations. The first one we need is the Fubini-Pick invariant F defined by

$$F = a_{ijk}a_{ijk} - \frac{3}{n+1}a_{iij}a_{jkk}\,. \tag{4.11}$$

The second one is the affine scalar curvature R defined by

$$R = \begin{cases} \frac{n-2}{n+1}a_{iijj} + \left(\frac{1}{4} - \frac{n-2}{n+1}\right)a_{ijk}a_{ijk} - \left(\frac{3}{4(n+1)} + \frac{(n-1)^2-1}{(n+1)^2}\right)a_{iij}a_{jkk}\,, & n \geqq 3 \\ a_{1111} & n=2\,. \end{cases} \tag{4.12}$$

The third one is the affine normal vector ν. We do not repeat its definition. But note that in the present case one has

$$\nu = \left(-\frac{1}{n+2}a_{ii1}, \cdots, -\frac{1}{n+2}a_{ii(n-1)}, 1\right) \tag{4.13}$$

at the point $(0, \cdots, 0, 1)$. On the other hand one can see that by certain unimodular transformation of the space $\boldsymbol{R}^n$, it is always possible to find a coordinate system (x^i) so that the defining function f has the property $a_{iik}=0$ for $1 \leqq k \leqq n-1$, namely so that the affine normal vector coincides with the euclidean normal vector at the point $(0, \cdots, 0, 1)$. Relative to this coordinate system, we have

$$\begin{aligned} &F = a_{ijk}a_{ijk} \\ &R = \frac{n-2}{n+1}a_{iijj} + \left(\frac{1}{4} - \frac{n-2}{n+1}\right)F \end{aligned} \tag{4.14}$$

for $n \geqq 3$. Hence

$$Q = \begin{cases} \frac{3(n+1)}{n-2}\left(R - \frac{F}{4}\right) + 7F & n \geqq 3 \\ 3R & n=2\,. \end{cases} \tag{4.15}$$

Now we have proved the following

THEOREM 6. *Fix $x_0 \in \Omega$ and $y_0 \in \partial\Omega$, Assume* (1) $d(x, y_0) = d(x, \partial\Omega)$ *for any point x on the line $\overline{x_0y_0}$,* (2) *the affine normal vector to $\partial\Omega$ at y_0 is equal to the euclidean normal vector and* (3) *the Gauss curvature of $\partial\Omega$ at y_0 is equal to* 1. *Then*

$$\begin{aligned} \chi_\Omega(x) = d_n d(x, \partial\Omega)^{-(n+1)/2} - \frac{2^{(n-3)/2}}{12}\pi^{(n-1)/2}\Gamma\left(\frac{n-1}{2}\right)Q d(x, \partial\Omega)^{-(n-1)/2} \\ + 0(B(d(x, \partial\Omega))) \end{aligned}$$

for $x \in \overline{x_0 y_0}$, *where* $B(t)=t^{-(n-3)/2}$ *for* $n \geqq 4$, $\log t$ *for* $n=3$ *and* 1 *for* $n=2$.

Appendix

A. The Einstein-Kähler metric on a tube domain and the Blaschke metric.

Let Ω be a bounded domain in $\boldsymbol{R}^n(x)$ and V_Ω be the cone over Ω given by $V_\Omega=\{(tx, t); x \in \Omega, t \in R^+\}$. Ω is identified with the set $\{t=1\}$. We denoted by u or u_Ω the unique negative convex solution of the equation (#). Associated with this we define a function v on V_Ω by

(A.1) $\quad v(tx, t)=-4b \log(-tu(x))+a; \; b=(n+1)/2(n+2)\,, \quad a=2b \log b\,.$

Let $D=V_\Omega+\sqrt{-1}\,\boldsymbol{R}^{n+1}$ be the tube domain over V_Ω. The aim of this Appendix A is to prove the following

THEOREM a. *The form* $i\partial\bar{\partial}v$ *defines a complete Einstein-Kähler metric on* D.

REMARK. Cheng and Yau have shown in [6] the unique existence of the complete Einstein-Kähler metric on domains in $\boldsymbol{C}^n$ belonging to a fairly large class; for example, C^2-weakly pseudoconvex bounded domains and tube domains of the form $B+\sqrt{-1}\,\boldsymbol{R}^n$, where B is bounded and convex. The existence proof of the solution is also due to them. Theorem a explains the relation between them in a special case.

Choose coordinates $(z^1, \cdots, z^{n+1})$ so that $\operatorname{Re} z^i=y^i$, $1 \leqq i \leqq n$, and $\operatorname{Re} z^{n+1}=t$. We have set $y^i=tx^i$. In the sequel $1 \leqq i, j \leqq n$ and $z=z^{n+1}$. Summation convention is used. Derivatives with respect to x^i are denoted simply with indices: $u_i=\partial u/\partial x^i$ and so on. Differentiating (A.1) we have

$$\begin{aligned}
&\partial^2 v/\partial z^i \partial \bar{z}^j=-b(\log u)_{ij}/t^2\\
&\partial^2 v/\partial z^i \partial \bar{z}=(b/t^2)(u_i/u+x^i(\log u)_{ij})\\
&\partial^2 v/\partial z \partial \bar{z}=(b/t^2)(1-2x^i u_i/u-x^i x^j(\log u)_{ij})\,.
\end{aligned}$$

These identities imply $i\partial\bar{\partial}v$ is positive definite and

$$\det \partial\bar{\partial}v=(-1)^n(b/t^2)^{n+1}u^{-n}\det u_{ij}\,.$$

On the other hand $e^{(n+2)v}=(-tu)^{-2(n+1)}e^{-(n+2)a}$. Since $a=(n+1)/(n+2)\times \log(n+1)/2(n+2)$ and $\det u_{ij}=(-u)^{-n-2}$ we have

$$\det \partial\bar{\partial} v = e^{(n+2)v} .$$

This shows $i\partial\bar{\partial}v$ is Einstein-Kähler.

Next we will see the completeness using the completeness of the Blaschke metric $-(1/u)d^2u$ ([4], [14]). Set $z^i = y^i + \sqrt{-1}w^i$ and $z = t + \sqrt{-1}s$. Since v is real we can devide $i\partial\bar{\partial}v$ into two parts: $i\partial\bar{\partial}v = ds_1^2 + ds_2^2$, ds_1^2 is containing only dy^i and ds_2^2 is containing only dw^i and ds, and both are positive semidefinite. Now suppose c is a divergent curve in D. We want to see the length of c is infinite. Let c_1 be the projection of c into V_Ω. Assume c_1 is bounded in V_Ω. Then $\operatorname{Im} c$ is unbounded and the coefficients of ds_2^2 are bounded since they are combinations of derivatives of u which are bounded near c_1. Hence the length of c is greater than the length of $\operatorname{Im} c$ relative to ds_2^2 which is infinite. Next assume c_1 is divergent. Let c_2 be the radial projection of c_1 into $\Omega = \{t=1\} \subset V_\Omega$. Calculations show

$$\text{(A.2)} \qquad \frac{1}{b} ds_1^2 = -\frac{u_{ij}}{u} dx^i dx^j + \left(\frac{u_i}{u} dx^i + \frac{dt}{t}\right)^2 .$$

If c_2 is divergent, then the length of c_2 is infinite by the completeness of $-u^{-1}d^2u$. If c_2 is bounded, then the t-coordinate of c_1 tends to 0 or ∞. Since $ds_1^2 \gtrsim b(dt/t)^2$, the length of c_1 is infinite in this case. Therefore in any case we have seen the length of c is infinite.

The equation (A.2) implies

COROLLARY b. *The hypersurface in V_Ω defined by $tu = -$constant with the induced metric is isometric to $(\Omega, -(b/u)d^2u)$.*

It would be a natural question to examine the Einstein-Kähler metric on a Siegel domain of the second kind in view of Theorem a. The result is Proposition d which we now describe. Let D be a domain in C^n. Suppose the existence of an Einstein-Kähler metric $i\partial\bar{\partial}v$, where v is a plurisubharmonic solution of the equation

$$\text{(A.3)} \qquad \det \partial\bar{\partial}v = e^{(N+1)v} \quad \text{on } D , \quad v = \infty \quad \text{on } \partial D .$$

By the uniqueness of a plurisubharmonic solution we know

LEMMA c. *Let A be an affine automorphism of D. Then*

$$v(Az) + (N+1)^{-1} \log |\operatorname{Jac} A|^2 = v(z) .$$

We will apply this lemma to a Siegel domain of the second kind. Let V be a non-degenerate convex cone in R^{n+1} and $F: C^m \times C^m \to C^{n+1}$ be a V-hermitian form, i.e. $F(w, w')$ is C-linear in w and $F(w, w')=\overline{F(w', w)}$, $F(w, w)\in \bar{V}$ and $F(w, w)=0$ if and only if $w=0$. Then the set $D=D(V, F)=\{(z, w)\in C^{n+1}\times C^{m}; \operatorname{Re} z-F(w, w)\in V\}$ is a Siegel domain of the second kind. The domain D has automorphisms τ_a, m_k and t_b defined by

$$\begin{aligned} &\tau_a(z, w)=(z+a, w) && a\in iR^{n+1} \\ &m_k(z, w)=(k^2 z, kw) && k\in R^{+} \\ &t_b(z, w)=(z+2F(w, b)+F(b, b), w+b) && b\in C^{m} . \end{aligned}$$

Since $\operatorname{Jac} t_b=1$, Lemma c shows $v(z, w)=v(z-F(w, w)), 0)$. $\operatorname{Jac}\tau_a=1$ implies v is independent of $\operatorname{Im} z$. Hence we can write $v(z, w)=U(\operatorname{Re} z-F(w, w))$ for some function U on the domain V. $\operatorname{Jac} m_k=k^{2n+2+m}$ shows that $U(\underline{y})=U(k\underline{y})+(2n+m+2)(n+m+2)^{-1}\log k$ for $\underline{y}\in V$. Set $U_1(\underline{y})=\exp(-(n+m+2)(2n+m+2)^{-1}U(\underline{y}))$. Then $U_1(k\underline{y})=kU_1(\underline{y})$. Now let Ω be a bounded hyperplane section of the cone V and choose a coordinate system $(y, t)\in R^{n+1}$ such that $V=\{(tx, t); x\in\Omega, t\in R^{+}\}$ as before. Then U_1 can be written as $U_1(y, t)=-tu(y/t)$ for a function $u(x)$ on Ω. All of the above imply

$$v(z, w)=-\left(\frac{2n+m+2}{n+m+2}\right)\log(-tu(y/t)) , \tag{A.4}$$

where $\operatorname{Re} z-F(w, w)=(y, t)$. The next thing to do is to rewrite the equation (A.3) in terms of u, which can be done routinely. Let us write the i-th component of F as $F(w, w')^i=a^i_{\alpha\bar\beta}w^\alpha\bar{w}^\beta$, $\overline{a^i_{\alpha\bar\beta}}=a^i_{\beta\bar\alpha}$. Then the result is

$$\begin{aligned} &\det u_{ij}=\left(\frac{n+m+2}{2n+m+2}\right)^{n+m+1}(-u)^{-n-2}\det(-u_i a^i_{\alpha\bar\beta}-(u-u_i x^i)a^{n+1}_{\alpha\bar\beta}) \quad \text{on } \Omega , \\ &u=0 \quad \text{on } \partial\Omega . \end{aligned} \tag{A.5}$$

Namely we have

PROPOSITION d. *The equation* (A.3) *on a Siegel domain* $D(V, F)$ *is reduced to the equation* (A.5) *on* Ω.

Hence the existence of a convex solution of (A.5), which seems to be not yet proved, implies the existence of the Einstein-Kähler metric on $D(V, F)$.

B. Affine invariants—Appendix to §4.

One can refer the book [1] for the affine geometry of surfaces in

R^3. To give a sketch of fundamental definitions in general dimension, we follow papers [3], [4], [10] and [7].

Let M be a hypersurface in R^{n+1}. Let $x: M \to R^{n+1}$ denote its embedding. R^{n+1} is equipped with the unimodular affine structure, especially the determinant $(\cdots)$. Let e_α, $1 \leqq \alpha \leqq n+1$, be a unimodular affine frame: $(e_1, \cdots, e_{n+1})=1$. Write

$$\text{(B.1)} \qquad \begin{aligned} dx &= \sum \omega^\alpha e_\alpha \\ de_\alpha &= \sum \omega_\alpha^\beta e_\beta \,. \end{aligned}$$

The structure equations are

$$\text{(B.2)} \qquad \begin{aligned} &\sum \omega_\alpha^\alpha = 0 \\ &d\omega^\alpha = \omega^\beta \wedge \omega_\beta^\alpha \\ &d\omega_\alpha^\beta = \omega_\alpha^\gamma \wedge \omega_\gamma^\beta \,. \end{aligned}$$

Choose $e_1, \cdots, e_n$ that are tangent to M. Then $\omega^{n+1}=0$ and $d\omega^{n+1}=\omega^i \wedge \omega_i^{n+1}=0$ along M. Here and later the range of indices is $1 \leqq i, j, \cdots \leqq n$. We have

$$\text{(B.3)} \qquad \omega_i^{n+1} = h_{ij}\omega^j$$

for some symmetric form h_{ij}. Now assume M is locally strongly convex so that the matrix $(h_{ij})>0$. Then, setting $H=\det h_{ij}$, we define

$$\text{(B.4)} \qquad II = H^{-1/n+2} h_{ij}\omega^i\omega^j \,.$$

This form is affinely invariant and defines a Riemannian structure on M, which is called the *affine metric* of M. We next choose e_{n+1} with the property that

$$(n+2)\omega_{n+1}^{n+1} + d\log H = 0 \,.$$

This is always possible, [8] p. 21, and we set

$$\text{(B.5)} \qquad \nu = H^{1/n+2} e_{n+1} \,.$$

This vector is also affinely invariant and is called the *affine normal vector*. With this choice of a frame, we have $d\omega_{n+1}^{n+1}=0$ and we can write

$$\text{(B.6)} \qquad \omega_{n+1}^i = -l^{ij}\omega_j^{n+1} \,.$$

for some symmetric l^{ij}. The quadratic form

$$\text{(B.7)} \qquad III = l_{ij}\omega^i\omega^j \;; \quad l_{ij} = h_{ik}l^{km}h_{mj} \,,$$

is called the *third fundamental form.* The scalar function

(B.8) $$L=\frac{1}{n}(\text{the trace of } III \text{ with respect to } II)$$

is called the *affine mean curvature.*

We next choose the frame so that $H=1$; see [10]. Hence $\omega_{n+1}^{n+1}=0$. Let us define a symmetric tensor h_{ijk} by

(B.9) $$h_{ijk}\omega^k=dh_{ij}-h_{ik}\omega_k^j-h_{jk}\omega_k^i\ .$$

Then the connection form $\bar{\omega}_i^j$ associated with the fundamental form II is

(B.10) $$\bar{\omega}_i^j=\omega_i^j+\frac{1}{2}h^{jk}h_{ikm}\omega^m\ ,$$

where (h^{jk}) is the inverse of (h_{ij}). Making use of this equality and above definitions we can compute the curvature tensor R_{ijkl}. The result is

(B.11) $$R_{ijkl}=\frac{1}{2}(l_{jl}h_{ik}-l_{il}h_{jk}-l_{jk}h_{il}+l_{ik}h_{jl})+\frac{1}{4}(h_{jkm}h_{iln}-h_{ikm}h_{jln})h^{mn}\ ,$$

see [3], [4]. From this, scalar curvature R is

(B.12) $$R=n(n-1)L+\frac{1}{4}F\ .$$

Here

(B.13) $$F=h_{ijk}h^{ijk}$$

is the *Fubini-Pick invariant.*

Now let us turn to the situation of §4. The hypersurface is defined by $x^{n+1}=f(x^1,\cdots,x^n)$. We are working around the point $y=(0,\cdots,0,1)$. Set $C=\det f_{ij}$ and $c=C^{-1/n(n+2)}$. Define a frame $(e_1,\cdots,e_{n+1})$ by

(B.14) $$\begin{aligned} e_i&=c(0,\cdots,0,\overset{i}{\check{1}},0,\cdots,f_i)\\ e_{n+1}&=c^{-n}(0,\cdots,0,1)\ . \end{aligned}$$

Then we have $h_{ij}=c^{n+2}f_{ij}$ and

(B.15) $$\nu=e_{n+1}+nc^{-n-2}c_jf^{jk}e_k$$

is the affine normal vector, where $c_j=\partial c/\partial x_j$ and (f^{jk}) is the inverse of

(f_{ij}). Moreover following definitions we have

$$l^{ij}=\frac{c^{-2n-2}}{n+2}((\log C)_k f^{ki})_m f^{jm}+\frac{c^{-2n-2}}{(n+2)^2}(\log C)_k(\log C)_m f^{ik}f^{jm}\,, \tag{B.16}$$

and

$$h_{ijk}=nc^{n+2}(c_i f_{jk}+c_j f_{ki}+c_k f_{ij})+c^{n+3}f_{ijk}\,. \tag{B.17}$$

We next evaluate the quantities at y. Since $f_{ij}(0)=\delta_{ij}$, we have $c(0)=1$ and $c_j(0)=-a_{iij}/n(n+2)$. So we have (4.13). The equalities (4.14) follow from (B.16) and (B.17).

We finally remark that we can always choose the frame so that the affine normal vector coincides with the euclidean normal vector. Consider the hypersurface $\{x^{n+1}=f(x^1,\cdots,x^n)\}$ with $f(x)=(1/2)a_{ij}x^ix^j+(1/6)a_{ijk}x^ix^jx^k+\cdots$ and introduce new coordinates $(X^1,\cdots,X^{n+1})$ by $x^i=p^iX^{n+1}+p^i_jX^j$ and $x^{n+1}=pX^{n+1}$. Then the surface is written as $\{X^{n+1}=g(X^1,\cdots,X^n)\}$, where $g(0)=g_i(0)=0$. Setting $g(X)=(1/2)b_{ij}X^iX^j+(1/6)b_{ijk}X^iX^jX^k+\cdots$, we have equalities

$$\begin{aligned} &pb_{ij}=a_{mn}p^m_ip^n_j\,,\\ &pb_{ijk}=\frac{2}{3}a_{nm}p^n(p^m_ib_{jk}+p^m_jb_{ik}+p^m_kb_{ij})+a_{lmn}p^l_ip^m_jp^n_k\,. \end{aligned} \tag{B.18}$$

From this, supposing $a_{ij}=\delta_{ij}$, we get

$$pb_{iik}=\left(\frac{n+2}{6}p^j+pa_{iij}\right)p^j_k\,. \tag{B.19}$$

Then it is always possible to find a transformation so as $b_{iik}=0$, $1\leqq k\leqq n$ and this proves the assertion.

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Added in Proof. The Problem 1 is solved: The metrics ω and κ are of asymptotically negative constant curvature. Relative to Problems 2 and 3, it is possible to expand the function k with respect to the function u up to the first order making use of the Fubini-Pick invariant. The precise statement will appear elsewhere.