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On Regular Fréchet-Lie Groups VIII

Primordial Operators and Fourier Integral Operators

Yoshiaki MAEDA, Hideki OMORI, Osamu KOBAYASHI
and Akira YOSHIOKA

*Keio University, Science University of Tokyo, Keio University
and Tokyo Metropolitan University*

In this paper, we prove that the group of invertible Fourier-integral operators of order 0 is a regular Fréchet-Lie group with the Lie algebra $\sqrt{-1}\mathscr{P}^1$, where $\mathscr{P}^1$ is the totality of pseudo-differential operators of order one with the real principal symbols. As stated in the preface of [8], this is the main purpose of this series. So, this paper is the final one of our series.

§1. Preliminaries and the statement of main theorem.

1.1. Notations.

Throughout this paper, we use mainly the same notations as in [8]. Let N be a closed C^∞ riemannian manifold and TN and T^*N be the tangent bundle and the cotangent bundle of N respectively. A point of TN (resp. T^*N) is denoted by $(x; X)$ (resp. $(x; \xi)$). Denote by $\mathring{T}^*N$ the complement of the zero section in T^*N, i.e., $T^*N-\{0\}$ in the notation of [8]. A symplectic diffeomorphism φ of T^*N is called to be *positively homogeneous* of degree one, if it commutes with multiplication by positive scalars. That is, if we write φ as $\varphi(x; \xi)=(\varphi_1(x; \xi); \varphi_2(x; \xi))$, then it satisfies $\varphi_1(x; r\xi)=\varphi_1(x; \xi)$, $\varphi_2(x; r\xi)=r\varphi_2(x; \xi)$, for any $r>0$.

Let $\mathscr{D}_\Omega^{(1)}$ be the totality of symplectic diffeomorphisms of $\mathring{T}^*N$ of positively homogeneous of degree one. Then, we have proved that $\mathscr{D}_\Omega^{(1)}$ *is naturally identified with* $\mathscr{D}_\omega(S^*N)$, *the group of all contact transformations on the unit sphere bundle* S^*N, *and* $\mathscr{D}_\Omega^{(1)}$ *is a regular Fréchet-Lie group* (cf. [6] and Theorem 6.4 in [11]).

Now, in this paper, all derivatives of functions, tensors, etc., on TN, T^*N and S^*N, etc. are taken by using a normal coordinate system at the considered point (cf. [8], §1, and [9], §1, (15)).

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1.2. Fourier-integral operators.

We have restricted our concern to Fourier-integral operators on N with the following expressions:

$$(1.1)\quad (F_\varphi u)(x)=\sum_\alpha \iint \lambda_\alpha a(x;\xi;X)e^{-i\langle\varphi_2(x;\xi)|X\rangle - i|\xi|A_\alpha(\varphi_1(x;\xi);X)}(\nu u)^\cdot(\varphi_1(x;\xi);X)dXd\xi \\ +(K\circ u)(x)\ ,$$

where we use the following notations:

(F.1) ν is a cut off function (cf. [8], p. 365) with the small breadth ε, $0<\varepsilon<r_1/12$, where r_1 is a small constant which depends only on the riemannian metric of N (cf. § 4.2). $(\nu u)^\cdot(x;X)=\nu(x,\cdot_x X)u(\cdot_x X)$ (cf. [8], p. 359).

(F.2) $a(x;\xi;X)$ is an element of $\tilde{\Sigma}^0_\varphi$, a class of amplitude functions (cf. [8], p. 366, (13)).

(F.3) $K\in C^\infty(N\times N)$ and $K\circ u$ is an integral operator with the kernel $K(x,y)$ (cf. [8], (12)).

$\{\lambda_\alpha(x;\xi)\}$ is an appropriate partition of unity on $\mathring{T}^*N$ (cf. [8], p. 373) such that $\lambda_\alpha(x;r\xi)=\lambda_\alpha(x;\xi)$ for any $r>0$, and $A_\alpha(y;X)$'s are quadratic forms written in the form $A_\alpha(y;X)=\sum_{i,j}A^{(\alpha)}_{ij}(y)X^iX^j$ added to $\langle\varphi_2(x;\xi)|X\rangle$ in order to make the phase function nondegenerate (cf. [8], pp. 366-368).

REMARK. There are in general a lot of ambiguities in the choice of $\{A_\alpha\}$ and hence $\{\lambda_\alpha\}$. The expression (1.1) is one of the way of describing operators whose wave front set is given by graph $\varphi\subset T^*(N\times N)$ (cf, [2], [3], [4], [14], [15]).

However, if φ is sufficiently close to the identity, one can set $A_\alpha=0$, hence (1.1) can be written in the form:

$$(1.2)\quad (F_\varphi u)(x)=\iint a(x;\xi;X)e^{-i\langle\varphi_2(x;\xi)|X\rangle}(\nu u)^\cdot(\varphi_1(x;\xi);X)dXd\xi+(K\circ u)(x)\ .$$

Moreover, we can always eliminate the variables X in the amplitude a (cf. [8], § 4 and Corrections). Thus, (1.2) can be rewritten as follows:

$$(1.3)\quad (F_\varphi u)(x)=\int_{T^*N} b(x;\xi)\widetilde{\nu u}(\varphi(x;\xi))d\xi+(K\circ u)(x)\ ,$$

where

$$(1.4)\quad \widetilde{\nu u}(y;\eta)=\int_N e^{-i\langle\eta|Y\rangle}\nu(y,z)u(z)dz\ ,\quad \cdot_y Y=z\ .$$

Now, the above expression (1.3) can be written as a composition of more "elementary operators". Remark that $\mathring{T}^*N$ is naturally diffeomorphic to $(0, \infty)\times S^*N$. We denote by $\mathscr{S}_N$ the space of all C^∞ functions f on $[0, \infty)\times S^*N$ such that $f(r, \omega)$ is rapidly decreasing as $r\to\infty$. In other words, by identifying $[0, \infty)$ with $[0, 1)$ (cf. [8], p. 364 (10)), $\mathscr{S}_N$ is the space of all C^∞ functions on $[0, 1]\times S^*N$ which are flat at $\{1\}\times S^*N$. Also, $\mathscr{S}_N$ is a Fréchet space and $\mathscr{D}_\Omega^{(1)}$ acts effectively and smoothly on $\mathscr{S}_N$ by $\varphi^*f(x;\xi)=f(\varphi(x;\xi))$, $\varphi\in\mathscr{D}_\Omega^{(1)}$, $f\in\mathscr{S}_N$. Note that the amplitude function $b(x;\xi)$ in (1.3) is an element of $\sum_C^0$ (cf. [8], p. 365). For each $b\in\sum_C^0$, we shall denote by $b\cdot$ the multiplication operator by b. Then, $b\cdot$ is a continuous linear operator of $\mathscr{S}_N$ into itself.

Define maps π: $\mathscr{S}_N\to C^\infty(N)$, and ι: $C^\infty(N)\to\mathscr{S}_N$ as follows:

$$\pi f(x)=\int_{T_x^*N} f(x;\xi)d\xi \ , \tag{1.5}$$

$$\iota u(x;\xi)=\nu\tilde{u}(x;\xi) \quad (\text{cf. } (1.4)) \ . \tag{1.6}$$

By the formula of Fourier transformation, we have

$$\pi\iota=\text{id} \ . \tag{1.7}$$

Using these operators (1.5) and (1.6), one can write (1.3) by

$$F_\varphi=\pi\circ b\cdot\circ\varphi^*\circ\iota+K\circ \ . \tag{1.8}$$

Remark. (i) The above expression (1.2) or (1.3) still have ambiguities. Using F_φ, one can only know φ and the *asymptotic expansion* of b. Namely, one can replace (b, K) by another (b', K') to obtain the same operator F_φ (cf. [8] and Corrections).

(ii) By (1.7), the operator $\iota\pi$: $\mathscr{S}_N\to\mathscr{S}_N$ is a projection operator, i.e., $(\iota\pi)^2=\iota\pi$.

1.3. Main theorem.

Now, we shall state the main theorem. Let $\mathfrak{U}$, V_1, U_0 be a connected neighborhood of the identity of $\mathscr{D}_\Omega^{(1)}$, a neighborhood of 1 in $\sum_C^0$, a neighborhood of 0 in $C^\infty(N\times N)$ respectively. Denote by $\mathfrak{N}(\mathfrak{U}, V_1, U_0)$ the set of all Fourier-integral operators of the form (1.8) such that $\varphi\in\mathfrak{U}$, $a\in V_1$, $K\in U_0$. Note that if $\mathfrak{U}$, V_1, U_0 are sufficiently small, then every element in $\mathfrak{N}(\mathfrak{U}, V_1, U_0)$ is invertible and the inverse is again in $\mathfrak{N}(\mathfrak{U}, V_1, U_0)$. Also, denote by $G\mathscr{F}_0^0$ the group generated by $\mathfrak{N}(\mathfrak{U}, V_1, U_0)$. Then, Theorem B in [9] shows that every element of $G\mathscr{F}_0^0$ can be written in the form (1.1).

Now, the goal of this paper is as follows:

THEOREM A. *$G\mathscr{F}_0^0$ is a regular Fréchet-Lie group.*

REMARK. Once a manifold structure is established on $G\mathscr{F}_0^0$, Proposition A in [10] shows that $\sqrt{-1}\mathscr{P}^1$ is its tangent space at the identity. Hence, by Lemma 2.2 in [11], $\sqrt{-1}\mathscr{P}^1$ is the Lie algebra of $G\mathscr{F}_0^0$.

§ 2. How to prove Theorem A.

2.1. Extensions of regular Fréchet-Lie groups.

Define a mapping $\Phi: G\mathscr{F}_0^0 \to \mathscr{D}_\Omega^{(1)}$ by

$$\Phi(F_\varphi)=\varphi^{-1}, \qquad \varphi\in\mathscr{D}_\Omega^{(1)}. \tag{2.1}$$

Then, in view of Theorem 5.5 in [9], Φ is a well-defined homomorphism, and the image of Φ is the identity component of $\mathscr{D}_\Omega^{(1)}$. The kernel of Φ is $G\mathscr{P}^0$, the group of invertible pseudo-differential operators of order 0 (cf. [8], (38)). Since $\mathscr{D}_\Omega^{(1)}$ is naturally isomorphic to $\mathscr{D}_\omega(S^*N)$, we have an exact sequence as follows:

$$1 \longrightarrow G\mathscr{P}^0 \longrightarrow G\mathscr{F}_0^0 \longrightarrow \mathscr{D}_\omega(S^*N) \dashrightarrow 1, \tag{2.2}$$

where the dotted arrow indicates that the image of Φ is an open subgroup.

We note here that *$\mathscr{D}_\omega(S^*N)$ is a regular Fréchet-Lie group* (cf. [6], [11]) *and also that $G\mathscr{P}^0$ is a regular Fréchet-Lie group.* Indeed, in [17], we have seen that $G\mathscr{P}^0_{(m)}$ is a regular Fréchet-Lie group for $m\leqq -\dim N-1$, and that $G\mathscr{P}^0$ is a regular Fréchet-Lie group obtained by the inverse limit of $\{G\mathscr{P}^0_{(m)};\, m\leqq -\dim N-1\}$.

REMARK. In view of the arguments in [16], we can easily check the following. *For every $m\leqq 0$, $G\mathscr{P}^0_{(m)}$ is an open subset of $\mathscr{P}^0_{(m)}$ and is an FL-group* (cf. [11]). The condition $m\leqq -\dim N-1$ is used only to ensure the convergence of product integrals.

2.2. Mappings r_γ, α_γ associated with the extension (2.2).

Now, we define a mapping $\gamma: \mathfrak{U}\to G\mathscr{F}_0^0$ by

$$\gamma(\varphi)=\pi\circ\varphi^{-1*}\circ\iota. \tag{2.3}$$

Obviously, $\Phi\circ\gamma=$id., and γ gives a local closs section of (2.2). Define a mapping r_γ by

$$r_\gamma(\varphi,\psi)=\gamma(\varphi\psi)^{-1}\gamma(\varphi)\gamma(\psi). \tag{2.4}$$

As $\Phi: G\mathscr{F}_0^0\to\mathscr{D}_\Omega^{(1)}$ is a homomorphism, r_γ is a mapping of $\mathfrak{U}\times\mathfrak{U}$ into $G\mathscr{P}^0$.

On the other hand, define $\alpha_\gamma(\varphi, A)$, for every $\varphi \in \mathfrak{U}$, $A \in G\mathscr{P}$ by

$$\alpha_\gamma(\varphi, A) = \gamma(\varphi)^{-1} A \gamma(\varphi) \in G\mathscr{P}^0 \ . \tag{2.5}$$

Recall that the topology of $G\mathscr{P}^0$ is obtained by the inverse limit of $\{G\mathscr{P}^0_{(m)}; m \leqq 0\}$. Hence recalling Proposition 5.2 and Theorem 5.4 in [11], to obtain Theorem A, we have only to show the following:

PROPOSITION 2.1. *The mappings r_γ and α_γ, defined by* (2.4) *and* (2.5) *respectively, have the following properties*:

(Ext. 1) $r_\gamma: \mathfrak{U} \times \mathfrak{U} \to G\mathscr{P}^0$ *is a C^∞ mapping of $\mathfrak{U} \times \mathfrak{U}$ into $G\mathscr{P}^0_{(m)}$ for every $m \leqq 0$.*

(Ext. 2) $\alpha_\gamma: \mathfrak{U} \times G\mathscr{P}^0 \to G\mathscr{P}^0$ *can be extended to a C^∞ mapping of $\mathfrak{U} \times G\mathscr{P}^0_{(m)}$ into $G\mathscr{P}^0_{(m)}$ for every $m \leqq 0$.*

REMARK. By the above proposition, we see also that there is $G\mathscr{P}^0_{(m)}$-extension of the identity component of $\mathscr{D}^{(1)}_\Omega$, which is an FL-group for each $m \leqq 0$, and a regular Fréchet-Lie group for $m \leqq -\dim N - 1$. We shall denote this extension by $G\mathscr{F}^0_{0\,(m)}$. $G\mathscr{F}^0_0$ is indeed the inverse limit of $\{G\mathscr{F}^0_{0\,(m)}; m \leqq 0\}$.

2.3. Key propositions.

To prove (Ext. 1-2) in Proposition 2.1, we have to know first the inverse of $\gamma(\varphi)$. To do that, set

$$\Xi(\varphi) = \pi(\varphi^* \circ \iota \circ \pi \circ \varphi^{*-1})\iota \ , \tag{2.6}$$

then we shall show the following in § 6:

PROPOSITION 2.2. *Notations being as above, we have*

(a) *if φ is sufficiently close to the identity, then $\Xi(\varphi)$ is a pseudo-differential operator of order zero, i.e., $\Xi(\varphi) \in \mathscr{P}^0(N)$.*

(b) $\Xi: \mathfrak{U} \to \mathscr{P}^0_{(m)}$, *defined by $\Xi(\varphi)$ in* (2.6), *is smooth for every $m \leqq 0$. Therefore, since $\Xi(\text{id.}) = 1$,*

(c) *$\Xi(\varphi)$ is invertible if φ is sufficiently close to the identity.*

By using Proposition 2.2, (c), we obtain for sufficiently small $\varphi \in \mathscr{D}^{(1)}_\Omega$,

$$\gamma(\varphi)^{-1} = \Xi(\varphi)^{-1} \gamma(\varphi^{-1}) \ . \tag{2.7}$$

Hence, if φ, ψ are sufficiently close to the identity, then $\Xi(\varphi\psi)$ is invertible. Thus, we obtain

$$r_\gamma(\varphi, \psi) = \Xi(\varphi\psi)^{-1} \pi\{(\varphi\psi)^* \iota \pi (\varphi\psi)^{*-1}\}(\varphi^* \iota \pi \psi^{*-1})\iota \ . \tag{2.8}$$

On the other hand, any $A\in G\mathscr{P}^0$ can be expressed as follows:

$$A=\pi a\cdot\iota+K\circ\,, \tag{2.9}$$

where $a\in\sum_c^0$ and $K\in C^\infty(N\times N)$. Hence, we have,

$$\begin{aligned}\alpha_\Gamma(\varphi, A)=&\,\Xi(\varphi)^{-1}\pi(\varphi^*\iota\pi\varphi^{*-1})(\varphi^*a\cdot\varphi^{*-1})(\varphi^*\iota\cdot\pi\varphi^{*-1})\iota\\&+\Xi(\varphi)^{-1}\pi(\varphi^*\iota K\circ\pi\varphi^{*-1})\iota\,.\end{aligned} \tag{2.10}$$

Note that $\varphi^*a\cdot\varphi^{*-1}=(\varphi^*a)\cdot$, and one may write

$$\begin{aligned}\alpha_\Gamma(\varphi, A)=&\,\Xi(\varphi)^{-1}\pi(\varphi^*\iota\pi\varphi^{*-1})(\varphi^*a)\cdot(\varphi^*\iota\pi\varphi^{*-1})\iota\\&+\Xi(\varphi)^{-1}\pi(\varphi^*\iota K\circ\pi\varphi^{*-1})\iota\,.\end{aligned} \tag{2.11}$$

The above computations show that operators of the form

$$\varphi^*\iota\pi\varphi^{*-1}\,,\qquad(\varphi^*a)\cdot\,,\qquad\varphi^*\iota K\circ\pi\varphi^{*-1}$$

and their composition rules play an important role in studying r_Γ and α_Γ. Thus, we shall set up a certain class of operators $\mathfrak{M}$, containing $\varphi^*\iota\pi\varphi^{*-1}$ for every φ which is sufficiently close to the identity. $\mathfrak{M}$ is indeed a C^∞ Fréchet manifold and a local semi-group with *smooth* semi-group operations (cf. §§ 4-6). Moreover, we shall see $\mathfrak{M}$ is closed under the multiplication by φ^*a. This is indeed smooth with respect to φ, a and $P\in\mathfrak{M}$ (cf. §§ 6-7). Next, we shall prove that the "projection": $\mathfrak{M}\to G\mathscr{P}^0_{(m)}$, $m\leqq 0$, $P\to\pi P\iota$ is smooth (cf. § 6, Proposition 6.2).

Denote by $E(\varphi)=\varphi^*\iota\pi\varphi^{*-1}$. Then, E can be regarded as a smooth mapping of $\mathfrak{U}$ into $\mathfrak{M}$ (cf. § 6, 6.6). Thus, by using these smoothness properties of $\mathfrak{M}$, we see that (2.8) and the first term of (2.11) are smooth. To treat the second term of (2.11), we shall need the following proposition which will be proved in § 6 as well as some other smoothness properties stated above:

Proposition 2.3. *For every* $\varphi\in\mathscr{D}_\Omega^{(1)}$, *and* $K\in C^\infty(N\times N)$, *put* $\Lambda(\varphi, K)=\pi(\varphi^*\iota K\circ\pi\varphi^{*-1})\iota$. *Then, we get*

(a) $\Lambda(\varphi, K)$ *is a linear operator on* $C^\infty(N)$ *with* C^∞ *kernel* $L(\varphi, K)$ (cf. Lemma 6.4).

(b) *The mapping* $L\colon\mathscr{D}_\Omega^{(1)}\times C^\infty(N\times N)\to C^\infty(N\times N)$ *is smooth.*

§ 3. Several properties of operators $\varphi^*\iota\pi\varphi^{*-1}$ and $\varphi^*\iota K\circ\pi\varphi^{*-1}$.

3.1. Kernel expression.

First, we shall compute the kernel of $\varphi^*\iota\pi\varphi^{*-1}$ and $\varphi^*\iota K\circ\pi\varphi^{*-1}$. Recall the definition of π and ι (cf. (1.5) and (1.6)). Then, we have

(3.1) $$(\iota\pi f)(x;\xi)=\iint \nu(x,y)e^{-\iota\langle\xi|\cdot^x y\rangle}f(y;\eta)đyđ\eta\ ,\quad f\in\mathcal{S}_N\ ,$$

where $\cdot^x y=Y$ implies $\cdot_x Y=y$. Let $\tau_0(x;\xi,y)$ be a smooth extension of $\langle\xi|\cdot^x y\rangle$ onto $\mathring{T}^*N\times\mathring{T}^*N$ such that

$$\tau_0(x;r\xi,y)=r\tau_0(x;\xi,y)\ ,\quad r>0\ .$$

Then, τ_0 has the following properties:

Lemma 3.1. *For given $r_1>0$ in* (F.1), *if $d(x,y)<2r_1/3$, then $\tau_0(x;\xi,y)$ has no critical point in $(x;\xi)$ for every y and $\tau_0(x;\xi,y)$ has no critical point in y for every $(x;\xi)\in\mathring{T}^*N$.*

Now, $\iota\pi$ can be regarded as an integral operator with smooth kernel $\nu(x,y)e^{-\iota\tau_0(x;\xi,y)}$, hence the kernel of $\varphi^*\iota\pi\varphi^{*-1}$ is given by $(\varphi^*\nu)e^{-\iota\varphi^*\tau_0}$ because of $\varphi^*đyđ\eta=đyđ\eta$, where $(\varphi^*\nu)(x;\xi,y;\eta)=\nu(\varphi_1(x;\xi),\varphi_1(y;\eta))$.

Similarly, the kernel of $\iota K\circ\pi$ is given by

(3.2) $$a_K(x;\xi,y)=\int\nu(x,z)K(z,y)e^{-\iota\tau_0(x;\xi,z)}đz\ .$$

Since $\nu(x,z)K(z,y)$ has a compact support in z, $a_K(x;\xi,y)$ is rapidly decreasing in $|\xi|$. Hence, the kernel of $\varphi^*\iota K\circ\pi\varphi^{*-1}$ is given by

(3.3) $$(\varphi^*a_K)(x;\xi,y;\eta)=a_K(\varphi(x;\xi),\varphi(y;\eta))\ .$$

3.2. A class of phase functions.

To unify τ_0 and $\varphi^*\tau_0$, we introduce a class of functions, which correspond to "phase functions" defined later.

Let $\tau(x;\xi,y;\eta)$ be a smooth function on $\mathring{T}^*N\times\mathring{T}^*N$ which satisfies

(P.1) $$\tau(x;r\xi,y;s\eta)=r\tau(x;\xi,y;\eta)\quad\text{for any}\quad r>0\ ,\quad s>0\ .$$

The above τ is considered as a smooth function on $[0,\infty)^2\times(S^*N)^2$ by putting

(3.4) $$\tilde{\tau}(r,s,x;\hat{\xi},y;\hat{\eta})=\tau(x;r\hat{\xi},y;s\hat{\eta})\ .$$

For above τ, define a subset $C(\tau)$ of $\mathring{T}^*N\times\mathring{T}^*N$ by

(3.5) $$C(\tau)=\{(x;\xi,y;\eta)\in\mathring{T}^*N\times\mathring{T}^*N;\ \nabla_{(x;\xi)}\tau=0\ \text{or}\ \nabla_{(y;\eta)}\tau=0\}\ .$$

Then, $C(\tau)$ is *conic*, that is, $(x;\xi,y;\eta)\in C(\tau)$ if and only if $(x;r\xi,y;s\eta)\in C(\tau)$ for every $r>0$, $s>0$.

Consider the following property for τ;

(P.2) $C(\tau)$ is bounded away from the diagonal set if $\tau \not\equiv 0$.

It is obvious that τ_0 satisfies (P.1) and (P.2), and that such properties are invariant under the action of $\mathscr{D}_\Omega^{(1)}$, hence $\varphi^*\tau_0$ satisfies (P.1–2) for every $\varphi \in \mathscr{D}_\Omega^{(1)}$.

However, what we shall need in the computation is not a general τ with (P.1–2) but $\varphi^*\tau_0$, $\varphi \in \mathscr{D}_\Omega^{(1)}$, or "0". Thus, we have to consider $\varphi^*\tau_0$ more precisely.

First of all recall that each $\varphi \in \mathscr{D}_\Omega^{(1)}$ leaves the canonical 1-form θ invariant, where θ is given locally by $\theta = \sum \xi_i dx^i$. This fact gives the following:

LEMMA 3.2. *For each* $\varphi \in \mathscr{D}_\Omega^{(1)}$, $\varphi^*\tau_0$ *can be written by*

$$\varphi^*\tau_0 = \tau_0 + Q(\tau)(x; \xi, y; \eta) , \tag{P.3}$$

and Q *vanishes at* $(x; \xi) = (y; \eta)$ *up to the first derivatives.*

PROOF. Use a normal coordinate system $(y^1, \cdots, y^n)$ at x and its dual coordinate system $(\xi_1, \cdots, \xi_n)$. Then, we get

$$\langle \xi \,|\, \cdot^x y \rangle = \xi_i y^i \quad \text{and} \quad \theta = \xi_i dy^i .$$

For $\varphi \in \mathscr{D}_\Omega^{(1)}$, we use a normal coordinate system and dual coordinate system at $\varphi_1(x; \xi)$. Denote $\varphi(y; \eta)$ by $(\bar{y}^1, \cdots, \bar{y}^n, \bar{\xi}_1, \cdots, \bar{\xi}_n)$. Then, letting $\bar{\xi} = \varphi_2(x; \xi)$, we have

$$\langle \varphi_2(x; \xi) |^{\varphi_1(x;\xi)} \cdot \varphi_1(y; \eta) \rangle = \bar{\xi}_i \bar{y}^i .$$

Remark that $\varphi^*\theta = \theta$ means that

$$\bar{\xi}_i \frac{\partial \bar{y}^i}{\partial y^j} = \xi_j , \qquad \bar{\xi}_i \frac{\partial \bar{y}^i}{\partial \eta_j} = 0 \qquad (\text{cf. [8], (25)}) .$$

Put

$$\bar{y}^i = \frac{\partial \bar{y}^i}{\partial y^j}(0; \xi) y^j + \frac{\partial \bar{y}^i}{\partial \eta_j}(0; \xi)(\eta_j - \xi_j) + H(\varphi)(y, \eta - \xi) ,$$

where $H(\varphi)(y, \eta - \xi)$ is the quadratic term with respect to y and $\eta - \xi$. Then, we have

$$\begin{aligned} \varphi^*\tau_0 &= \langle \varphi_2(x; \xi) |^{\varphi_1(x;\xi)} \cdot \varphi_1(y; \eta) \rangle \\ &= \xi_i y^i + \langle \xi \,|\, H(\varphi)(y, \eta - \xi) \rangle . \end{aligned}$$

So, $Q(\varphi)$ is given by the last term of the above equality. □

3.3. A class $\mathscr{B}(\tau)$ of amplitude functions.

We define amplitude functions associated with τ in 3.2. Let τ be a C^∞ function on $\mathring{T}^*N\times\mathring{T}^*N$ which satisfies (P.1–2) in 3.2. (Remark that the Property (P.3) is not used in this section.)

For above τ, we denote by $\mathscr{B}(\tau)$ the linear space of smooth functions h on $\mathring{T}^*N\times\mathring{T}^*N$ such that

(B.1) h is a C^∞ function on $[0, \infty)^2\times(S^*N)^2$ and all derivatives of h are bounded.

(B.2) There exists a conic neighborhood V_h of $C(\tau)$ on which $h(r, x; \hat{\xi}, y; \eta)=h(x; r\hat{\xi}, y; \eta)$, $r\hat{\xi}=\xi$, is rapidly decreasing as $r\to\infty$.

Recall the kernels obtained in 3.1 and we know the significance of the following:

LEMMA 3.3. (a) $\nu(x, y)\in\mathscr{B}(\tau_0)$. (b) $\varphi^*\nu\in\mathscr{B}(\varphi^*\tau_0)$ *for any* $\varphi\in\mathscr{D}_\Omega^{(1)}$. (c) $\varphi^*a_K(x; \xi, y; \eta)\in\mathscr{B}(0)$.

PROOF. Since $\nu=0$ on a neighborhood $C(\tau_0)$, we get (a). (b) and (c) are easily obtained by a direct computation of derivatives. □

3.4. Primordial operators.

Let τ satisfy (P.1–2) in 3.2 and let $a\in\mathscr{B}(\tau)$. Consider the following operator

$$P(a, \tau)f(x; \xi)=\iint a(x; \xi, y; \eta)e^{-i\tau(x;\xi,y;\eta)}f(y; \eta)đyđ\eta\ , \quad f\in\mathscr{S}_N\ . \tag{3.6}$$

By Lemmas 3.2–3, $\varphi^*\iota\pi\varphi^{*-1}$, and $\varphi^*\iota K\circ\pi\varphi^{*-1}$ are written in the above form (3.6), which will be called *primordial operators* in this paper.

Now, we can give a rigid meaning of (3.6) as an operator as follows:

PROPOSITION 3.4. *Let* τ *satisfy* (P.1–2) *and let* $a\in\mathscr{B}(\tau)$. *Then,* $P(a, \tau)$ *in* (3.6) *defines a linear operator on* $\mathscr{S}_N$ *into itself.*

PROOF. Let $\phi(x; \xi, y; \eta)$ be a smooth function such that $\phi(x; r\xi, y; s\eta)=\phi(x; \xi, y; \eta)$, $r>0$, $s>0$, and $\phi\equiv 1$ on a neighborhood of $C(\tau)$ and $\operatorname{supp}\phi\subset V_a$ (cf. (B.2) for the notation V_a).

Divide (3.6) into two parts:

$$\begin{aligned}P(a, \tau)f(x; \xi)&=\iint \phi a e^{-i\tau}f\,đyđ\eta+\iint (1-\phi)ae^{-i\tau}f\,đyđ\eta\\&=P_1+P_2\ .\end{aligned}\tag{3.7}$$

Since $\phi ae^{-i\tau}$ is rapidly decreasing in $|\xi|$, we see that $P_1\in\mathscr{S}_N$ for every

$f \in \mathcal{S}_N$. Now, consider P_2. Remark that on the support of $(1-\phi)a$, τ has no critical point in $(y;\eta)$. So, let

$$L_\tau = \frac{1 + ir(\nabla_{\hat{\eta}}\tau \cdot \nabla_{\hat{\eta}} + \nabla_y \tau \cdot \nabla_y)}{1 + r^2 |\nabla_{(y;\hat{\eta})}\tau|^2}, \quad r = |\xi|, \quad \tau = \tau(x;\hat{\xi}, y;\hat{\eta}).$$

Then, $L_\tau e^{-i\tau} = e^{-i\tau}$ and the coefficients of the operator L_τ can be bounded by r^{-1} for sufficiently large $r > 0$. So, P_2 can be written as

$$\text{(3.8)} \qquad P_2(x;\xi) = \int_{S^*N}\int_0^\infty (1-\phi)a(x; r\hat{\xi}, y; s\hat{\eta})(L_\tau)^e e^{-ir\tau(x;\hat{\xi}, y;\hat{\eta})} f(y; s\hat{\eta}) s^{n-1} \mathrm{đ}y \mathrm{đ}s \mathrm{đ}\hat{\eta}.$$

Repeating the integration by parts, we see that $P_2(x;\xi)$ is rapidly decreasing in $|\xi|$. Smoothness at $r=0$ of $P_2(x;\xi)$ follows from those of $\phi(x; r\hat{\xi}, y;\eta)$ and $a(x; r\hat{\xi}, y;\eta)$ at $r=0$. □

Finally, we remark that in what follows we shall restrict our concern to much narrower class of amplitudes. The main reason to do so is that $\mathscr{B}(\tau)$ is not invariant under $\mathscr{D}_\Omega^{(1)}$. The restricted class is invariant under $\mathscr{D}_\Omega^{(1)}$ and contains $\sum_C^0$, though $\varphi^* a \notin \sum_C^0$ even if $a \in \sum_C^0$.

§ 4. Phase functions.

Now, to fix the restricted class of primordial operators, we shall introduce a class of phase functions and study the properties of compositions of phase functions induced by the composition of primordial operators.

4.1. Definition of phase functions.

Let ϑ be the space of all C^∞ functions τ on $\mathring{T}^*N \times \mathring{T}^*N$ satisfying (P.1) in 3.2. Since such τ is uniquely determined by the values on $S^*N \times S^*N$, we shall give a topology for ϑ by using the C^∞ topology on $S^*N \times S^*N$. Denote by ϑ_0 the closed affine subspace of ϑ defined by

(4.1) $\vartheta_0 = \{\tau \in \vartheta;\ \tau - \tau_0$ vanishes on the diagonal set up to the first derivatives$\}$.

Remark that every $\tau \in \vartheta_0$ satisfies (P.2) and (P.3) in Lemma 3.2. $\mathscr{D}_\Omega^{(1)}$ acts on ϑ by the following: Given $\varphi \in \mathscr{D}_\Omega^{(1)}$, $\tau \in \vartheta$, we define a mapping $ev: \mathscr{D}_\Omega^{(1)} \times \vartheta \to \vartheta$ by

$$\text{(4.2)} \qquad ev(\varphi, \tau) = \varphi^*\tau(x;\xi, y;\eta) = \tau(\varphi(x;\xi), \varphi(y;\eta)).$$

Then, we have

LEMMA 4.1. *The mapping* $ev: \mathscr{D}_\Omega^{(1)} \times \vartheta \to \vartheta$ *is a* C^∞ *mapping and* *it leaves ϑ_0 invariant.*

PROOF. The smoothness of ev is obvious by that of composition of mappings (cf. [1], [12], [5]). The invariance of ϑ_0 follows from Lemma 3.1. □

4.2. Composition of phase functions.

Given $\tau_1, \tau_2 \in \vartheta$, we define a composition $\tau_1 \boxplus \tau_2$ as a function on $\mathring{T}^*N \times \mathring{T}^*N \times \mathring{T}^*N$ by

$$\tau_1 \boxplus \tau_2(x; \xi, y; \eta, z; \zeta) = \tau_1(x; \xi, y; \eta) + \tau_2(y; \eta, z; \zeta) . \tag{4.3}$$

For a later use, we have to know at first the critical point and the critical value of (4.3) with respect to $(y; \eta)$. However, this is not so easy in general. Thus, we shall do this under the assumption that $\tau_1, \tau_2 \in \vartheta_0$ and they are sufficiently close to τ_0. Moreover, we shall restrict the domain of $\tau_1 \boxplus \tau_2$ onto $d(x, y) \leqq r_1/2$, $d(x, z) \leqq r_1/4$, where r_1 is constant depending only on the riemannian structure of N, which will be given below.

On this restricted domain, one may set $y = \cdot_x X$, $z = \cdot_y Y = \cdot_x Z$ and $(y; Y, \eta) = \cdot_x(X, Y', \eta')$ by using the normal coordinate system at x. Y' is given by $Y' = \tilde{S}(x; Z, X) = \tilde{S}_1(x; Z, X)(Z - X)$ (cf. [8], p. 360, (3)). The constant r_1 is defined by the supremum of r such that $\partial_X \tilde{S}|_{X=0}$ and $\tilde{S}_1(x; Z, X)$ are invertible matrices whenever $d(x, z) \leqq r$. For the standard sphere, $r_1 = \pi/2$ and for many riemannian manifolds, r_1 is given as a half of the injectivity radius.

Set $\tau_i = \tau_0 + Q_i$ $(i = 1, 2)$. Then, $\tau_1 \boxplus \tau_2$ can be written in the form

$$\langle \xi | X \rangle + \langle \eta' | \tilde{S}(x; Z, X) \rangle + Q_1(x; \xi, \cdot_x(X, \eta')) + Q_2(\cdot_x(X, \eta'), z; \zeta) . \tag{4.4}$$

Thus, consider the equations

$$\partial_X(\tau_1 \boxplus \tau_2) = \xi + \langle \eta' | \partial_X \tilde{S} \rangle + \partial_X Q_1 + \partial_X Q_2 = 0 , \tag{4.5}$$

$$\partial_{\eta'}(\tau_1 \boxplus \tau_2) = \tilde{S}(x; Z, X) + \partial_{\eta'} Q_1 + \partial_{\eta'} Q_2 = 0 . \tag{4.6}$$

LEMMA 4.2. *Suppose τ_1 and τ_2 are sufficiently close to τ_0 in ϑ_0 and suppose $d(x, y) \leqq r_1/2$, $d(x, z) \leqq r_1/4$. Then, we obtain the following:*

(i) *The equation $\partial_y(\tau_1 \boxplus \tau_2) = 0$ can be solved uniquely with respect to η. Let $\bar{\eta}$ be its solution. Then, $\bar{\eta} = \bar{\eta}(x; \xi, y, z; \zeta)$ is C^∞ and $\bar{\eta}(x; r\xi, y, z; s\zeta) = r\bar{\eta}(x; \xi, y, z; \zeta)$ for any $r > 0$, $s > 0$.*

(ii) *There are constants $C > 0$, $M > 0$ such that*

$$|\partial_y(\tau_1 \boxplus \tau_2)| \geqq M(|\xi| + |\eta|) \quad \textit{if} \quad |\eta| \geqq C|\xi| \quad \textit{or} \quad |\eta| \leqq C^{-1}|\xi| . \tag{4.7}$$

PROOF. One may assume that there are small $\delta>0$ and a constant $K>0$ such that $|\partial_X(Q_1+Q_2)|\leqq\delta(|\xi|+|\eta'|)$, $K^{-1}\leqq|\partial_X\tilde{S}|\leqq K$, $K^{-1}\leqq|(\partial_X\tilde{S})^{-1}|\leqq K$. By (4.5), we see easily that if $\bar{\eta}$ exists then $\bar{\eta}$ must satisfy

$$\frac{1}{2}C^{-1}|\xi|\leqq|\bar{\eta}'|\leqq 2C|\xi| , \qquad \cdot_x(X, \bar{\eta}')=(y; \bar{\eta})$$

for some constant $C\geqq 2$. Moreover, on this domain one may set

$$|\partial_{\eta'}\partial_X(\tau_1\boxplus\tau_2)-\partial_X\tilde{S}|\leqq\delta\Big(\frac{|\eta'|}{|\xi|}+1\Big)\leqq\delta(2C+1) .$$

It follows that $\partial_{\eta'}\partial_X(\tau_1\boxplus\tau_2)$ is non-singular matrix on the conical domain: $d(x, y)\leqq r_1/2$, $d(x, z)\leqq r_1/4$, $(1/2)C^{-1}|\xi|\leqq|\eta|\leqq 2C|\xi|$.

Suppose $Q_1=Q_2=0$ in (4.5). Then, it has the unique solution $\bar{\eta}=-\xi(\partial_X\tilde{S})^{-1}$. By means of the implicit function theorem (cf. [12], Lemma 4.9) on the above conical domain, we obtain the unique existence of $\bar{\eta}$. Smoothness of $\bar{\eta}$ follows from the regularity of $\partial_y\partial_{\eta'}(\tau_1\boxplus\tau_2)$, and the homogeneity of $\bar{\eta}$ follows from those of τ_1, τ_2.

Now, suppose $|\eta|\geqq C|\xi|$ or $|\eta|\leqq C^{-1}|\xi|$. Then, $\partial_y(\tau_1\boxplus\tau_2)$ cannot attain 0. Hence, there must be a constant M such that $|\partial_y(\tau_1\boxplus\tau_2)|\geqq M(|\xi|+|\eta|)$. □

PROPOSITION 4.3. *Suppose τ_1, τ_2 are sufficiently close to τ_0 in ϑ_0. If $d(x, y)\leqq r_1/2$, $d(x, z)\leqq r_1/4$, then*

(i) *the function $\tau_1\boxplus\tau_2$ has only one critical point $(y_c; \eta_c)$, which is non-degenerate;*

(ii) *the critical point $(y_c; \eta_c)$ depends smoothly on $(x; \xi, z; \zeta)$ and satisfies*

$$\text{(4.8)}\qquad \begin{cases} y_c(x; r\xi, z; s\zeta)=y_c(x; \xi, z; \zeta) \\ \eta_c(x; r\xi, z; s\zeta)=r\eta_c(x; \xi, z; \zeta) \end{cases} \quad r>0 , \quad s>0 ;$$

(iii) *the critical value $\tau_{12}=(\tau_1\boxplus\tau_2)(x; \xi, y_c; \eta_c, z; \zeta)$ has the properties (P.1–3) in the variables $(x; \xi, z; \zeta)$.*

PROOF. We substitute $\eta'=\bar{\eta}'(x; \xi, y, z; \zeta)$ into (4.6). Note that $\partial_{\eta'}(\tau_1\boxplus\tau_2)(x; \xi, y; \bar{\eta}, z; \zeta)$ is homogeneous of degree zero with respect to ξ. Suppose $Q_1=Q_2=0$. Then, (4.6) has the unique solution $X_c=Z$, i.e., $y_c=z$. Recall that $\partial_X\partial_{\eta'}(\tau_1\boxplus\tau_2)$ is invertible. Hence, the implicit function theorem (cf. [12], Lemma 4.9) implies (i). The uniqueness of $(y_c; \eta_c)$ and the homogeneity of τ_1, τ_2 yields (ii), which indicates that τ_{12} satisfies (P.1). As for (P.2), (P.3) in (iii), we may consider near the diagonal set. Put $(z; \zeta)=(x; \xi)$, i.e., $\cdot_x(Z, \zeta')=\cdot_x(0, \xi)$ in (4.5) and (4.6). Then, the first derivatives of Q_1, Q_2 vanish at $(y;\eta)=(x;\xi)$, so we get $\cdot_x(X_c,\eta'_c)=\cdot_x(0,\xi)=(x;\xi)$. Hence, the Taylor expansion of (X_c,η'_c) with respect to (Z,ζ') at $(0,\xi)$ is

$$\begin{cases} X_c=aZ+b(\xi-\eta')+\cdots\ , \\ \eta'_c=\xi+cZ+d(\xi-\zeta')+\cdots\ . \end{cases}$$

Substituting this into τ_{12}, we see that τ_{12} has the properties (P.2) and (P.3). □

4.3. Properties of the critical value of $\tau_1 \boxplus \tau_2$.

Next, we shall observe the critical value τ_{12} more carefully. Choose a C^∞ function ψ on $\boldsymbol{R}$ such that $\psi\equiv 1$ on $|t|\leqq r_1/5$ and $\psi\equiv 0$ on $|t|\geqq r_1/4$, and define a function $c(\tau_1,\tau_2)$ by

$$c(\tau_1,\tau_2)=\psi(d(x,z))\tau_{12}(x;\xi,z;\zeta)+(1-\psi(d(x,z)))\tau_0(x;\xi,z)\ . \tag{4.9}$$

c can be regarded as a function of τ_1, τ_2. By Proposition 4.3, we see also

LEMMA 4.4. *$c(\tau_1,\tau_2)\in\vartheta_0$ for τ_1, τ_2 sufficiently close to τ_0. c is a C^∞ mapping of $U_{\tau_0}\times U_{\tau_0}$ into ϑ_0, such that $c(\tau_0,\tau_0)=\tau_0$, where U_{τ_0} is a small neighborhood of τ_0 in ϑ_0.*

PROOF. The desired smoothness follows from the implicit function theorem (cf. [12], Lemma 4.9). The property $c(\tau_0,\tau_0)=\tau_0$ is obtained by the computations in the case $Q_1=Q_2=0$. □

The following is a special case of Proposition 4.3.

COROLLARY 4.5. *Let $\tau\in\vartheta_0$ be sufficiently close to τ_0. Then, $c(\tau_1,\tau_0)$ does not involve the ζ-variable, i.e., $c(\tau,\tau_0)=c(\tau,\tau_0)(x;\xi,z)$. Moreover, it is written in the form $\tau_0(x;\xi,z)+Q(x;\xi,z)$, where Q satisfies $Q(x;\xi,x)=0$, $(\partial Q/\partial Z)_{Z=0}(x;\xi,\cdot_x Z)=0$.*

Now, set $T=\tau_1\boxplus\tau_2-c(\tau_1,\tau_2)$. Using Proposition 4.3 and Lemma 4.2, we have the following properties of T.

COROLLARY 4.6. *With the same notations as in Proposition 4.3, T has the following properties:*

(T.1) *$T(x;\xi,y;\eta,z;\zeta)$ is positively homogeneous of degree 1 in $\theta=(\xi,\eta)$ and degree zero in ζ.*

(T.2) *There are constants $C>0$, $M>0$ such that $|\partial_y T|\geqq M(|\xi|+|\eta|)$ if $|\eta|\leqq C^{-1}|\xi|$ or $|\eta|\geqq C|\xi|$.*

(T.3) *If $(1/2)C^{-1}|\xi|\leqq|\eta|\leqq 2C|\xi|$, then on any conical subset in $\mathring{T}^*N\times\mathring{T}^*N\times\mathring{T}^*N$ bounded away from the critical set $\{(x;\xi,y_c;\eta_c,z;\zeta)\}$, there is*

$\delta>0$ *such that* $|\nabla_{(y;\eta)}\hat{T}|\geqq\delta$ *on* $(1/2)C^{-1}\leqq|\eta|\leqq 2C$, *where* $\hat{T}=T(x;\hat{\xi},y;\eta,z;\hat{\zeta})$.

PROOF. We have only to show (T.3). Since T has no critical point on the considered domain and $(x;\hat{\xi},y;\eta,z;\hat{\zeta})$ moves in a compact set, we see the existence of $\delta>0$. □

4.4. Normal form of T.

We continue to assume that τ_1, τ_2 are sufficiently close to τ_0 in ϑ_0, and let r_1 be as in 4.2. Let $(y_c;\eta_c)$ be the critical point in the domain $d(x,y)\leqq r_1/2$, $d(x,z)\leqq r_1/4$. Recall that if $Q_1=Q_2=0$, then $(y_c;\eta_c)=\cdot_x(Z,-\xi(\partial_X\tilde{S})^{-1}|_{X=Z})$. Therefore, one may assume that there is $\delta>0$ such that $|X_c-Z|\leqq\delta$, $|\eta'_c+\xi(\partial_X\tilde{S})^{-1}|\leqq\delta|\xi|$ in general, whenever τ_1, τ_2 are sufficiently close to τ_0.

Denote by D_δ the domain given by

$$D_\delta=\{(x;\xi,\cdot_x(X,\eta'),\cdot_x(Z,\zeta'));|X|\leqq r_1/2,\ |Z|\leqq r_1/4,\ |\eta'+\xi(\partial_X\tilde{S})^{-1}|\leqq\delta|\xi|\}\ . \tag{4.10}$$

Obviously, $(x;\xi,\cdot_x(X_c,\eta'_c),z;\zeta)\in D_\delta$. Moreover, the index of the critical point $(y_c;\eta_c)$ is the same as that of $\tau_0\boxplus\tau_0$ and hence 0. Thus, by a suitable change of coordinate on a neighborhood of $(y_c;\eta_c)$, T can be expressed in the form $-\langle\eta'-\eta'_c|X-X_c\rangle$. This is known as the Morse lemma. However, the proof of the Morse lemma shows more precisely the following:

PROPOSITION 4.7. *Suppose that* $\delta>0$ *is sufficiently small. There are an open neighborhood* D' *of* D_δ *and a* C^∞ *diffeomorphism* $\tilde{\Psi}$ *of* D' *into* $(\mathring{T}^*N)^3$ *such that* $\tilde{\Psi}(x;\xi,y;\eta,z;\zeta)=(x;\xi,\tilde{\Psi}_1(*);\tilde{\Psi}_2(*),z;\zeta)$ *and satisfy the following*

(i) $\tilde{\Psi}(D')\supset D_\delta$.

(ii) $\tilde{\Psi}_1(x;r\xi,y;r\eta,z;s\zeta)=\tilde{\Psi}_1(x;\xi,y;\eta,z;\zeta)$,
$\tilde{\Psi}_2(x;r\xi,y;r\eta,z;s\zeta)=r\tilde{\Psi}_2(x;\xi,y;\eta,z;\zeta)$ *for any* $r>0$, $s>0$.

(iii) $\tilde{\Psi}$ *depends smoothly on* τ_1, τ_2.

(iv) $\tilde{\Psi}^*T=-\langle\eta'-\eta'_c|X-X_c\rangle$.

The above proposition will be proved in several lemmas below. At first, denote $T_0=\tau_0\boxplus\tau_0-c(\tau_0,\tau_0)$. Since the critical point $(y_c;\eta_c)=\cdot_x(X_c,\eta'_c)$ in this case, is given by $(X_c,\eta'_c)=(Z,-\xi(\partial_X\tilde{S})^{-1}|_{X=Z})$, we see that

$$\begin{aligned}T_0(x;\xi,\cdot_x(X,\eta'),z;\zeta)&=\langle\xi-\eta'\tilde{S}_1(x;Z,X)\,|\,X-Z\rangle\\&=\langle-\eta'_c(\partial_X\tilde{S})|_{X=Z}-\eta'\tilde{S}_1(x;Z,X)\,|\,X-X_c\rangle\ ,\end{aligned}$$

where $\tilde{S}(x;Z,X)=\tilde{S}_1(x;Z,X)(Z-X)$. Using $\tilde{S}_1(x;Z,Z)=-\partial_X\tilde{S}|_{X=Z}$, we see that T_0 can be written in the form

$$T_0 = -\langle \eta' - \eta'_c \,|\, X - X_c \rangle + S_c (X - X_c)^2 ,$$

where $S_c = S_c(x; \xi, \cdot_x(X, \eta'), z; \zeta)$ and $S_c = O(|\eta'|)$.

LEMMA 4.8. *On a neighborhood D' of D_δ, $T(x; \xi, \cdot_x(X, \eta'), z; \zeta)$ can be written as*

$$T = A_1(\eta' - \eta'_c)^2 + (-I + A_2)(\eta' - \eta'_c)(X - X_c) + (S_c + A_3)(X - X_c)^2 ,$$

where $A_i = A_i(x; \xi, \cdot_x(X, \eta'), z; \zeta)$ and A_1, A_2, A_3 are positively homogeneous of degree -1, 0, 1, respectively with respect to the combined variable $\theta = (\xi, \eta')$ and of degree 0 with respect to ζ. Moreover, if $\tau_1 \to \tau_0$, $\tau_2 \to \tau_0$, then $|A_1||\theta| \to 0$, $|A_2| \to 0$, $|A_3|/|\theta| \to 0$ uniformly on D'.

Proof is easy by using Taylor's theorem at (X_c, η'_c).

Now, consider a quadratic form $h(\xi, X)$ on $\boldsymbol{R}^n \times \boldsymbol{R}^n$ such that

$$h = P^{ij}\xi_i\xi_j + (\delta^i_j + l^i_j)\xi_i X^j + R_{ij} X^i X^j ,$$

where (δ^i_j) is the identity matrix.

LEMMA 4.9. *Suppose $|l^i_j|$ and $|P^{ij}R_{kl}|$ are sufficiently small for all i, j, k, l. Then there are matrices (a^i_j), (f^i_j) depending smoothly on (P^{ij}), (l^i_j), (R_{ij}) such that*

$$h = (\xi_i + a_{ij}X^j)(f^i_k X^k + P^{il}\xi_l)$$

and $|f^i_k - \delta^i_k|$ are sufficiently small.

Proof has been done by using the implicit function theorem (cf. [10], pp. 243-244). We have only to solve

$$f^l_j + P^{li}(f^{-1})^k_i R_{kj} = \delta^l_j + l^l_j , \tag{4.11}$$

and set $a_{ij} = (f^{-1})^k_i R_{kj}$. □

Set $(a_{ij}) = \Phi(P, l, R)$, $(f^i_j) = \Psi(P, l, R)$ and apply the above lemma to our $-T$, then, we have the following:

LEMMA 4.10. *On the domain D', $-T$ can be expressed in the form*

$$-T = \langle \eta' - \eta'_c + \Phi(A_1, A_2, A_3)(X - X_c) \,|\, \Psi(A_1, A_2, A_3)(X - X_c) - A_1(\eta' - \eta'_c) \rangle .$$

Moreover, $\Phi(A_1, A_2, A_3)$ (resp. $\Psi(A_1, A_2, A_3)$) is positively homogeneous of degree 1 (resp. 0) in the variable θ and Φ, Ψ are positively homogeneous of degree 0 in the variable ζ'.

PROOF. We have only to show the second statement. Recall the homogeneity property of A_i. Since $(P^{ij})=-A_1$, $(l^i_j)=-A_2$, $(R_{ij})=-(S_o+A_3)$, the equation (4.11) shows that (f^i_j) is positively homogeneous of degree 0 with respect to θ. Hence, by the equality $a=f^{-1}R$, we get the desired property. □

PROOF OF PROPOSITION 4.7. Now, set

$$(4.12)\qquad \begin{cases} \bar{\eta}'-\eta'_c=\eta'-\eta'_c+\Phi(A_1, A_2, A_3)(X-X_c)\ , \\ \bar{X}-X_c=\Psi(A_1, A_2, A_3)(X-X_c)-A_1(\eta'-\eta'_c)\ . \end{cases}$$

The estimates for A_i's in Lemma 4.8 yield that the Jacobian $D(\bar{\eta}, \bar{X})/D(\eta', X)$ never vanishes. So the above equation can be solved reversely with respect to (X, η') by using the implicit function theorem. Moreover, by the implicit function theorem given in [12], Lemma 4.9, we see that

$$\begin{cases} \eta'=\eta'(x; \xi, \cdot_x(\bar{X}, \bar{\eta}'), z; \zeta; A_1, A_2, A_3) \\ X=X(x; \xi, \cdot_x(\bar{X}, \bar{\eta}'), z; \zeta; A_1, A_2, A_3) \end{cases}$$

are smooth. Thus, remarking that A_i's depend smoothly on τ_1, τ_2, we see η', X depend smoothly on $(x; \xi, \cdot_x(\bar{X}, \bar{\eta}'), z; \zeta, \tau_1, \tau_2)$. Since τ_1, τ_2 are sufficiently close to τ_0, one may assume that the domain of η', X contains D_δ. □

§ 5. Amplitude functions (blown up symbols).

In this section, we shall fix a class of amplitude functions of primordial operators. Roughly speaking, functions in such a class are obtained by the blowing up of usual amplitude functions. The main reason for using such functions is to make the class invariant under the natural action of $\mathscr{D}^{(1)}_\Omega$ and to make it closed under the multiplication.

5.1. Compactification of $\mathring{T}^*N$.

Recall that $\mathring{T}^*N$ is naturally diffeomorphic to $\boldsymbol{R}_+\times S^*N$, where $\boldsymbol{R}_+=(0, \infty)$. Hence for a positive integer k,

$$(\mathring{T}^*N)^k=\underbrace{\mathring{T}^*N\times\cdots\times\mathring{T}^*N}_{k}$$

can be viewed as $\boldsymbol{R}^k_+\times(S^*N)^k$. Here, we shall give a compactification of $\boldsymbol{R}^k_+$.

Take a positive constant K, $K>1$. For each integer l, $0\leqq l\leqq k$, and each l-tuple of ordered integers $I=(i_1, \cdots, i_l)$, $1\leqq i_1, \cdots, i_l\leqq k$, which are mutually distinct. (If $l=0$, we write simply by $I=\emptyset$.) We define a subset $\Delta_{k,I}$ by

$$\Delta_{k,I}=\{(s_1, \cdots, s_k)\in \mathbf{R}_+^k;\ s_{i_1}\geqq K^{-1},\ s_{i_j}\geqq K^{-1}s_{i_{j-1}},\ j=2, \cdots, l, \text{ and } 0<s_j\leqq K \text{ for } j\notin I\}\ .$$

Then, it is easily seen that $\bigcup_{\substack{I:\text{ all ordering}\\ 0\leqq l\leqq k}}\Delta_{k,I}=\mathbf{R}_+^k$. Define maps $i_{k,I}\colon \Delta_{k,I}\to [0, K]^k$ for $I=(i_1, \cdots, i_l)\neq\emptyset$, by

$$i_{k,I}(s_1, \cdots, s_k)=(r^{-1}, t_1, \cdots, \check{t}_{i_1}, \cdots, t_k)\ , \tag{5.1}$$

where

$$\begin{cases} r=s_{i_1},\ t_{i_2}=s_{i_1}/s_{i_2}, \cdots, t_{i_l}=s_{i_{l-1}}/s_{i_l}\ , & \text{and} \\ s_j=t_j \quad \text{for} \quad j\neq i_1, \cdots, i_l\ . \end{cases} \tag{5.2}$$

Moreover, for $I=\emptyset$, we define $i_{k,\emptyset}$ by

$$i_{k,\emptyset}(s_1, \cdots, s_k)=(t_1, \cdots, t_k)\ ,\quad s_j=t_j \quad \text{for} \quad j=1, \cdots, k\ . \tag{5.3}$$

REMARK. (i) We put a coordinate on $\Delta_{k,I}$ by using variables r^{-1}, $t_1, \cdots, t_k$. But one of these is not used for each I (see List 5.1). (ii) To give a compactification of $\mathbf{R}_+^k$, we use the variable r^{-1} instead of r.

To simplify the notation, we often write $(t_1, \cdots, t_k)$ by t, a point $(x_1; \xi_1, \cdots, x_k; \xi_k)$ of $(T^*N)^k$ by $(\boldsymbol{x}; \boldsymbol{\xi})$ and point $(x_1; \hat{\xi}_1, \cdots, x_k; \hat{\xi}_k)$ of $(S^*N)^k$ by $(\boldsymbol{x}; \hat{\boldsymbol{\xi}})$, respectively.

By attaching $r^{-1}=0$, $t_1=\cdots=t_k=0$, we obtain a compactification of $\mathbf{R}_+^k$. Remark that the above compactification of $\mathbf{R}_+$ is natural two points compactification $[0, \infty]$.

Since our compactification is complicated, we shall list up the exact domains and used variables of $\Delta_{k,I}$ for the case $k=2, 3$ for our later use:

LIST 5.1. (A) $k=2$;
$\Delta_{2,\phi}=\{(t_1, t_2);\ 0<t_1, t_2\leqq K\}$,
$\Delta_{2,(1)}=\{(r, t_2);\ 0<r^{-1}, t_2\leqq K\}$,
$\Delta_{2,(2)}=\{(t_1, r);\ 0<r^{-1}, t_1\leqq K\}$,
$\Delta_{2,(1,2)}=\{(r, r/t_2);\ 0<r^{-1}, t_2\leqq K\}$,
$\Delta_{2,(2,1)}=\{(r/t_1, r);\ 0<r^{-1}, t_1\leqq K\}$.
(B) $k=3$;
$\Delta_{3,\emptyset}=\{(t_1, t_2, t_3);\ 0<t_i\leqq K,\ i=1, 2, 3\}$,
$\Delta_{3,(1)}=\{(r, t_2, t_3);\ 0<r^{-1}, t_2, t_3\leqq K\}$,
$\Delta_{3,(2)}=\{(t_1, r, t_3);\ 0<r^{-1}, t_1, t_3\leqq K\}$,
$\Delta_{3,(3)}=\{(t_1, t_2, r);\ 0<r^{-1}, t_1, t_2\leqq K\}$,

$$\Delta_{3,(1,2)}=\{(r,\ r/t_2,\ t_3);\ 0<r^{-1},\ t_2,\ t_3\leqq K\},$$
$$\Delta_{3,(2,1)}=\{(r/t_1,\ r,\ t_3);\ 0<r^{-1},\ t_1,\ t_3\leqq K\},$$
$$\Delta_{3,(2,3)}=\{(t_1,\ r,\ r/t_3);\ 0<r^{-1},\ t_1,\ t_3\leqq K\},$$
$$\Delta_{3,(3,2)}=\{(t_1,\ r/t_2,\ r);\ 0<r^{-1},\ t_1,\ t_2\leqq K\},$$
$$\Delta_{3,(3,1)}=\{(r/t_1,\ t_2,\ r);\ 0<r^{-1},\ t_1,\ t_2\leqq K\},$$
$$\Delta_{3,(1,3)}=\{(r,\ t_2,\ r/t_3);\ 0<r^{-1},\ t_2,\ t_3\leqq K\},$$
$$\Delta_{3,(1,2,3)}=\{(r,\ r/t_2,\ r/t_2t_3);\ 0<r^{-1},\ t_2,\ t_3\leqq K\},$$
$$\Delta_{3,(1,3,2)}=\{(r,\ r/t_2t_3,\ r/t_3);\ 0<r^{-1},\ t_2,\ t_3\leqq K\},$$
$$\Delta_{3,(2,1,3)}=\{(r/t_1,\ r,\ r/t_1t_3);\ 0<r^{-1},\ t_1,\ t_3\leqq K\},$$
$$\Delta_{3,(2,3,1)}=\{(r/t_1t_3,\ r,\ r/t_3);\ 0<r^{-1},\ t_1,\ t_3\leqq K\},$$
$$\Delta_{3,(3,1,2)}=\{(r/t_1,\ r/t_1t_2,\ r);\ 0<r^{-1},\ t_1,\ t_2\leqq K\},$$
$$\Delta_{3,(3,2,1)}=\{(r/t_1t_2,\ r/t_2,\ r);\ 0<r^{-1},\ t_1,\ t_2\leqq K\}.$$

Now, by the identification $(\mathring{T}^*N)^k=\boldsymbol{R}_+^k\times(S^*N)^k$ and the above compactification of $\boldsymbol{R}_+^k$, we get a compactification of $(\mathring{T}^*N)^k$. Namely, for $I=(i_1,\cdots,i_l)$, $l\leqq k$, we use a set $\Delta_{k,I}\times(S^*N)^k$, and a map $i_{k,I}\times\mathrm{id.}$: $\Delta_{k,I}\times(S^*N)^k\to[0,K]^k\times(S^*N)^k$ and compactify $(\mathring{T}^*N)^k$. Hereafter, we shall use the same notations $\Delta_{k,I}$ and $i_{k,I}$ instead of $\Delta_{k,I}\times(S^*N)^k$ and $i_{k,I}\times\mathrm{id.}$.

5.2. The class of amplitude functions $\mathfrak{a}^k$.

Now, each C^∞ function f on $(\mathring{T}^*N)^k$ can be regarded as a function on $\boldsymbol{R}_+^k\times(S^*N)^k$ and therefore, we write it by the same letter f if it is not confused, i.e.,

$$f(\boldsymbol{s},\boldsymbol{x};\hat{\boldsymbol{\xi}})=f(x_1;s_1\hat{\xi}_1,\cdots,x_k;s_k\hat{\xi}_k)\,, \tag{5.4}$$
$$\boldsymbol{s}=(s_1,\cdots,s_k)\,,\quad(\boldsymbol{x};\hat{\boldsymbol{\xi}})=(x_1;\hat{\xi}_1,\cdots,x_k;\hat{\xi}_k)\,.$$

For $I=(i_1,\cdots,i_l)$, $l\leqq k$, consider $i_{k,I}^{-1*}(f\,|\,\Delta_{k,I})$, where $i_{k,I}$ is defined by (5.2) and (5.3). We often write by $\tilde{f}_{k,I}$ instead of $i_{k,I}^{-1*}(f\,|\,\Delta_{k,I})$ for the sake of simplicity.

DEFINITION 5.2. $f\in C^\infty((\mathring{T}^*N)^k)$ is called an *amplitude function*, if the following conditions are satisfied:

(A.1) For each $I=(i_1,\cdots,i_l)$, $0\leqq l\leqq k$, $\tilde{f}_{k,I}$ can be extended smoothly at $t_j=0$ $(j=1,\cdots,k)$.
(A.2) For each $I=(i_1,\cdots,i_l)$, $0<l\leqq k$, $\tilde{f}_{k,I}$ has an asymptotic expansion as follows:

$$\tilde{f}_{k,I}(r^{-1},\boldsymbol{t},\boldsymbol{x};\hat{\boldsymbol{\xi}})\sim\sum_{j\leqq 0}A_j(\boldsymbol{t},\boldsymbol{x};\hat{\boldsymbol{\xi}})r^j\,, \tag{5.5}$$

where $A_j(\boldsymbol{t},\boldsymbol{x};\hat{\boldsymbol{\xi}})$ are C^∞ functions on $[0,K]^{k-1}\times(S^*N)^k$.

REMARK. The condition (5.5) means that $\tilde{f}_{k,I}$ is smooth at $r=\infty$.

DEFINITION 5.3. (i) We denote by $\mathfrak{a}^k$ the totality of amplitude functions which satisfy (A.1) and (A.2) in the above definition.

(ii) For each $I=(i_1, \cdots, i_l)$, $0<l\leqq k$, we denote by $\tilde{\mathfrak{a}}^k_{I,m}$ the totality of C^∞ functions $\tilde{f}_{k,I}$ on $[0, K]^k\times(S^*N)^k$ such that for non-positive integer m, $\tilde{f}_{k,I}$ has the following asymptotic expansion:

(5.6) $$\tilde{f}_{k,I}\sim\sum_{j\leqq m} A_j(t, x; \hat{\xi})r^j\ , \quad A_j\in C^\infty([0, K]^{k-1}\times(S^*N)^k)\ .$$

(iii) For small $\varepsilon_1, \cdots, \varepsilon_{k-1}>0$, denote by $\mathfrak{a}^k(\varepsilon_1, \cdots, \varepsilon_{k-1})$ the space of all functions $f\in\mathfrak{a}^k$ such that

(5.7) $$f(x; \xi)\equiv 0 \quad \text{if} \quad d(x_i, x_{i+1})>\varepsilon_i \quad \text{for some} \quad i=1, \cdots, k-1\ .$$

REMARK. By Definition 5.3 and the remark in 5.1, $f(x; \xi)\in\mathfrak{a}^1$ if and only if it satisfies, for $t>0$

(a) $\tilde{f}(t, x; \hat{\xi})=f(x; t\hat{\xi})$ can be extended smoothly on $[0, \infty)\times S^*N$;

(b) f has an asymptotic expansion, for large $r>0$, $\tilde{f}\sim\sum_{j\leqq 0} A_j(x; \hat{\xi})r^j$.

Next, we shall put a system of norms on $\mathfrak{a}^k$. Let $f\in\mathfrak{a}^k$. Then, for every $I=(i_1, \cdots, i_l)$, $I\neq\emptyset$, and any non-positive integer m, $\tilde{f}_{k,I}=i^{-1*}_{k,I}(f\,|\,\Delta_{k,I})$ in (5.5) can be written in the following form: For fixed C^∞ function $\phi(r)$ such that $\phi(r)\equiv 0$ on $0\leqq r\leqq 2K^{-1}$, and $\equiv 1$ on $r\geqq 3K^{-1}$, we have

(5.8) $$\tilde{f}_{k,I}=\phi(r)\Big(\sum_{m\leqq j\leqq 0} A_j(t, x; \hat{\xi})r^j\Big)+\tilde{f}_{I,m-1}(r^{-1}, t, x; \xi)\ ,$$

where $\tilde{f}_{I,m-1}\in\tilde{\mathfrak{a}}^k_{I,m-1}$.

Let $|A_j|_s$ be the C^s-norm on $[0, K]^{k-1}\times(S^*N)^k$.

DEFINITION 5.4. For each function $\tilde{f}_{k,I}$ on $[0, K]^k\times(S^*N)^k$, we define a norm $\|\tilde{f}_{k,I}\|_{m,s}$, $s\geqq 0$, $m\leqq 0$, $I\neq\emptyset$, as follows:

(5.9) $$\|\tilde{f}_{k,I}\|_{m,s}=\sum_{m\leqq j\leqq 0}|A_j|_s+\square\tilde{f}_{I,m-1}\square_s\ ;$$

(5.10) $$\|\tilde{f}_{I,m-1}\|_s=\sup_{\substack{r>0,\, p+|\alpha|=s\\ (x;\hat{\xi})\in(S^*N)^k}}|(1+r)^{-m+p+1}(\partial/\partial r)^p D^\alpha_{(t,x;\hat{\xi})}\tilde{f}_{I,m-1}|\ ,$$

where $D_{(t,x;\hat{\xi})}$ is the derivative on $[0, K]^{k-1}\times(S^*N)^k$ by using a normal coordinate system.

DEFINITION 5.5. For each $f\in\mathfrak{a}^k$, we define a norm $\|f\|_{m,s}$, $s\geqq 0$, $m\leqq 0$ by

(5.11) $$\|f\|_{m,s}=\sum_{\substack{I=\{i_1,\cdots,i_l\}\\ 1\leqq l\leqq k}}\|\tilde{f}_{k,I}\|_{m,s}+\|\tilde{f}_\emptyset\|_s\ ,$$

where $\|\tilde{f}_\emptyset\|_s$ is the C^s-norm of $\tilde{f}_\emptyset=i^{-1*}_{k,\emptyset}(f\,|\,\Delta_{k,\emptyset})$ on $[0, K]^k\times(S^*N)^k$, and the summation of the first term of (5.11) is taken by all l-tuple of mutually distinct indices in $\{1, \cdots, k\}$.

For every $m \leqq 0$, the system of norms $\{\|\cdot\|_{m,s}; s=0, 1, 2, \cdots\}$ gives a topology T_m on $\mathfrak{a}^k$. We denote by $\mathfrak{a}^k_{(m)}$ the completion of $(\mathfrak{a}^k, T_m)$. An element of $\mathfrak{a}^k_{(m)}$ will be called an *extended amplitude function* on $(T^*N)^k$. It is not hard to see that $\cap_m \mathfrak{a}^k_{(m)} = \mathfrak{a}^k$. Thus, we define the inverse limit topology for $\mathfrak{a}^k$. As a result, $\mathfrak{a}^k$ has a *Fréchet structure by above system of norms*. Also, we denote by $\mathfrak{a}^k_{(m)}(\varepsilon_1, \cdots, \varepsilon_{k-1})$, for $m \leqq 0$, the closure of $\mathfrak{a}^k(\varepsilon_1, \cdots, \varepsilon_{k-1})$ in $\mathfrak{a}^k_{(m)}$.

REMARK. By the definition of amplitude functions, it is easily seen that $\mathfrak{a}^k$ is invariant under any permutation of variables.

5.3. Properties of $\mathfrak{a}^k$.

In the following, we shall investigate the differentiability of some operations on $\mathfrak{a}^k$.

Given $f, g \in \mathfrak{a}^k$, denote by $f \cdot g$ the natural pointwise multiplication of f and g. Then, it is easily seen that $f \cdot g \in \mathfrak{a}^k$. Moreover, we have the following:

LEMMA 5.6. *The multiplication map $M: \mathfrak{a}^k \times \mathfrak{a}^k \to \mathfrak{a}^k$, defined by $M(f, g) = f \cdot g$, can be extended to a continuous bilinear mapping of $\mathfrak{a}^k_{(m)} \times \mathfrak{a}^k_{(m)}$ into $\mathfrak{a}^k_{(m)}$, for every $m \leqq 0$.*

For each $\mathfrak{a}^k$, $k \geqq 1$, $\mathfrak{a}^{k-1}$ can be embedded smoothly in $\mathfrak{a}^k$ as follows: Let $p_j: (\mathring{T}^*N)^k \to (\mathring{T}^*N)^{k-1}$ be the projection defined by, for $j = 1, \cdots, k$,

$$p_j(x; \xi) = (x_1; \xi_1, \cdots, \check{x}_j; \check{\xi}_j, \cdots, x_k; \xi_k) , \tag{5.12}$$

where $\check{x}_j$, $\check{\xi}_j$ mean that x_j, ξ_j are omitted.

For p_j, we have the following:

LEMMA 5.7. *Given $f \in \mathfrak{a}^{k-1}$, $p_j^* f \in \mathfrak{a}^k$. Moreover, the mapping p_j^*: $\mathfrak{a}^{k-1} \to \mathfrak{a}^k$ can be extended to a continuous linear mapping of $\mathfrak{a}^{k-1}_{(m)}$ into $\mathfrak{a}^k_{(m)}$ for every $m \leqq 0$.*

PROOF. Let $I = (i_1, \cdots, i_l)$ be l-tuple of indices. By the remark in 5.2, we may assume that $i_1 < i_2 < \cdots < i_l$. Then, for any $f \in \mathfrak{a}^{k-1}$, we have

$$(\widetilde{p_j^* f})_{k,I} = \begin{cases} f_{k-1,I}(r^{-1}, t_1, \cdots, \check{t}_j, \cdots, t_k, x_1; \hat{\xi}_1, \cdots, \check{x}_j; \check{\xi}_j, \cdots, x_k; \hat{\xi}_k) , & j \notin I ; \\ f_{k-1,I}((r/t_{i_2})^{-1}, t_1, \cdots, \check{t}_j, \cdots, t_k, x_1; \hat{\xi}_1, \cdots, \check{x}_j; \check{\xi}_j, \cdots, x_k; \hat{\xi}_k) , & j = i_1 ; \\ f_{k-1,I}(r^{-1}, t_1, \cdots, t_{i_{m-1}}, \check{t}_j, t_{i_m} \cdot t_{i_{m+1}}, \cdots, t_k, x_1; \hat{\xi}_1, \cdots, \check{x}_j; \check{\xi}_j, \cdots, \\ \qquad x_k; \hat{\xi}_k) , & j = i_m \quad (m \geqq 2) . \end{cases} \tag{5.13}$$

From this, we get the lemma. □

Next, we give a diagonalized operation. Given positive integer i, $1\leqq i\leqq k-1$, define a map d_i: $(\mathring{T}^*N)^k\times(\mathring{T}^*N)^{k+1}$ by

(5.14) $$d_i(\boldsymbol{x};\boldsymbol{\xi})=(x_1;\xi_1,\cdots,x_i;\xi_i,x_i;\xi_i,x_{i+1};\xi_{i+1},\cdots,x_k;\xi_k)\,.$$

Denote by $d_i^*\colon\mathfrak{a}^{k+1}\to\mathfrak{a}^k$ the pull-back mapping induced from d_i. By a similar computation as above, we get the following:

LEMMA 5.8. *For every $f\in\mathfrak{a}^{k+1}$, $d_i^*f\in\mathfrak{a}^k$ $(i=1,\cdots,k-1)$. The mapping $d_i^*\colon\mathfrak{a}^{k+1}\to\mathfrak{a}^k$ can be extended to a continuous linear mapping of $\mathfrak{a}^{k+1}_{(m)}$ into $\mathfrak{a}^k_{(m)}$ for every $m\leqq 0$.*

Now, for $f\in\mathfrak{a}^k$, $g\in\mathfrak{a}^{k'}$, define a map $\boxtimes\colon\mathfrak{a}^k\times\mathfrak{a}^{k'}\to\mathfrak{a}^{k+k'-1}$ by

(5.15) $$\begin{aligned}f\boxtimes g(x_1;\xi_1,\cdots,x_{k+k'-1};\xi_{k+k'-1})\\=f(x_1;\xi_1,\cdots,x_k;\xi_k)g(x_k;\xi_k,\cdots,x_{k+k'-1};\xi_{k+k'-1})\,.\end{aligned}$$

Namely,

(5.16) $$f\boxtimes g=d_k^*M(p^*_{k,k'+k}f,\,p^*_{k',k'+k}g)\,.$$

Hence, from Lemmas 5.7–8, we have the following:

COROLLARY 5.9. *The mapping $\boxtimes\colon\mathfrak{a}^k\times\mathfrak{a}^{k'}\to\mathfrak{a}^{k+k'-1}$ can be extended to a continuous bilinear mapping of $\mathfrak{a}^k_{(m)}\times\mathfrak{a}^{k'}_{(m)}$ into $\mathfrak{a}^{k+k'-1}_{(m)}$ for every $m\leqq 0$.*

Finally, we shall state the differentiability of the action of $\mathscr{D}^{(1)}_\Omega$ on $\mathfrak{a}^k$. Namely, we get the following:

LEMMA 5.10. *For each $\varphi\in\mathscr{D}^{(1)}_\Omega$ and $f\in\mathfrak{a}^k$, φ^*f is an element of $\mathfrak{a}^k$. Moreover, the mapping $ev\colon\mathscr{D}^{(1)}_\Omega\times\mathfrak{a}^k\to\mathfrak{a}^k$, defined by $ev(\varphi,f)=\varphi^*f$ can be extended to a C^∞ mapping of $\mathscr{D}^{(1)}_\Omega\times\mathfrak{a}^k_{(m)}$ into $\mathfrak{a}^k_{(m)}$ for every $m\leqq 0$.*

PROOF. Let $\varphi\in\mathscr{D}^{(1)}_\Omega$. Write $\varphi(x;\xi)$ by $(\varphi_1(x;\xi);\varphi_2(x;\xi))$. Putting $\mu(x;\hat{\xi})=|\varphi_2(x;\hat{\xi})|$, we see $\mu(x;\hat{\xi})>0$ and φ maps $(r,x;\hat{\xi})$ to $(\mu(x;\hat{\xi})r$, $\hat{\varphi}(x;\hat{\xi}))$, where $\hat{\varphi}(x;\hat{\xi})=(\varphi_1(x;\hat{\xi});\mu(x;\hat{\xi})^{-1}\varphi_2(x;\hat{\xi}))$. Hence, we have

(5.17) $$\begin{aligned}\varphi^*f(x_1;r_1\hat{\xi}_1,\cdots,x_k;r_k\hat{\xi}_k)=f(\varphi_1(x_1;\hat{\xi}_1);\mu(x_1;\hat{\xi}_1)r_1\cdot\hat{\varphi}_2(x_1;\hat{\xi}_1),\cdots,\\ \varphi_1(x_k;\hat{\xi}_k);\mu(x_k;\hat{\xi}_k)r_k\cdot\hat{\varphi}_2(x_k;\hat{\xi}_k))\,.\end{aligned}$$

Thus, for any $\mathrm{I}=(i_1,\cdots,i_l)$, $i_1<\cdots<i_l$, we have

(5.18) $$(\widetilde{\varphi^*f})_{k,I}(r^{-1},\boldsymbol{t},\boldsymbol{x};\hat{\boldsymbol{\xi}})=\tilde{f}_{k,I}((\mu(x_{i_1};\hat{\xi}_{i_1})r)^{-1};\boldsymbol{t}',\hat{\varphi}(x_1;\hat{\xi}_1),\cdots,\hat{\varphi}(x_k;\hat{\xi}_k))\,,$$

where $t'_i=t_i$ for $i\notin I$, and $t'_{i_2}=\mu(x_{i_1};\hat{\xi}_{i_1})t_{i_2}/\mu(x_{i_2};\hat{\xi}_{i_2}),\cdots,t'_{i_l}=\mu(x_{i_{l-1}};$

$\hat{\xi}_{i_{l-1}})t_{i_l}/\mu(x_{i_l};\hat{\xi}_{i_l})$. For the other case of I, the computation is smilar. By the differentiability of (5.18) for each I, we obtain the desired results. □

5.4. Local amplitudes.

For our later use in § 7, we shall modify Lemma 5.10 to a certain local form. First of all, we remark the following:

LEMMA 5.11. *Suppose* $f \in C^\infty((\mathring{T}^*N)^2)$ *satisfies the following conditions:*

(LA.1) $f \equiv 0$ *if* $|\xi|^2+|\eta|^2 \leqq R^2$, *or* $|\xi|/|\eta| \geqq C$ *or* $|\eta|/|\xi| \geqq C$, $C \geqq 1$;

(LA.2) *Put* $F(r, \theta, x; \hat{\xi}, y; \hat{\eta}) = f(x; r(\cos\theta)\hat{\xi}, y; r(\sin\theta)\hat{\eta})$. *Then* F *has an asymptotic expansion*

$$F \sim \sum_{j \leqq 0} A_j(\theta, x; \hat{\xi}, y; \hat{\eta}) r^j \quad (r \gg 0) .$$

Then, $f \in \mathfrak{a}^2$.

PROOF. Set $\bar{f}(r_1, r_2, x; \hat{\xi}, y; \hat{\eta}) = f(x; r_1\hat{\xi}, y; r_2\hat{\eta})$, and recall List 5.1. On $\Delta_{2,\varnothing}$, $\Delta_{2,(1)}$, $\Delta_{2,(2)}$, there is no problem because $\bar{f}_{2,I}$ on each domain is identically zero. Hence, we have only to check that $\bar{f}(r/t, r, x; \hat{\xi}, y; \hat{\eta})$ and $\bar{f}(r, r/t, x; \hat{\xi}, y; \hat{\eta})$ have asymptotic expansions requested in Definition 5.2. However, these functions are zero whenever $t \leqq C^{-1}$ or $t \geqq C$. Thus, we have the desired expansion by using (LA.2). □

Now, denote by $D_{\varepsilon,\delta}$ the domain $\{(x; \xi, \cdot_x(X, \eta')); |X| \leqq \varepsilon, |\xi - \eta'| \leqq \delta(|\xi| + |\eta'|)\}$, where $\varepsilon > 0$, $\delta > 0$. We consider C^∞ functions f such that $\operatorname{supp} f \subset D_{\varepsilon,\delta}$ and f satisfies (LA.1-2) in the above lemma. For such a class of functions, we give the restricted topology of $\mathfrak{a}^2_{(m)}$.

Let Ψ be a C^∞ diffeomorphism of $D_{\varepsilon,\delta}$ into an open neighborhood of $D_{\varepsilon,\delta}$ such that $\Psi(x; r\xi, y; r\eta) = (\Psi_1; r\Psi_2, \Psi_3; r\Psi_4)$ where $\Psi_i = \Psi_i(x; \xi, y; \eta)$. Such a class of diffeomorphisms can be topologized by the standard C^∞ topology by which it turns out to be an open set of a Fréchet space. For such Ψ, and for such f defined above, $\Psi^* f$ is again a C^∞ function on $(\mathring{T}^*N)^2$ satisfying (LA.1-2). Moreover, by the smoothness of compositions, we have

LEMMA 5.12. *Notations and assumptions being as above,* $\Psi^* f$ *is smooth with respect to* Ψ *and* f *for every* $m \leqq 0$.

§ 6. Proof of Theorem A.

In this section, we shall prove Propositions 2.2-3 and finally give the proof of Theorem A by assuming the smoothness property of some oscillatory integral (cf. Proposition 6.1). This smoothness property will be proved in the next section.

6.1. Contraction integrals.

(a) Choose $\varepsilon_1, \varepsilon_2>0$ so that $\varepsilon_i<r_1/4$, where r_1 is given in § 4. Recall the definition of $\mathfrak{a}^3(\varepsilon_1, \varepsilon_2)$ and $\mathfrak{a}^3_{(m)}(\varepsilon_1, \varepsilon_2)$ (cf. Definitions 5.3 and 5.5). Let τ_1, τ_2 be elements of ϑ_0 which are sufficiently close to τ_0. Given $a \in \mathfrak{a}^3(\varepsilon_1, \varepsilon_2)$, consider the following integral

$$\langle ae^{-i\tau_1\boxplus\tau_2}\rangle(x; \xi, z; \zeta)=Os-\iint a(x; \xi, y; \eta, z; \zeta)e^{-i\tau_1\boxplus\tau_2(x;\xi,y;\eta,z;\zeta)}\text{đ}y\text{đ}\eta\ . \tag{6.1}$$

The above integral can be defined as the oscillatory integral for any fixed $(x; \xi)$, $(z; \zeta)$ and it will be called the *contraction integral of a by* $\tau_1\boxplus\tau_2$.

First of all, we state the following, which will be proved in § 7:

PROPOSITION 6.1. (i) *For* $\tau_1, \tau_2 \in \vartheta_0$, *sufficiently close to* τ_0, *and* $a \in \mathfrak{a}^3(\varepsilon_1, \varepsilon_2)$, $\langle ae^{-i\tau_1\boxplus\tau_2}\rangle$ *can be written by*

$$\langle ae^{-i\tau_1\boxplus\tau_2}\rangle(x; \xi, z; \zeta)=b(x; \xi, z; \zeta)e^{-ic(\tau_1,\tau_2)(x;\xi,z;\zeta)}\ , \tag{6.2}$$

where $b \in \mathfrak{a}^2(\varepsilon_1+\varepsilon_2)$ *and* $c(\tau_1, \tau_2)$ *is defined in* (4.9).

(ii) *For a sufficiently small neighborhood* U_{τ_0} *of* τ_0 *in* ϑ_0, *the mapping* $A(a, \tau_1, \tau_2)=b$ *can be extended to a* C^∞ *mapping of* $\mathfrak{a}^3_{(m)}(\varepsilon_1, \varepsilon_2)\times U_{\tau_0}\times U_{\tau_0}$ *into* $\mathfrak{a}^2_{(m)}(\varepsilon_1+\varepsilon_2)$ *for every* $m\leqq 0$.

(b) Next integral is much simpler than the above case (a). Now, we denote by $\mathfrak{a}^3(\infty, \varepsilon_2)$ the totality of $a \in \mathfrak{a}^3$ such that

$$a(x; \xi, y; \eta, z; \zeta)\equiv 0 \quad \text{for} \quad d(y, z)>\varepsilon_2\ . \tag{6.3}$$

Denote by $\mathfrak{a}^3_{(m)}(\infty, \varepsilon_2)$ the closure of $\mathfrak{a}^3(\infty, \varepsilon_2)$ in $\mathfrak{a}^3_{(m)}$ for each $m\leqq 0$. For $a \in \mathfrak{a}^3(\infty, \varepsilon_2)$, we consider the following integral

$$\langle ae^{-i\tau_0}\rangle(x; \xi, z; \zeta)=Os-\iint a(x; \xi, y; \eta, z; \zeta)e^{-i\tau_0(y;\eta,z)}\text{đ}y\text{đ}\eta\ . \tag{6.4}$$

As in the case of $\langle ae^{-i\tau_1\boxplus\tau_2}\rangle$, (6.4) is well-defined as an oscillatory integral, which will be called also the *contraction integral of a by* τ_0. This integral has the following property:

PROPOSITION 6.2. (i) *For every* $a \in \mathfrak{a}^3(\infty, \varepsilon_2)$, $\langle ae^{-i\tau_0}\rangle$ *is contained in* $\mathfrak{a}^2$.

(ii) *The mapping* $\langle *e^{-i\tau_0}\rangle: \mathfrak{a}^3(\infty, \varepsilon_2)\to\mathfrak{a}^2$ *can be extended to a continuous linear mapping of* $\mathfrak{a}^3_{(m)}(\infty, \varepsilon_2)$ *into* $\mathfrak{a}^2_{(m)}$ *for every* $m\leqq 0$.

(iii) *If* $a \in \mathfrak{a}^3(\infty, \varepsilon_2)$ *is rapidly decreasing in* $|\xi|$, *then so is* $\langle ae^{-i\tau_0}\rangle$ *in* $|\xi|$.

PROOF. (iii) is trivial, since $\langle ae^{-i\tau_0}\rangle$ is defined as an oscillatory integral. To prove (i), (ii), we have only to repeat the standard technique on each local coordinate system $\Delta_{s,I}$ (cf. List 5.1), by finding operators L such that $Le^{-i\tau_0}=e^{-i\tau_0}$ and repeating the integration by parts. We omit here the precise procedure of these, for these will be discussed again more precisely in the next section § 7. □

6.2. Connection between $\mathfrak{a}^1$ and $\sum_C^0$.

In the previous papers, pseudo-differential operators of order 0 have been defined as operators with symbols contained in $\sum_C^0$ (cf. [16]). Here, we shall remark the same operators can be defined by using $a\in\mathfrak{a}^1$ instead of $a\in\sum_C^0$.

Recall the definition of $\mathfrak{a}^1$ and the remark in 5.2. Given $a\in\mathfrak{a}^1$, we define a linear operator $Q(a)$ on $C^\infty(N)$ as follows:

$$(Q(a)u)(x)=Os-\iint a(x;\xi)\nu(x,y)e^{-i\tau_0(x;\xi,y)}u(y)dy d\xi\ . \tag{6.5}$$

Now, fix a C^∞ function $\phi(x;\xi)$ on $\mathring{T}^*N$ such that $\phi\equiv1$ on $|\xi|\leqq K$ and $\phi\equiv0$ on $|\xi|\geqq 2K$ where K is a positive constant. Divide (6.5) into two parts:

$$\begin{aligned}(Q(a)u)(x)&=\iint \phi a\nu e^{-i\tau_0}u\,dy d\xi+\iint(1-\phi)a\nu e^{-i\tau_0}u\,dy d\xi\\ &=Q_1+Q_2\ .\end{aligned} \tag{6.6}$$

Since $(1-\phi)a\nu\in\tilde{\sum}_C^0$ (cf. [8], p. 365), Q_2 is a pseudo-differential operator of order 0, because $\tau_0(x;\xi,y)=\langle\xi|\cdot^x y\rangle$ on $\operatorname{supp}(1-\phi)a\nu$. By Kuranishi's technique (cf. [10], p. 269), we can eliminate the y-variable in the amplitude $(1-\phi)a\nu$ and obtain a pseudo-differential operator with the amplitude contained in $\sum_C^0$.

On the other hand, Q_1 is smoothing operator with the kernel

$$K_{Q_1}(x,y)=\int\phi(x;\xi)a(x;\xi)\nu(x,y)e^{-i\tau_0(x;\xi,y)}d\xi\ , \tag{6.7}$$

which is obviously smooth. Hence recalling how we defined the norm $\|\ \|_{m,s}$ on the space $\mathscr{P}^0$ and using Lemma 1 in [16], we obtain easily the following:

LEMMA 6.3. *Let $a\in\mathfrak{a}^1$. Then, $Q(a)$ is a pseudo-differential operator of order zero on N and the mapping $Q\colon\mathfrak{a}^1\to\mathscr{P}^0$ can be extended to a continuous linear mapping from $\mathfrak{a}^1_{(m)}$ into $\mathscr{P}^0_{(m)}$ for every $m\leqq0$.*

6.3. Proof of Proposition 2.3.

For $K\in C^\infty(N\times N)$ and $\varphi\in\mathscr{D}^{(1)}_\Omega$, we shall consider the following operator

$$\lambda(\varphi, K)=\varphi^*\iota K\circ\pi\varphi^{*-1}: \mathscr{S}_N\to\mathscr{S}_N\ . \tag{6.8}$$

Then, recalling the statement of Proposition 2.3, we have $\Lambda(\varphi, K)=\pi\circ\lambda(\varphi, K)\circ\iota$. By (3.3), we have

$$(\lambda(\varphi, K)f)(x;\xi)=\iint(\varphi^*a_K)(x;\xi, y;\eta)f(y;\eta)đyđ\eta\ , \tag{6.9}$$

where

$$a_K(x;\xi, y)=\iint\nu(x, z)K(z, y)e^{-i\langle\xi|\cdot x_z\rangle}đz\ .$$

First, we compute $\lambda(\varphi, K)\iota$. Then, we have for $u\in C^\infty(N)$ that

$$(\lambda(\varphi, K)\iota u)(x;\xi)=\iiint A(\varphi, K)(x;\xi, y;\eta, z)e^{-i\tau_0(y;\eta, z)}u(z)đyđ\eta đz\ , \tag{6.10}$$

where

$$A(\varphi, K)(x;\xi, y;\eta, z)=(\varphi^*a_K)(x;\xi, y;\eta)\nu(y, z)\ . \tag{6.11}$$

By Corollary 5.9 and Lemma 5.10, we see $A(\varphi, K)\in\mathfrak{a}^8(\infty, \varepsilon)$ and $A(\varphi, K)$ is rapidly decreasing in $|\xi|$, for so is a_K. Hence by Proposition 6.2, we have $\langle A(\varphi, K)e^{-i\tau_0}\rangle\in\mathfrak{a}^2$ and rapidly decreasing in $|\xi|$. Moreover this is smooth with respect to φ and K.

Since $\Lambda(\varphi, K)=\pi\circ\lambda(\varphi, K)\circ\iota$, the kernel of $\Lambda(\varphi, K)$ is given by

$$L(\varphi, K)(x, z)=\int\langle A(\varphi, K)e^{-i\tau_0}\rangle(x;\xi, z)đ\xi\ , \tag{6.12}$$

which is obviously smooth on $N\times N$. Thus, we get the following, which proves Proposition 2.3:

LEMMA 6.4. *Let $K\in C^\infty(N\times N)$ and $\varphi\in\mathscr{D}^{(1)}_\Omega$. Then, $\Lambda(\varphi, K)=\pi\circ\lambda(\varphi, K)\circ\iota$ is an linear operator with a smooth kernel $L(\varphi, K)$ defined by* (6.12). *Moreover the mapping $L:\mathscr{D}^{(1)}_\Omega\times C^\infty(N\times N)\to C^\infty(N\times N)$ is a smooth mapping.*

6.4. Reduction form of some oscillatory integrals.

Before proving Proposition 2.2, we shall remark some properties of a certain oscillatory integral. Namely, consider the following linear operator on $C^\infty(N)$:

$$(6.13)\qquad (\mu(a,\tau)u)(x)=\iint a(x;\xi,y)e^{-i\tau(x;\xi,y)}u(y)dyd\xi\ ,$$

where $a(x;\xi,y)\in \mathfrak{a}^2(\varepsilon)$ and $\tau\in\vartheta_0$ do not involve η-variable and τ is sufficiently close to τ_0.

Remark that on the support of a, $\tau(x;\xi,\cdot_x Y)$ can be expressed as

$$(6.14)\qquad \begin{aligned}\tau(x;\xi,\cdot_x Y)&=\langle\xi|Y\rangle+\langle\xi|Q(x;\xi,Y)Y^2\rangle=\langle\xi|Y+QY^2\rangle\\&=\langle\xi|(I+QY)Y\rangle\ .\end{aligned}$$

Since $\tau-\tau_0$ is small and $|Y|<\varepsilon$, one may assume that $I+Q(x;\xi,Y)Y$ is an invertible matrix. Set $\xi'=\xi(I+QY)$. Then by the implicit function theorem (cf. [12]), ξ can be expressed as a C^∞ function $\Psi_\tau(x;\xi',y)$ depending smoothly on τ. Let $D(\varepsilon)$ be the domain $\{(x;\xi,y)\in(\mathring{T}^*N)\times N; d(x,y)\leqq\varepsilon\}$. Then Ψ_τ is actually a C^∞ diffeomorphism of $D(\varepsilon)$ onto itself and positively homogeneous of degree 1. Hence, we have the following:

LEMMA 6.5. *For $\tau\in\vartheta_0$, sufficiently close to τ_0, there exists a C^∞ diffeomorphism Ψ_τ of $D(\varepsilon)$ onto itself such that $\Psi_\tau^*\tau=\tau_0$ and Ψ_τ is positively homogeneous of degree one. Moreover, Ψ_τ is smooth with respect to τ under the C^∞ topology for Ψ_τ.*

Now, using the above lemma, we rewrite (6.13) as follows:

$$(6.15)\qquad (\mu(a,\tau)u)(x)=\iint(\Psi_\tau^*a)(x;\xi,y)|\det D\Psi_\tau|e^{-i\tau_0(x;\xi,y)}u(y)dyd\xi\ ,$$

where we see easily that $(\Psi_\tau^*a)|\det D\Psi_\tau|\in\mathfrak{a}^2(\varepsilon)$ and does not involve η-variable, and $(\Psi_\tau^*a)|\det D\Psi_\tau|$ depends smoothly on τ (cf. Lemma 5.10).

Thus, using Kuranishi's technique, one can eliminate the y-variable in the amplitude $(\Psi_\tau^*a)|\det D\Psi_\tau|$. Thus, by the same computation as in 6.2, we obtain the following:

LEMMA 6.6. (i) *For $\tau\in\vartheta_0$, sufficiently close to τ_0, and $a\in\mathfrak{a}^2(\varepsilon)$ which do not contain η-variable, $\mu(a,\tau)$ is a pseudo-differential operator of order* 0. (ii) *The mapping $\mu:\mathfrak{a}^2_\eta(\varepsilon)\times\vartheta_0\to\mathscr{P}^0$ can be extended to a C^∞ mapping of $\mathfrak{a}^2_{\eta,(m)}(\varepsilon)\times\vartheta_0$ into $\mathscr{P}^0_{(m)}$, where $\mathfrak{a}^2_\eta(\varepsilon)$ is the totality of $a\in\mathfrak{a}^2(\varepsilon)$ which does not involve η-variable and $\mathfrak{a}^2_{\eta,(m)}(\varepsilon)$ is its closure in $\mathfrak{a}^2_{(m)}(\varepsilon)$.*

6.5. Proof of Proposition 2.2.

Denote by $E(\varphi)$, for $\varphi\in\mathscr{D}^{(1)}_\Omega$, the linear operator on $\mathscr{S}_N$

$$(6.16)\qquad E(\varphi)=\varphi^*\iota\pi\varphi^{*-1}\quad(\text{cf. } 2.3)\ .$$

Recall the argument in 3.1. $E(\varphi)$ is an integral operator with a smooth kernel $\varphi^*\nu e^{-i\varphi^*\tau_0}$. By Lemma 4.1, $\varphi^*\tau_0 \in \vartheta_0$, and by Lemma 5.10, $\varphi^*\nu \in \mathfrak{a}^2(\varepsilon)$ if φ is sufficiently close to the identity. Moreover, $\Xi(\varphi)$ of (2.6) is written as $\pi E(\varphi)\iota$, hence we have

$$(\Xi(\varphi)u)(x)=\iint B(\varphi)(x;\xi,z)u(z)dzd\xi\ , \tag{6.17}$$

where

$$B(\varphi)(x;\xi,z)=\langle\varphi^*\nu\boxtimes\nu e^{-i\varphi^*\tau_0\boxplus\tau_0}\rangle\ . \tag{6.18}$$

Note that $\varphi^*\nu\boxtimes\nu\in\mathfrak{a}^3(\varepsilon,\varepsilon)$ and does not involve ζ-variable. Then, using Proposition 6.1, we have

$$B(\varphi)(x;\xi,z)=b(\varphi)(x;\xi,z)e^{-ic(\varphi^*\tau_0,\tau_0)}\ ,\quad b(\varphi)\in\mathfrak{a}^2(2\varepsilon). \tag{6.19}$$

It is easy to see that $b(\varphi)$ does not involve ζ-variable, because $\varphi^*\nu\boxtimes\nu$ and $\varphi^*\tau_0\boxplus\tau_0$ do not.

Also by Lemma 4.4, we have $c(\varphi^*\tau_0,\tau_0)\in\vartheta_0$ is sufficiently close to $\tau_0(x;\xi,z)$, if φ is sufficiently close to the identity. Thus, by Lemma 6.6, we get Proposition 2.2, (a), (b). Proposition 2.2, (c) is obvious, because $\Xi(\text{id.})=\text{id.}$ and $G\mathscr{P}^0_{(m)}$ is an open subset of $\mathscr{P}^0_{(m)}$ for every $m\leqq 0$.

6.6. Proof of Theorem A.

Now, we shall give the proof of the main theorem. As in 2.3, recall the operators r_γ, α_γ in (2.8), (2.11). We denote by $\mathfrak{M}$ the pairs (a,τ) where $a\in\mathfrak{a}^2(\varepsilon)$, $\tau\in U_{\tau_0}$, a sufficiently small open neighborhood of τ_0 in ϑ_0. By using Fréchet structures on $\mathfrak{a}^2(\varepsilon)$ and ϑ_0, $\mathfrak{M}$ captures a structure as an open set of a Fréchet space. Associating with $a\in\mathfrak{a}^2(\varepsilon)$ and $\tau\in U_{\tau_0}$, we consider, a primordial operator on $\mathscr{S}_N$ of the form

$$(P(a,\tau)f)(x;\xi)=\iint a(x;\xi,y;\eta)e^{-i\tau(x;\xi,y;\eta)}f(y;\eta)dyd\eta\ , \tag{6.20}$$

and this plays an important role in the observation of r_γ and α_γ. Namely remark that

$$r_\gamma(\varphi,\psi)=\Xi(\varphi\psi)^{-1}\pi E(\varphi\psi)E(\varphi)\iota\ , \tag{6.21}$$

$$\alpha_\gamma(\varphi,A)=\Xi(\varphi)^{-1}\pi E(\varphi)\varphi^*a\cdot E(\varphi)\iota+\Xi(\varphi)^{-1}\Lambda(\varphi,K)\ , \tag{6.22}$$

where $A=\pi a\iota+K\circ$, $a\in\sum^0_C$, $K\in C^\infty(N\times N)$. Remark also that $E(\varphi\psi)$, $E(\varphi)$ are primordial operators written in the form (6.20).

First of all, we shall observe (6.21). Remark that

$$(6.23)\qquad (E(\varphi\psi)E(\varphi)f)(x;\xi) = \iint ((\varphi\psi)^*\nu\boxtimes\varphi^*\nu)(x;\xi,y;\eta,z;\zeta)e^{-i((\varphi\psi)^*\tau_0\boxplus\varphi^*\tau_0)}f(z;\zeta)\,dy\,d\eta\,dz\,d\zeta\,.$$

By the result in § 4–5, we have $(\varphi\psi)^*\nu\boxtimes\varphi^*\nu\in\mathfrak{a}^3(\varepsilon,\varepsilon)$ and $(\varphi\psi)^*\tau_0$, $\varphi^*\tau_0\in\vartheta_0$ if φ, ψ are sufficiently close to the identity. Therefore, Proposition 6.1 can be applied in this case, and the kernel of (6.23) is given by the contraction integral

$$(6.24)\qquad \langle(\varphi\psi)^*\nu\boxtimes\varphi^*\nu e^{-i((\varphi\psi)^*\tau_0\boxplus\varphi^*\tau_0)}\rangle = b(\varphi,\psi)e^{-ic((\varphi\psi)^*\tau_0,\varphi^*\tau_0)}\,,$$

for some $b(\varphi,\psi)\in\mathfrak{a}^2(2\varepsilon)$. Thus, by Lemma 4.4, Proposition 6.1 and Lemma 5.10, we obtain the following:

LEMMA 6.7. *There exists a neighborhood V of the identity in $\mathscr{D}_\Omega^{(1)}$ such that the mapping of $V\times V$ into $\mathfrak{a}^2(2\varepsilon)\times\vartheta_0$ defined by $(\varphi,\psi)\rightsquigarrow(b(\varphi,\psi), c((\varphi\psi)^*\tau_0,\varphi^*\tau_0))$ in (6.24) is a smooth mapping of $V\times V$ into $\mathfrak{a}^2_{(m)}(2\varepsilon)\times\vartheta_0$ for every $m\leqq 0$.*

Now, we shall compute $\pi E(\varphi\psi)E(\varphi)\iota$. Set $\tau'=c((\varphi\psi)^*\tau_0,\varphi^*\tau_0)$ in the above notation. Then, we obtain

$$(6.25)\qquad (E(\varphi\psi)E(\varphi)\iota u)(x) = \iint b(\varphi,\psi)(x;\xi,y;\eta)\nu(y,z)e^{-i(\tau'\boxplus\tau_0)(x;\xi,y;\eta,z)}u(z)\,dz\,dy\,d\eta\,.$$

Thus

$$(6.26)\qquad (\pi E(\varphi\psi)E(\varphi)\iota u)(x) = \iint \langle b(\varphi,\psi)\boxtimes\nu e^{-i\tau'\boxplus\tau_0}\rangle(x;\xi,z)u(z)\,dz\,d\xi\,.$$

Since τ' is sufficiently close to τ_0, one can apply Proposition 6.1 again and obtain

$$(6.27)\qquad \langle b(\varphi,\psi)\boxtimes\nu e^{-i\tau'\boxplus\tau_0}\rangle = \tilde{b}(\varphi,\psi)e^{-ic(\tau',\tau_0)}\,,\quad \tilde{b}(\varphi,\psi)\in\mathfrak{a}^2(3\varepsilon)\,.$$

Remark that $\tilde{b}(\varphi,\psi)$ does not involve ζ-variable. Hence, by Lemma 6.6 we see that $\pi E(\varphi\psi)E(\varphi)\iota$ is a pseudo-differential operator of order 0 and the amplitude depends smoothly on (φ,ψ). This proves the smoothness of $r_7(\varphi,\psi)$, because the smoothness of $\Xi(\varphi\psi)^{-1}$ has been already obtained in 6.5.

Next, we shall consider (6.22). The smoothness of the second term has been given in 6.3 combined with Proposition 2.2. Thus, we have only to consider the first term. However, the smooth dependence of $E(\varphi)\varphi^*a\cdot E(\varphi)$ can be easily seen by the similar way as the above argument. Hence, we complete the proof of Theorem A. □

Now, what remains to be proved is only Proposition 6.1. Though the proof of Proposition 6.2 is not precisely given, the detail of the computations on each coordinate neighborhood can be naturally understood from the computations in the next section.

§ 7. Contraction integrals.

Our goal in this section is to prove Proposition 6.1 in § 6.

7.1. Contraction integral $\langle ae^{-i\tau_1 \boxplus \tau_2}\rangle$.

Let τ_1, τ_2 be elements of ϑ_0 and are sufficiently close to τ_0. Given $a \in \mathfrak{a}^3(\varepsilon_1, \varepsilon_2)$, recall the following integral:

$$(7.1) \qquad \langle ae^{-i\tau_1 \boxplus \tau_2}\rangle = Os-\iint a(x;\xi, y;\eta, z;\zeta) e^{-i\tau_1 \boxplus \tau_2(x;\xi,y;\eta,z;\zeta)} đy d\eta \ .$$

The above integral is defined as the oscillatory integral. Now, we shall show Proposition 6.1 by several steps as below.

Put as in 4.3

$$(7.2) \qquad T(x;\xi, y;\eta, z;\zeta) = \tau_1 \boxplus \tau_2(x;\xi, y;\eta, z;\zeta) - c(\tau_1, \tau_2)(x;\xi, z;\zeta) \ .$$

Let $\vartheta^{(2)}$ be the set of all functions T such that

$$T(x;\xi, y;\eta, z;\zeta) = \tau_1 \boxplus \tau_2(x;\xi, y;\eta, z;\zeta) - \tau_3(x;\xi, z;\zeta) \ ,$$

where $\tau_i \in \vartheta_0$ $(i=1, 2, 3)$. One can define the factor topology on $\vartheta^{(2)}$ by using that of ϑ.

Also, rewrite (7.1) by the following:

$$(7.3) \qquad \langle ae^{-i\tau_1 \boxplus \tau_2}\rangle = A(a, \tau_1, \tau_2)(x;\xi, z;\zeta) e^{-ic(\tau_1,\tau_2)(x;\xi,z;\zeta)} \ ,$$

where

$$(7.4) \qquad A(a, \tau_1, \tau_2) = Os-\iint a(x;\xi, y;\eta, z;\zeta) e^{-iT(x;\xi,y;\eta,z;\zeta)} đy d\eta \ .$$

Therefore, to prove Proposition 6.1, we may prove the following:

PROPOSITION 7.1. *Notations being as above, we have*

(i) *For $\tau_1, \tau_2 \in \vartheta_0$, sufficiently close to τ_0, and $a \in \mathfrak{a}^3(\varepsilon_1, \varepsilon_2)$, the integral $A(a, \tau_1, \tau_2) \in \mathfrak{a}^2(\varepsilon_1+\varepsilon_2)$.*

(ii) *The mapping $A: \mathfrak{a}^3(\varepsilon_1, \varepsilon_2) \times U_{\tau_0} \times U_{\tau_0} \to \mathfrak{a}^2(\varepsilon_1+\varepsilon_2)$, defined by (7.4) can be extended to a C^∞ mapping from $\mathfrak{a}^3_{(m)}(\varepsilon_1, \varepsilon_2) \times U_{\tau_0} \times U_{\tau_0}$ into $\mathfrak{a}^2_{(m)}(\varepsilon_1+\varepsilon_2)$ for every $m \leq 0$.*

As an easy remark, if the integral (7.4) can be defined, then it is easily obtained that $A\equiv 0$ on $d(x, z)>\varepsilon_1+\varepsilon_2$.

The above proposition will be proved by dividing the integral into several domains $D_{(j)}$ and by expressing A by $A_{(j)}$. So, in what follows, we shall denote by Lem. $A_{(j)}$ the same statement as in Proposition 7.1 replacing A by $A_{(j)}$. If Lem. $A_{(j)}$ holds for every j, then so does Proposition 7.1.

First, we take a positive constant R and fix it. Let ω_R be a C^∞ function on $(T^*N)^2$ such that $\omega_R\geqq 0$,

$$(7.5)\qquad \omega_R(x;\xi, y;\eta)\equiv 1 \quad\text{on}\quad d(x, y)\leqq\varepsilon_1 \quad\text{and}\quad |\xi|^2+|\eta'|^2\leqq R^2/2\,,$$

where $\cdot_x(X, \eta')=(y;\eta)$ and

$$(7.6)\qquad \operatorname{supp}\omega_R\subset\{(x;\xi, y;\eta)\in(T^*N)^2;\ d(x, y)\leqq 2\varepsilon_1,\ |\xi|^2+|\eta'|^2\leqq R^2\}\,.$$

Using ω_R, we divide (7.4) into two parts;

$$(7.7)\qquad \begin{aligned} A(a, \tau_1, \tau_2)&=Os-\iint(1-\omega_R)ae^{-iT}+\iint\omega_R ae^{-iT}\\ &=A_{(0)}+A_{(-1)}\,. \end{aligned}$$

Remark that the second term $A_{(-1)}$ in (7.7) is integrable in the usual sense. Hence, a direct computation shows that Lem. $A_{(-1)}$ holds.

REMARK. In fact, $A_{(-1)}(x;\xi, z;\zeta)$ is bounded in $|\zeta|$ and rapidly decreasing in $|\xi|$.

Next, we divide $A_{(0)}$ in (7.7) into several parts. First, let $\phi(x;\xi, y;\eta, z;\zeta)$ be a C^∞ function on $(\mathring{T}^*N)^3$ satisfying

(i) $\operatorname{supp}\phi\subset\{(x;\xi, y;\eta, z;\zeta);\ d(x, y)\leqq r_1\}$;

(ii) $\phi\geqq 0$ and $\phi\equiv 1$ on $|\eta-\eta_c|\leqq\delta_1|\xi|/2$ and $\equiv 0$ on $|\eta-\eta_c|\geqq\delta_1|\xi|$, where $(y_c;\eta_c)$ is the critical point given by Proposition 4.3 and δ_1 is chosen to be a sufficiently small constant.

(iii) $\phi(x;r\xi, y;r\eta, z;s\zeta)=\phi(x;\xi, y;\eta, z;\zeta)$ for any $r, s>0$.

Then, it is easily obtained that the critical point of T which is the same as that of $\tau_1\boxplus\tau_2$, obtained in Lemma 4.3, is contained in $\operatorname{supp}\phi$. Therefore, we get

$$(7.8)\qquad \begin{aligned} A_{(0)}(a, \tau_1, \tau_2)&=\iint\phi a'e^{-iT}+Os-\iint(1-\phi)a'e^{-iT}\\ &=A^1+A^2\,, \end{aligned}$$

where $a'=(1-\omega_R)a$. Easily, $\phi a'$, $(1-\phi)a'\in\mathfrak{a}^8(\varepsilon_1, \varepsilon_2)$ by Lemmas 5.6-7. Moreover, we divide A^2 in (7.8) by using a partition of unity: Namely we choose functions ψ_i $(i=1, 2, 3)$ with the following properties:

$$\sum_{i=1}^{3} \psi_i \equiv 1 \ , \quad \psi_i \in \mathfrak{a}^3 \ ; \tag{7.9}$$

and

$$\begin{cases} \operatorname{supp} \psi_1 \subset \{(x; \xi, y; \eta, z; \zeta) \in (T^*N)^3; |\eta| \leqq C^{-1}|\xi|\} \ , \\ \operatorname{supp} \psi_2 \subset \left\{(x; \xi, y; \eta, z; \zeta) \in (T^*N)^3; \frac{1}{2}C^{-1}|\xi| \leqq |\eta| \leqq 2C|\xi|\right\} \ , \\ \operatorname{supp} \psi_3 \subset \{(x; \xi, y; \eta, z; \zeta) \in (T^*N)^3; |\eta| \geqq C|\xi|\} \ , \end{cases} \tag{7.10}$$

where C is chosen in Corollary 4.6. Now, we put

$$\begin{aligned} A^2 &= Os-\iint \psi_1(1-\phi)a'e^{-iT} + Os-\iint \psi_2(1-\phi)a'e^{-iT} \\ &\quad + Os-\iint \psi_3(1-\phi)a'e^{-iT} \\ &= A^{2,1} + A^{2,2} + A^{2,3} \ . \end{aligned} \tag{7.11}$$

Using Lemmas 5.7–8, we summarize the following:

LEMMA 7.2. *Suppose that* $\varepsilon_1, \varepsilon_2 < r_1/4$ *and fix functions* ω_R, ϕ, ψ_i $(i=1, 2, 3)$ *defined as above. Then we have*

(i) *The mapping* $a \mapsto a' = (1-\omega_R)a$ *can be extended to a* C^∞ *mapping on* $\mathfrak{a}^3_{(m)}(\varepsilon_1, \varepsilon_2)$ *for every* $m \leqq 0$.

(ii) $A_{(0)}(a, \tau_1, \tau_2)$ *in* (7.8) *can be written by*

$$A_{(0)}(a, \tau_1, \tau_2) = A^1 + A^{2,1} + A^{2,2} + A^{2,3} \ , \tag{7.12}$$

where

$$A^1 = \iint c_1(x; \xi, y; \eta, z; \zeta) e^{-iT(x;\xi,y;\eta,z;\zeta)} đy đ\eta \ , \tag{7.13}$$

$$A^{2,i} = Os-\iint c_{2,i}(x; \xi, y; \eta, z; \zeta) e^{-iT(x;\xi,y;\eta,z;\zeta)} đy đ\eta \ , \quad (i=1, 2, 3) \ , \tag{7.14}$$

where $c_1 = \phi(1-\omega_R)a$, $c_{2,i} = (1-\omega_R)(1-\phi)\psi_i a$ *are elements in* $\mathfrak{a}^3(\varepsilon_1, \varepsilon_2)$ *respectively and*

$$\begin{aligned} \operatorname{supp} c_1 \subset D_1 = \{&(x; \xi, y; \eta, z; \zeta) \in (T^*N)^3; d(x, y) \leqq \varepsilon_1, d(y, z) \leqq \varepsilon_2, \\ &|\xi|^2 + |\eta'|^2 \geqq R^2/2, |\eta - \eta_c| \leqq \delta_1 |\xi|\} \ , \end{aligned} \tag{7.15}$$

$$\operatorname{supp} c_{2,1} \subset D_{2,1} = \Big\{(x; \xi, y; \eta, z; \zeta) \in (T^*N)^3; d(x, y) \leqq \varepsilon_1, d(y, z) \leqq \varepsilon_2, \tag{7.16}$$

$$|\xi|^2+|\eta'|^2 \geqq R^2/2,\ |\eta-\eta_c| \geqq \frac{1}{2}\delta_1|\xi|,\ |\eta| \leqq C^{-1}|\xi| \Big\}\ ,$$

(7.17) $$\operatorname{supp} c_{2,2} \subset D_{2,2} = \Big\{(x;\xi,y;\eta,z;\zeta) \in (T^*N)^3;\ d(x,y) \leqq \varepsilon_1,\ d(y,z) \leqq \varepsilon_2,$$
$$|\xi|^2+|\eta'|^2 \geqq R^2/2,\ |\eta-\eta_c| \geqq \frac{1}{2}\delta_1|\xi|,$$
$$\frac{1}{2}C^{-1}|\xi| \leqq |\eta| \leqq 2C|\xi| \Big\}\ ,$$

(7.18) $$\operatorname{supp} c_{2,3} \subset D_{2,3} = \Big\{(x;\xi,y;\eta,z;\zeta) \in (T^*N)^3;\ d(x,y) \leqq \varepsilon_1,\ d(y,z) \leqq \varepsilon_2,$$
$$|\xi|^2+|\eta'|^2 \geqq R^2/2,\ |\eta-\eta_c| \geqq \frac{1}{2}\delta_1|\xi|,\ |\eta| \geqq C|\xi| \Big\}\ .$$

(iii) *Moreover,* c_1, $c_{2,i}$ $(i=1, 2, 3) \in \mathfrak{a}^3(\varepsilon_1, \varepsilon_2)$ *depend continuous-linearly on* a *in* $\mathfrak{a}^3_{(m)}$ *topology.*

7.2. Lem. A^1.

First, we remark that A^1 is integrable in the usual sense. We shall check the conditions (A.1-2) in Definition 5.2 by using coordinate system $\{\Delta_{2,I}\}$ (cf. List 5.1 for $k=2$). Now, we may take R by $R>4K$. Then, $\Delta_{2,\varnothing} \cap \operatorname{supp} c_1 = \varnothing$ and we have only to investigate the four cases: $\Delta_{2,(1)}$, $\Delta_{2,(2)}$, $\Delta_{2,(1,2)}$, $\Delta_{2,(2,1)}$.

On $\Delta_{2,(1)}$: By using the variables (r^{-1}, t) in $\Delta_{2,(1)}$, we have

$$(\tilde{A}^1)_{2,(1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta}) = \iint_{D_1} c_1(x; r\hat{\xi}, y; \eta, z; t\hat{\zeta}) e^{-iT(x;r\hat{\xi},y;\eta,z;t\hat{\zeta})} dy d\eta\ .$$

By Proposition 4.7, if we take δ_1 as a sufficiently small constant, we get

(7.19) $$(\tilde{A}^1)_{2,(1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta}) = \iint_{D_1'} c_1'(x; r\hat{\xi}, y; \eta, z; t\hat{\zeta}) e^{i\langle \eta'-\eta_c' | X-X_c\rangle} dy d\eta\ ,$$

where $c_1' = (c \circ \tilde{\Phi})|\det D\tilde{\Phi}|$, $D_1' = \tilde{\Phi}^{-1} D_1$. Setting $\tilde{\eta} = (1/r)\eta$, we get

$$(\tilde{A}^1)_{2,(1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta})$$
$$= \iint r^n c_1'(x; r\hat{\xi}, y; r\tilde{\eta}, z; t\hat{\zeta}) e^{ir\langle \tilde{\eta}'-\eta_c'(x;\hat{\xi},z;\hat{\zeta}) | X-X_c(x;\hat{\xi},z;\hat{\zeta})\rangle} dy d\tilde{\eta}\ ,$$

where $\cdot_*(X, \tilde{\eta}') = (y; \tilde{\eta})$. Consider the function $c_1'(x; r\hat{\xi}, y; r\tilde{\eta}, z; t\hat{\zeta})$. By using Proposition 4.7 (ii), we have

$$c_1'(x; r\hat{\xi}, y; r\tilde{\eta}, z; t\hat{\zeta}) = c_1(x; r\hat{\xi}, \tilde{\Phi}_1(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta}); r\tilde{\Phi}_2(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta}), z; t\hat{\zeta})$$
$$\times |\det D\tilde{\Phi}(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta})|\ ,$$

where $\tilde{\Phi}^{-1}(y; r\tilde{\eta})$ moves in D_1. Thus, using (4.12), we have $C^{-1}\|\partial_X\tilde{S}^{-1}\| - N\delta_1 \leqq |\tilde{\eta}'| \leqq C\|\partial_X\tilde{S}^{-1}\| + N\delta_1$ for certain constants C, $N>0$ depending on the riemannian structure. Since X, Z are sufficiently small, we put

$$\tilde{r}=r|\tilde{\eta}| , \qquad \tilde{t}_1=|\tilde{\eta}| , \qquad \tilde{t}_3=t , \tag{7.20}$$

and define a function ρ by

$$\rho(\tilde{r}^{-1}, \tilde{t}_1, \tilde{t}_3, x; \hat{\xi}, y; \hat{\tilde{\eta}}, z; \hat{\zeta})=c_1'(x; (\tilde{r}/\tilde{t}_1)\hat{\xi};, y; \tilde{r}\hat{\tilde{\eta}}, z; \tilde{t}_3\hat{\zeta}) .$$

Then, ρ is C^∞ on $\Delta_{3,(2,1)}$. By putting $\tilde{\eta}''=\tilde{\eta}'-\eta_c'$ and $X''=X-X_c$, we get

$$(\tilde{A}^1)_{2,(1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta})$$
$$=\iint \rho((r|\tilde{\eta}|)^{-1}, |\tilde{\eta}|, t, x; \hat{\xi}, \cdot_x(X, \hat{\tilde{\eta}}'), z; \hat{\zeta}))e^{ir\langle\tilde{\eta}''|X''\rangle}r^n d\tilde{\eta}'' dX'' .$$

Using the Taylor expansion of $\rho(\cdots)$ with respect to X'' and integration by parts, we get

$$(\tilde{A}^1)_{2,(1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta}) \tag{7.21}$$
$$=\sum \frac{1}{\alpha!}\partial^\alpha_{\tilde{\eta}''}\partial^\alpha_{X''}\rho((r|\tilde{\eta}|)^{-1}, |\tilde{\eta}|, t, x; \hat{\xi}, \cdot_x(X, \hat{\tilde{\eta}}'), z; \hat{\zeta}))\Big|_{\substack{X''=0\\ \tilde{\eta}''=0}}(-ir)^{-|\alpha|}$$
$$+R_{m-1} ,$$

where R_{m-1} is the remainder term obtained by Taylor expansion and is of order $O(r^{m-1})$. Moreover, use that ρ can have the asymptotic expansion with respect to r. Then (A.1-2) are obvious.

On $\Delta_{2,(2)}$: Also, use the variables in $\Delta_{2,(2)}$ and the notation as in above. Then, we have

$$(\tilde{A}^1)_{2,(2)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta})=\iint_{D_1} c_1(x; t\hat{\xi}, y; \eta, z; r\hat{\zeta})e^{-iT(x;t\hat{\xi},y;\eta,z;r\hat{\zeta})}dy d\eta . \tag{7.22}$$

By Proposition 4.7, we have

$$(\tilde{A}^1)_{2,(2)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta})=\iint_{D_1'} c_1'(x; t\hat{\xi}, y; \eta, z; r\hat{\zeta})e^{i\langle\eta'-\eta_c'(x;t\hat{\xi},z;\hat{\zeta})|X-X_c(x;t\hat{\xi},z;\hat{\zeta})\rangle}dy d\eta ,$$

where $c_1'=(c_1\circ\tilde{\Phi})|(\mathrm{Det}\, D\tilde{\Phi})|$. Also, by using Proposition 4.7 (ii), we get

$$c_1'(x; t\hat{\xi}, y; \eta, z; r\hat{\zeta})=c_1(x; \xi, \tilde{\Phi}_1(x; t\hat{\xi}, y; \eta, z; \hat{\zeta}); \tilde{\Phi}_2(x; t\hat{\xi}, y; \eta, z; \hat{\zeta}), z; r\hat{\zeta})$$
$$\times|\det D\Phi(x; \hat{\xi}, y; \eta, z; \zeta)| .$$

Since $|\eta|$ may be estimated by $|\eta|\leqq K$, put

$$\tilde{r}=r , \qquad \tilde{t}_1=t , \qquad \tilde{t}_2=|\eta| . \tag{7.23}$$

We see that

$$\rho(\tilde{r}^{-1}, \tilde{t}_1, \tilde{t}_2, x; \hat{\xi}, y; \hat{\eta}, z; \hat{\zeta})=c_1'(x; \tilde{t}_1\hat{\xi}, y; \tilde{t}_2\hat{\eta}, z; \tilde{r}\hat{\zeta})$$

is smooth on $\Delta_{3,(3)}$. So by using Taylor expansion, we get (A.1-2).

On $\Delta_{2,(1,2)}$: Using variables in $\Delta_{2,(1,2)}$ and using Proposition 4.7, we have

$$\begin{aligned}(\tilde{A}^1)_{2,(1,2)}&(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta})\\ &=\iint_{D_1'} c_1'(x; r\hat{\xi}, y; \eta, z; (r/t)\hat{\zeta})e^{i\langle\eta'-r\eta_c'(x;\hat{\xi},z;\hat{\zeta})|X-X_c(x;\hat{\xi},z;\hat{\zeta})\rangle}dy d\eta .\end{aligned}$$

By changing variables $\tilde{\eta}=(1/r)\eta$, we have

$$\begin{aligned}(7.24)\qquad (\tilde{A}^1)_{2,(1,2)}&(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta})\\ &=\iint c_1'(x; r\hat{\xi}, y; r\tilde{\eta}, z; (r/t)\hat{\zeta})e^{ir\langle\tilde{\eta}'-\eta_c'(x;\hat{\xi},z;\hat{\zeta})|X-X_c(x;\hat{\xi},z;\hat{\zeta})\rangle}r^n dy d\tilde{\eta} ,\end{aligned}$$

where $C^{-1}\|\partial_X\tilde{S}^{-1}\|-N\delta_1\leqq|\tilde{\eta}|\leqq C\|\partial_X\tilde{S}^{-1}\|+N\delta_1$. Put

$$(7.25)\qquad \tilde{r}=r|\tilde{\eta}| , \qquad \tilde{t}_1=|\tilde{\eta}| , \qquad \tilde{t}_3=t .$$

Then, we get

$$\rho(r^{-1}, \tilde{t}_1, \tilde{t}_3, x; \hat{\xi}, y; \hat{\tilde{\eta}}, z; \hat{\zeta})=c_1'(x; (\tilde{r}/\tilde{t}_1)\hat{\xi}, y; \tilde{r}\hat{\tilde{\eta}}, z; (\tilde{r}/\tilde{t}_1\tilde{t}_3)\hat{\zeta}) ,$$

is smooth on $\Delta_{3,(2,1,3)}$, where

$$\begin{aligned}c_1'(x; r\hat{\xi}, y; r\tilde{\eta}, z; (r/t)\hat{\zeta})=c_1(&x; r\hat{\xi}, \tilde{\Phi}_1(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta}); r\tilde{\Phi}_2(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta}),\\ &z; (r/t)\hat{\zeta})|\det D\tilde{\Phi}(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta})| .\end{aligned}$$

Therefore, by using Taylor expansion of ρ, we get (A.1-2) in $\Delta_{2,(1,2)}$.

On $\Delta_{2,(2,1)}$: Using variables in $\Delta_{2,(2,1)}$ and using Proposition 4.7, we have

$$\begin{aligned}(\tilde{A}^1)_{2,(2,1)}&(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta})\\ &=\iint_{D_1'} c_1'(x; (r/t)\hat{\xi}, y; \eta, z; r\hat{\zeta})e^{i\langle\eta'-(r/t)\eta_c'(x;\hat{\xi},z;\hat{\zeta})|X-X_c(x;\hat{\xi},z;\hat{\zeta})\rangle}dy d\eta ,\end{aligned}$$

where $c_1'=(c_1\circ\tilde{\Phi})|(\det D\tilde{\Phi})|$. By changing variables $(r/t)\tilde{\eta}=\eta$, we have

$$\begin{aligned}(7.26)\qquad (\tilde{A}^1)_{2,(2,1)}&(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta})\\ &=\iint c_1'(x; (r/t)\hat{\xi}, y; (r/t)\tilde{\eta}, z; r\hat{\zeta})e^{i(r/t)\langle\tilde{\eta}'-\eta_c'(x;\hat{\xi},z;\hat{\zeta})|X-X_c(x;\hat{\xi},z;\hat{\zeta})\rangle}\\ &\quad\times(r/t)^n dy d\tilde{\eta} ,\end{aligned}$$

where $C^{-1}\|\partial_X\tilde{S}^{-1}\|-N\delta_1\leqq|\tilde{\eta}|\leqq C\|\partial_X\tilde{S}^{-1}\|+N\delta_1$.

If $|\tilde{\eta}|/t\leqq K$, then we put

(7.27) $$\tilde{r}=r|\tilde{\eta}|/t\,,\qquad \tilde{t}_1=t\,,\qquad \tilde{t}_3=|\tilde{\eta}|/t\,.$$

Therefore,

(7.28) $$\rho(\tilde{r}^{-1},\ \tilde{t}_1,\ \tilde{t}_3,\ x;\ \hat{\xi},\ y;\ \hat{\tilde{\eta}},\ z;\ \hat{\zeta})=c_1'(x;\ (\tilde{r}/\tilde{t}_1\tilde{t}_3)\hat{\xi},\ y;\ \tilde{r}\hat{\tilde{\eta}},\ z;\ (\tilde{r}/\tilde{t}_3)\hat{\zeta})$$

is smooth on $\Delta_{3,(2,3,1)}$. Also, if $|\tilde{\eta}|/t\geqq K^{-1}$, then we put

(7.29) $$\tilde{r}=r\,,\qquad \tilde{t}_1=|\tilde{\eta}|\,,\qquad \tilde{t}_2=t/|\tilde{\eta}|\,.$$

Therefore

(7.30) $$\rho(\tilde{r}^{-1},\ \tilde{t}_1,\ \tilde{t}_2,\ x;\ \hat{\xi},\ y;\ \hat{\tilde{\eta}},\ z;\ \hat{\zeta})=c_1'(x;\ (\tilde{r}/\tilde{t}_1\tilde{t}_2)\hat{\xi},\ y;\ (\tilde{r}/\tilde{t}_2)\hat{\tilde{\eta}},\ z;\ \tilde{r}\hat{\zeta})$$

is smooth on $\Delta_{3,(3,2,1)}$. Here

$$c_1'(x;\ (r/t)\hat{\xi},\ y;\ (r/t)\tilde{\eta},\ z;\ r\hat{\zeta})=c_1(x;\ (r/t)\hat{\xi},\ \tilde{\Phi}_1(x;\ \hat{\xi},\ y;\ \tilde{\eta},\ z;\ \hat{\zeta});\ (r/t)\tilde{\Phi}_2(x;\ \hat{\xi},\ y;\ \tilde{\eta},\ z;\ \hat{\zeta}),\ z;\ r\hat{\zeta})|\det D\tilde{\Phi}(x;\ \hat{\xi},\ y;\ \tilde{\eta},\ z;\ \hat{\zeta})|\,.$$

Applying Taylor expansion for both cases (7.28) and (7.30), we get (A.1-2) in $\Delta_{2,(2,1)}$.

Lastly, we have to check the differentiability of A^1 with respect to α, τ_1 and τ_2. It is easily seen by differentiating (7.19), (7.21), (7.24) and (7.26) directly with respect to α, τ_1, τ_2 and by the same computations as above. □

7.3. Lem. $A^{2,1}$.

We shall consider the integral $A^{2,1}$ in Lemma 7.4, i.e.,

$$A^{2,1}(x;\ \xi,\ z;\ \zeta)=Os-\iint c_{2,1}(x;\ \xi,\ y;\ \eta,\ z;\ \zeta)e^{-iT(x;\xi,y;\eta,z;\zeta)}dyd\eta\,,$$

where $c_{2,1}$ is defined by (7.14). To check the differentiability of $A^{2,1}$ with respect to α, τ_1 and τ_2, we shall formally differentiate $A^{2,1}$ with respect to α, τ_1 and τ_2. Then, it is easily seen that these derivatives can be written by the sum of the following integrals, for $|l|\geqq 0$,

(7.31) $$\nabla^l A^{2,1}(x;\ \xi,\ z;\ \zeta)=Os-\iint c_{2,1}^{l,T'}(x;\ \xi,\ y;\ \eta,\ z;\ \zeta)e^{-iT(x;\xi,y;\eta,z;\zeta)}dyd\eta\,,$$

where $c_{2,1}^{l,T'}$ can be described as follows:

(a) $c_{2,1}^{l,T'}=\tilde{c}_{2,1}(T')^l$,

(b) $(T')^l=(T_1')^{l_1}\cdots(T_k')^{l_k}$, $T_i'\in\vartheta^{(2)}$,

(c) $\tilde{c}_{2,1}\in\mathfrak{a}^3(\varepsilon_1,\ \varepsilon_2)$ and satisfies the same conditions for $c_{2,1}$ in (7.16).

Now, we shall prove that Lem. $A^{2,1}$ holds.

We shall observe $A^{2,1}$ (or $\nabla^l A^{2,1}$) for each chart $\{\Delta_{2,I}\}$. Remark that on support of $c_{2,1}$, we have $|\xi|^2 \geqq R^2/2(1+C^{-2})$. Therefore, if we take $R \geqq \sqrt{2K(1+C^{-2})}$, then $\operatorname{supp} c_{2,1} \cap \Delta_{2,\varnothing} = \varnothing$ and $\operatorname{supp} c_{2,1} \cap \Delta_{2,(2)} = \varnothing$. So, we shall only investigate $\nabla^l A^{2,1}$ for the cases $\Delta_{2,(1)}$, $\Delta_{2,(1,2)}$ and $\Delta_{2,(2,1)}$.

On $\Delta_{2,(1)}$: Use the coordinate on $\Delta_{2,(1)}$. Then, we have

$$(7.32)\qquad (\widetilde{\nabla^l A^{2,1}})_{2,(1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta}) = Os-\iint_{D_{2,1}} c_{2,1}^{l;T'}(x; r\hat{\xi}, y; \eta, z; t\hat{\zeta}) e^{-iT(x;r\hat{\xi},y;\eta,z;t\hat{\zeta})} dy d\eta \,.$$

Setting $\eta = r\tilde{\eta}$, we get

$$(\widetilde{\nabla^l A^{2,1}})_{2,(1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta}) = \iint_{|\tilde{\eta}| \leq C^{-1}} c_{2,1}^{l;T'}(x; r\hat{\xi}, y; r\tilde{\eta}, z; t\hat{\zeta}) e^{-irT(x;\hat{\xi},y;\tilde{\eta},z;\hat{\zeta})} r^n dy d\tilde{\eta} \,,$$

where

$$c_{2,1}^{l;T'}(x; r\hat{\xi}, y; r\tilde{\eta}, z; t\hat{\zeta}) = \tilde{c}_{2,1}(x; r\hat{\xi}, y; r\tilde{\eta}, z; t\hat{\zeta}) r^{|l|} (T'(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta}))^l \,.$$

By Corollary 4.6, (2), we put

$$(7.33)\qquad L_T = \frac{i\partial_y T(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta}) \cdot \partial_y}{r|\partial_y T(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta})|^2} \,.$$

Then, $L_T e^{-irT} = e^{-irT}$ and $|\partial_y T| \geqq M$. Using the integration by parts, we get

$$(\widetilde{\nabla^l A^{2,1}})_{2,(1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta}) = \iint_{|\tilde{\eta}| \leq C^{-1}} r^{n+|l|} (L_T^*)^m [\tilde{c}_{2,1}(x; r\hat{\xi}, y; r\tilde{\eta}, z; t\hat{\zeta}) (T'(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta}))^l] \times e^{-irT(x;\hat{\xi},y;\tilde{\eta},z;\hat{\zeta})} dy d\tilde{\eta} \,,$$

where L_T^* is the adjoint operator of L_T. Since K can be chosen as sufficiently large number, we put

$$(7.34)\qquad \tilde{r} = r|\tilde{\eta}| \,, \qquad \tilde{t}_1 = |\tilde{\eta}| \,, \qquad \tilde{t}_3 = t \,.$$

Then, the function ρ defined by

$$\rho(\tilde{r}^{-1}, \tilde{t}_1, \tilde{t}_2, x; \hat{\xi}, y; \hat{\tilde{\eta}}, z; \hat{\zeta}) = \tilde{c}_{2,1}(x; (\tilde{r}/\tilde{t}_1)\hat{\xi}, y; \tilde{r}\hat{\tilde{\eta}}, z; \tilde{t}_3\hat{\zeta})$$

is smooth on $\Delta_{3,(2,1)}$. By choosing m sufficiently large, we have

$$(7.35)\qquad (\widetilde{\nabla^l A^{2,1}})_{2,(1)} = O(r^{-N}) \quad \text{for any} \quad N \geqq 0 \,.$$

Also, by a similar computation, we get

$$(7.36)\qquad D^{\alpha}_{(r,t,x;\hat{\xi},z;\hat{\zeta})} (\widetilde{\nabla^l A^{2,1}})_{2,(1)} = O(r^{-N}) \quad \text{for any} \quad N \geqq 0 \,.$$

On $\Delta_{2,(1,2)}$: Use the coordinate on $\Delta_{2,(1,2)}$: Then, we have

$$(7.37)\quad (\widetilde{\nabla^l A^{2,1}})_{2,(1,2)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta}) = Os-\iint c_{2,1}^{l,T'}(x; r\hat{\xi}, y; \eta, z; (r/t)\hat{\zeta})e^{-iT(x;r\hat{\xi},y;\eta,z;(r/t)\hat{\zeta})}\,đy\,đ\eta\,.$$

Setting $\eta = r\hat{\eta}$, we have

$$(\widetilde{\nabla^l A^{2,1}})_{2,(1,2)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta}) = \iint_{|\hat{\eta}|\leq C^{-1}} c_{2,1}^{l,T'}(x; r\hat{\xi}, y; r\hat{\eta}, z; (r/t)\hat{\zeta})e^{-irT(x;\hat{\xi},y;\hat{\eta},z;\hat{\zeta})}r^n\,đy\,đ\hat{\eta}\,,$$

where

$$c_{2,1}^{l,T'}(x; r\hat{\xi}, y; r\hat{\eta}, z; (r/t)\hat{\zeta}) = \tilde{c}_{2,1}(x; r\hat{\xi}, y; r\hat{\eta}, z; (r/t)\hat{\zeta})r^{|l|}(T'(x; \hat{\xi}, y; \hat{\eta}, z; \hat{\zeta}))^l\,.$$

By Corollary 4.6 (2), we use the operator L_T in (7.33). Remark that by putting

$$(7.38)\qquad \tilde{r} = r|\hat{\eta}|\,, \qquad \tilde{t}_1 = |\hat{\eta}|\,, \qquad \tilde{t}_3 = t\,,$$

the function

$$\rho(\tilde{r}^{-1}, \tilde{t}_1, \tilde{t}_3, x; \hat{\xi}, y; \hat{\hat{\eta}}, z; \hat{\zeta}) = \tilde{c}_{2,1}(x; (\tilde{r}/\tilde{t}_1)\hat{\xi}, y; \tilde{r}\hat{\hat{\eta}}, z; (\tilde{r}/\tilde{t}_1\tilde{t}_3)\hat{\zeta})$$

is smooth on $\Delta_{3,(2,1,3)}$. The same computation as in (7.35) gives

$$(7.39)\qquad (\widetilde{\nabla^l A^{2,1}})_{2,(1,2)} = O(r^{-N})\,,$$

$$(7.40)\qquad D^{\alpha}_{(r,t,x;\hat{\xi},z;\hat{\zeta})}(\widetilde{\nabla^l A^{2,1}})_{2,(1,2)} = O(r^{-N})\,,$$

for any $N \geqq 0$.

On $\Delta_{2,(2,1)}$: Use the coordinate on $\Delta_{2,(2,1)}$. Then, we get

$$(7.41)\quad (\widetilde{\nabla^l A^{2,1}})_{2,(2,1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta}) = Os-\iint_{D_{2,1}} c_{2,1}^{l,T'}(x; (r/t)\hat{\xi}, y; \eta, z; r\hat{\zeta})e^{-iT(x;(r/t)\hat{\xi},y;\eta,z;r\hat{\zeta})}\,đy\,đ\eta\,.$$

Set $\eta = (r/t)\hat{\eta}$, and we have

$$(\widetilde{\nabla^l A^{2,1}})_{2,(2,1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta}) = \iint_{|\hat{\eta}|\leq C^{-1}} c_{2,1}^{l,T'}(x; (r/t)\hat{\xi}, y; (r/t)\hat{\eta}, z; r\hat{\zeta})e^{-i(r/t)T(x;\hat{\xi},y;\hat{\eta},z;\hat{\zeta})}(r/t)^n\,đy\,đ\hat{\eta}\,,$$

where

$$c_{2,1}^{l;T'}(x;(r/t)\hat{\xi}, y; (r/t)\tilde{\eta}, z; r\hat{\zeta})$$
$$=\tilde{c}_{2,1}(x; (r/t)\hat{\xi}, y; (r/t)\tilde{\eta}, z; r\hat{\zeta})(r/t)^{|l|}(T'(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta}))^l .$$

Put

$$L'_T=\frac{it\partial_y T(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta})\cdot\partial_y}{r|\partial_y T(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta})|^2} . \tag{7.42}$$

Then, $L'_T e^{-i(r/t)T}=e^{-i(r/t)T}$ and $|\partial_y T(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta})|\geqq M$ on $\operatorname{supp} c_{2,1}$ by Corollary 4.6. Thus, we get, for any $m\geqq 0$,

$$(\widetilde{\nabla^l A^{2,1}})_{2,(2,1)}=\iint_{|\tilde{\eta}|\leq C^{-1}}(L'^*_T)^m[c_{2,1}^{l;T'}(x; (r/t)\hat{\xi}, y; (r/t)\tilde{\eta}, z; r\hat{\zeta})]$$
$$\times(r/t)^n e^{-i(r/t)T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\zeta})}dyd\tilde{\eta} .$$

Put as in (7.37)-(7.40), we get

$$(\widetilde{\nabla^l A^{2,1}})_{2,(2,1)}=O(r^{-N}t^N) \quad \text{for any} \quad N\geqq 0 . \tag{7.43}$$

and similarly

$$D^{\alpha}_{(r,t,x;\hat{\xi},z;\hat{\zeta})}(\widetilde{\nabla^l A^{2,1}})_{2,(2,1)}=O(r^{-N}t^N) \tag{7.44}$$

for any $N\geqq 0$.

What we have shown in the above argument is any formal differentials of $A^{2,1}$ with respect to (a, τ_1, τ_2) are well-defined in $\mathfrak{a}^2(\varepsilon_1, \varepsilon_2)$ and these differentials are continuous. To prove the differentiability of $A^{2,1}(a, \tau_1, \tau_2)$, we have to take the formal Taylor expansion and compute the remainder term (cf. [12]). However, the estimation of the remainder term can be obtained by the oscillatory integrals of the remainder term of Taylor expansion of the integrand $A^{2,1}$ by similar computations as above. Thus, we obtain Lem. $A^{2,1}$. □

7.4. Lem. $A^{2,3}$.

Now, we shall consider the integral $A^{2,3}$ in Lemma 7.4. As in 7.3, to consider the differentiability with respect to a, τ_1, τ_2, we have only to consider the following integral, for $|l|\geqq 0$,

$$\nabla^l A^{2,3}(x; \xi, z; \zeta)=Os-\iint_{D_{2,3}} c_{2,3}^{l;T'}(x; \xi, y; \eta, z; \zeta)e^{-iT(x;\xi,y;\eta,z;\zeta)}dyd\eta , \tag{7.45}$$

where $c_{2,3}^{l;T'}$ can be described as follows:

(a) $c_{2,3}^{l;T'}=\tilde{c}_{2,3}(T')^l$,

(b) $(T')^l=(T'_1)^{l_1}\cdots(T'_k)^{l_k}$, $T'_i\in\vartheta^{(2)}$,

(c) $\tilde{c}_{2,3}\in\mathfrak{a}^3(\varepsilon_1, \varepsilon_2)$ and satisfies the same condition for $c_{2,3}$ in (7.18).

Now, we shall prove that Lem. $A^{2,3}$ holds by observing $A^{2,3}$ on each chart $\{\Delta_{2,I}\}$.

On $\Delta_{2,\emptyset}$: Use the coordinate on $\Delta_{2,\emptyset}$. Then, we have

$$(\widetilde{\nabla^l A^{2,3}})_{2,\emptyset}(t_1, t_2, x; \hat{\xi}, z; \hat{\zeta}) = Os-\iint c_{2,3}^{l,T'}(x; t_1\hat{\xi}, y; \eta, z; t_2\hat{\zeta})e^{-iT(x;t_1\hat{\xi},y;\eta,z;t_2\hat{\zeta})}đyđ\eta\ .$$

Now, we divide the above integral into two parts. Namely, by using a cut off function $\psi(y;\eta)$ such that $\text{supp}\,\psi(y;\eta)\subset\{(y;\eta)\in \mathring{T}^*N; |\eta|\leqq K\}$, we get

$$(7.46)\qquad (\widetilde{\nabla^l A^{2,3}})_{2,\emptyset}(t_1, t_2, x; \hat{\xi}, z; \hat{\zeta})$$
$$=\iint \psi c_{2,3}^{l,T'}(x; t_1\hat{\xi}, y; \eta, z; t_2\hat{\zeta})e^{-iT(x;t_1\hat{\xi},y;\eta,z;t_2\hat{\zeta})}đyđ\eta$$
$$+Os-\iint (1-\psi)c_{2,3}^{l,T'}(x; t_1\hat{\xi}, y; \eta, z; t_2\hat{\zeta})e^{-iT(x;t_1\hat{\xi},y;\eta,z;t_2\hat{\zeta})}đyđ\eta$$
$$=(\widetilde{\nabla^l A^{2,3}})^1_{2,\emptyset}+(\widetilde{\nabla^l A^{2,3}})^2_{2,\emptyset}.$$

It is easily seen that the first integral $(\widetilde{\nabla^l A^{2,3}})^1_{2,\emptyset}$ is differentiable function in $(t_1, t_2, x; \hat{\xi}, z; \hat{\zeta})$ by using the coordinate $\Delta_{3,\emptyset}$. For $(\widetilde{\nabla^l A^{2,3}})^2_{2,\emptyset}$, use the operator

$$(7.47)\qquad L''_T=\frac{i\partial_y T(x; t_1\hat{\xi}, y; \eta, z; \hat{\zeta})\cdot\partial_y}{|\partial_y T(x; t_1\hat{\xi}, y; \eta, z; \hat{\zeta})|^2}\ .$$

Then, $L''_T e^{-iT}=e^{-iT}$ and $|\partial_y T(x; t_1\hat{\xi}, y; \eta, z; \hat{\zeta})|\geqq M|\eta'|>0$ on $\Delta_{3,(2)}$. By using L''_T and the fact that $\tilde{c}_{2,3}(x; t_1\hat{\xi}, y; \eta, z; t_2\hat{\zeta})$ is differentiable on $\Delta_{3,(2)}$, we get $(\widetilde{\nabla^l A^{2,3}})^2_{2,\emptyset}$ is differentiable on $(t_1, t_2, x; \hat{\xi}, z; \hat{\zeta})$ and therefore $(\widetilde{\nabla^l A^{2,3}})_{2,\emptyset}$ is differentiable on $(t_1, t_2, x; \hat{\xi}, z; \hat{\zeta})$.

On $\Delta_{2,(1)}$: Use the coordinate on $\Delta_{2,(1)}$. Then, we get

$$(\widetilde{\nabla^l A^{2,3}})_{2,(1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta})$$
$$=Os-\iint c_{2,3}^{l,T'}(x; r\hat{\xi}, y; \eta, z; t\hat{\zeta})e^{-iT(x;r\hat{\xi},y;\eta,z;t\hat{\zeta})}đyđ\eta\ .$$

Putting $\eta=r\tilde{\eta}$, we have

$$(7.48)\qquad (\widetilde{\nabla^l A^{2,3}})_{2,(1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta})$$
$$=Os-\iint c_{2,3}^{l,T'}(x; r\hat{\xi}, y; r\tilde{\eta}, z; t\hat{\zeta})e^{-irT(x;\hat{\xi},y;\tilde{\eta},z;\hat{\zeta})}r^n đyđ\tilde{\eta}\ ,$$

where

$$c_{2,3}^{l,T'}(x; r\hat{\xi}, y; r\tilde{\eta}, z; t\hat{\zeta})=\tilde{c}_{2,3}(x; r\hat{\xi}, y; r\tilde{\eta}, z; t\hat{\zeta})(T'(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta}))^l r^{|l|}\ ,$$

and $|\tilde{\eta}'|\geqq C>0$. Now, we use L_T in (7.33). Integrating by parts, we have

$$(\widetilde{\nabla^l A^{2,3}})_{2,(1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta})$$
$$=Os-\iint (L_T^*)^m(c_{2,3}^{l,T'}(x; r\hat{\xi}, y; r\tilde{\eta}, z; t\hat{\zeta}))e^{-irT(x;\hat{\xi},y;\tilde{\eta},z;\hat{\zeta})}r^n dy d\tilde{\eta} .$$

Put

(7.49) $$\tilde{r}=r , \quad \tilde{t}_2=1/|\tilde{\eta}| , \quad \tilde{t}_3=t .$$

Then, $\tilde{c}_{2,3}(x; r\hat{\xi}, y; r\tilde{\eta}, z; t\hat{\zeta})$ is smooth on $\Delta_{3,(1,2)}$. Therefore, we get

(7.50) $$(\widetilde{\nabla^l A^{2,3}})_{2,(1)}=O(r^{-N}) \quad \text{for any} \quad N\geqq 0 .$$

Similarly, we have

(7.51) $$D^{\alpha}_{(r,t,x;\hat{\xi},z;\hat{\zeta})}(\widetilde{\nabla^l A^{2,3}})_{2,(1)}=O(r^{-N}) \quad \text{for any} \quad N\geqq 0 .$$

On $\Delta_{2,(2)}$: Use the coordinate in $\Delta_{2,(2)}$. We have

$$(\widetilde{\nabla^l A^{2,3}})_{2,(2)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta})$$
$$=Os-\iint c_{2,3}^{l,T'}(x; t\hat{\xi}, y; \eta, z; r\hat{\zeta})e^{-iT(x;t\hat{\xi},y;\eta,z;\hat{\zeta})}dy d\eta ,$$

where $|\eta|\geqq Ct$. By using a cut off function $\psi(y; \eta)$ defined for $\Delta_{2,\emptyset}$ in (7.46), we divide the above integration as

(7.52) $$(\nabla^l A^{2,3})_{2,(2)}=\iint_{Ct\leq|\eta|\leq K} \psi\cdot c_{2,3}^{l,T'}e^{-iT}+Os-\iint_{|\eta|\geq K^{-1}}(1-\psi)c_{2,3}^{l,T'}e^{-iT}$$
$$=(\nabla^l A^{2,3})^1_{2,(2)}+(\nabla^l A^{2,3})^2_{2,(2)} .$$

Use the fact that $\tilde{c}_{2,3}(x; t\hat{\xi}, y; \eta, z; r\hat{\zeta})$ is smooth with respect to $(r^{-1}, |\eta|, t, x; \hat{\xi}, y; \hat{\eta}, z; \hat{\zeta})$ on $\Delta_{3,(3)}$ if $|\eta|\leqq K$. Then, $(\widetilde{\nabla^l A^{2,3}})^1_{2,(2)}$ is smooth and is $O(1)$. Next, consider $(\widetilde{\nabla^l A^{2,3}})^2_{2,(2)}$. Put, if $|\eta|/r\leqq K$,

(7.53) $$\tilde{r}=|\eta| , \quad \tilde{t}_1=t , \quad \tilde{t}_3=|\eta|/r ;$$

and if $r/|\eta|\leqq K$,

(7.54) $$\tilde{r}=r , \quad \tilde{t}_1=\tilde{t} , \quad t_3=r/|\eta| .$$

Then, $\tilde{c}_{2,3}(x; t\hat{\xi}, y; \eta, z; r\hat{\zeta})$ is smooth on $\Delta_{3,(2,3)}$ and $\Delta_{3,(3,2)}$. Use also L_T'' defined in (7.47), and we get

(7.55) $$(\widetilde{\nabla^l A^{2,3}})^2_{2,(2)}=O(1) .$$

Therefore, $(\widetilde{\nabla^l A^{2,3}})_{2,(2)}=O(1)$. Similarly, we get

(7.56) $$D^{\alpha}_{(t,x;\hat{\xi},z;\hat{\zeta})}(\nabla^l A^{2,3})_{2,(2)} = O(1) .$$

On $\Delta_{2,(1,2)}$: Use the coordinate in $\Delta_{2,(1,2)}$. We have

$$(\widetilde{\nabla^l A^{2,3}})_{2,(1,2)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta})$$
$$= Os - \iint c^{l,T'}_{2,3}(x; r\hat{\xi}, y; \eta, z; (r/t)\hat{\zeta}) e^{-iT(x;r\hat{\xi},y;\eta,z;(r/t)\hat{\zeta})} đyđ\eta .$$

Setting $\eta = r\tilde{\eta}$, we get

(7.57) $$(\widetilde{\nabla^l A^{2,3}})_{2,(1,2)}$$
$$= Os - \iint_{|\eta| \geq C} c^{l,T'}_{2,3}(x; r\hat{\xi}, y; r\tilde{\eta}, z; (r/t)\hat{\zeta}) r^n e^{-irT(x;\hat{\xi},y;\tilde{\eta},z;\hat{\zeta})} đyđ\eta ,$$

where

$$c^{l,T'}_{2,3}(x; r\hat{\xi}, y; r\tilde{\eta}, z; (r/t)\hat{\zeta}) = \tilde{c}_{2,3}(x; r\hat{\xi}, y; r\tilde{\eta}, z; (r/t)\hat{\zeta}) r^{|l|} (T')^l(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta}) .$$

Use L_T in (7.33) and note that $|\partial_y T(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta})| \geq M(1+|\tilde{\eta}|)$ because of Corollary 4.6 and also $|\partial^{\alpha}_y T(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta})| \leq C_{\alpha}(1+|\tilde{\eta}|)$. Therefore, integrating by parts, we have

(7.58) $$(\widetilde{\nabla^l A^{2,3}})_{2,(1,2)}(x; r\hat{\xi}, z; (r/t)\hat{\zeta})$$
$$= \iint (L^*_T)^m [\tilde{c}_{2,3}(x; r\hat{\xi}, y; r\tilde{\eta}, z; (r/t)\hat{\zeta}) r^{n+|l|} (T')^l(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta})]$$
$$\times e^{-irT(x;\hat{\xi},y;\tilde{\eta},z;\hat{\zeta})} đyđ\tilde{\eta} .$$

Put, if $|\tilde{\eta}| t \leq K$,

(7.59) $$\tilde{r} = r , \qquad \tilde{t}_2 = 1/|\tilde{\eta}| , \qquad \tilde{t}_3 = t|\tilde{\eta}| ;$$

and if $|\tilde{\eta}| t \geq K^{-1}$,

(7.60) $$\tilde{r} = r , \qquad \tilde{t}_2 = 1/|\tilde{\eta}| t , \qquad \tilde{t}_3 = t .$$

The amplitude function $\tilde{c}_{2,3}$ in (7.58) is smooth in (r, t) on $\Delta_{3,(1,2,3)}$ and $\Delta_{3,(1,3,2)}$ for the cases (7.59) and (7.60), respectively. We have

(7.61) $$\begin{cases} \partial_r = \partial_{\tilde{r}} , & \partial_t = |\tilde{\eta}| \partial_{\tilde{t}_3} \quad \text{for (7.59)} \\ \partial_r = \partial_{\tilde{r}} , & \partial_t = -\dfrac{1}{t} \tilde{t}_2 \partial_{\tilde{t}_2} + \partial_{\tilde{t}_3} \quad \text{for (7.60)} . \end{cases}$$

Remark that for the case (7.60), we get $(1/t) \leq K|\tilde{\eta}|$. Differentiating (7.58) in $(r, t, x; \hat{\xi}, z; \hat{\zeta})$, we get, by taking m so large,

(7.62) $$D^{\alpha}_{(r,t,x;\hat{\xi},z;\hat{\zeta})}(\widetilde{\nabla^l A^{2,3}})_{2,(1,2)} = O(r^{-N}) \quad \text{for any} \quad N \geq 0 .$$

On $\Delta_{2,(2,1)}$: Use the coordinate on $\Delta_{2,(2,1)}$. Then, we get

$$(\widetilde{\nabla^l A^{2,3}})_{2,(2,1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta}) = Os-\iint c_{2,3}^{l;T'}(x; (r/t)\hat{\xi}, y; \eta, z; r\hat{\zeta}) e^{-iT(x;(r/t)\hat{\xi},y;\eta,z;r\hat{\zeta})} đy đ\eta .$$

Set $\eta=(r/t)\tilde{\eta}$, and we have

$$(7.63)\quad (\widetilde{\nabla^l A^{2,3}})_{2,(2,1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta}) = Os-\iint_{|\tilde{\eta}|\geq C} c_{2,3}^{l;T'}(x; (r/t)\hat{\xi}, y; (r/t)\tilde{\eta}, z; r\hat{\zeta}) e^{-i(r/t)T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\zeta})} \times (r/t)^n đy đ\tilde{\eta} ,$$

where

$$c_{2,3}^{l;T'}(x; (r/t)\hat{\xi}, y; (r/t)\tilde{\eta}, z; r\hat{\zeta}) = \tilde{c}_{2,3}(x; (r/t)\hat{\xi}, y; (r/t)\tilde{\eta}, z; r\hat{\zeta}) T'(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta})^l (r/t)^{|l|} .$$

Now, put

$$(7.64)\quad \tilde{r}=r , \qquad \tilde{t}_1=t , \qquad \tilde{t}_2=1/|\tilde{\eta}| .$$

Then, the function

$$\rho(\tilde{r}^{-1}, \tilde{t}_1, \tilde{t}_2, x; \hat{\xi}, y; \hat{\tilde{\eta}}, z; \hat{\zeta}) = \tilde{c}_{2,3}(x; (\tilde{r}/\tilde{t}_1)\hat{\xi}, y; (\tilde{r}/\tilde{t}_1\tilde{t}_2)\hat{\tilde{\eta}}, z; \tilde{r}\hat{\zeta})$$

is smooth on $\Delta_{3,(3,1,2)}$. Since $\partial_r=\partial_{\tilde{r}}$ and $\partial_t=\partial_{\tilde{t}_1}$, we have

$$(7.65)\quad |\partial_r^a \partial_t^b D_{(x;\hat{\xi},z;\hat{\zeta})} \tilde{c}_{2,3}| \leq c_{a,b,\alpha} \quad \text{for some constant } c_{a,b,\alpha} .$$

Use L'_T in (7.42), and the fact that if $|\tilde{\eta}|\geq C$,

$$|\partial_y T(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta})| \geq M|\tilde{\eta}| ,$$
$$|\partial_y^\alpha T(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta})| \leq C_\alpha(1+|\tilde{\eta}|) , \quad |\alpha|\geq 1 ,$$

for some constants C_α. Integrating by parts with L'_T, we have

$$(7.66)\quad (\widetilde{\nabla^l A^{2,3}})_{2,(2,1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta}) = O(r^{-N}t^N)$$

and

$$(7.67)\quad D^\alpha_{(r,t,x;\hat{\xi},z;\hat{\zeta})} (\widetilde{\nabla^l A^{2,3}})_{2,(2,1)} = O(r^{-N}t^N) ,$$

for any $N\geq 0$.

By the same reasoning as in the last paragraph of Lem. $A^{2,1}$, we can see the differentiability of $A^{2,3}$ with respect to (a, τ_1, τ_2). □

7.5. Lem. $A^{2,2}$.

Lastly, we shall consider the integral $A^{2,2}$ in Lemma 7.4. As in 7.3-4, to consider the differentiability with respect to α, τ_1, τ_2, we have to consider the following integral; for $|l|\geqq 0$,

$$\nabla^l A^{2,2}(x;\xi,z;\zeta)=Os-\iint_{D_{2,2}} c_{2,2}^{l,T'}(x;\xi,y;\eta,z;\zeta)e^{-iT(x;\xi,y;\eta,z;\zeta)}đyđ\eta\ , \tag{7.68}$$

where $c_{2,2}^{l,T'}$ can be described as follows:

(a) $c_{2,2}^{l,T'}=\tilde{c}_{2,2}(T')^l$,

(b) $(T')^l=(T_1')^{l_1}\cdots(T_k')^{l_k}$, $T_i'\in\vartheta^{(2)}$,

(c) $\tilde{c}_{2,2}\in\alpha^8(\varepsilon_1,\varepsilon_2)$ and satisfies the same condition for $c_{2,2}$ in (7.17).

Now, we shall prove that Lem. $A^{2,2}$ holds:

Remark that on support of $c_{2,2}$, we have $|\xi|^2\geqq R^2/2(1+4C^2)$. Therefore, if we take $R\geqq\sqrt{2(1+4C^2)K}$, $\operatorname{supp} c_{2,2}\cap\varDelta_{2,\varnothing}=\varnothing$ and $\operatorname{supp} c_{2,2}\cap\varDelta_{2,(2)}=\varnothing$. So, we have only to consider $A^{2,2}$ on the domains $\varDelta_{2,(1)}$, $\varDelta_{2,(1,2)}$ and $\varDelta_{2,(2,1)}$.

On $\varDelta_{2,(1)}$: Use the coordinate on $\varDelta_{2,(1)}$. Then, we have

$$(\widetilde{\nabla^l A^{2,2}})_{2,(1)}(r^{-1},t,x;\hat{\xi},z;\hat{\zeta})=Os-\iint c_{2,2}^{l,T'}(x;r\hat{\xi},y;\eta,z;t\hat{\zeta})e^{-iT(x;r\hat{\xi},y;\eta,z;t\hat{\zeta})}đyđ\eta\ .$$

Set $\eta=r\tilde{\eta}$, and we have

$$\begin{aligned}(\widetilde{\nabla^l A^{2,2}})_{2,(1)}&(r^{-1},t,x;\hat{\xi},z;\hat{\zeta})\\ &=\iint_{(1/2)C^{-1}\leqq|\tilde{\eta}|\leqq 2C} r^n c_{2,2}^{l,T'}(x;r\hat{\xi},y;r\tilde{\eta},z;t\hat{\zeta})e^{-irT(x;\hat{\xi},y;\tilde{\eta},z;\hat{\zeta})}đyđ\tilde{\eta}\ ,\end{aligned}\tag{7.69}$$

where

$$c_{2,2}^{l,T'}(x;r\hat{\xi},y;r\tilde{\eta},z;t\hat{\zeta})=\tilde{c}_{2,2}(x;r\hat{\xi},y;r\tilde{\eta},z;t\hat{\zeta})(T')^l(x;\hat{\xi},y;\tilde{\eta},z;\hat{\zeta})r^{|l|}\ .$$

Remark that the above integral is well-defined. Now, put

$$L_T^{(3)}=\frac{i[\partial_y T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\zeta})\partial_y+\partial_{\tilde{\eta}}T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\zeta})\partial_{\tilde{\eta}}]}{r[|\partial_y T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\zeta})|^2+|\partial_{\tilde{\eta}}T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\zeta})|^2]}\ . \tag{7.70}$$

By Corollary 4.6, $|\partial_y T|^2+|\partial_{\tilde{\eta}}T|^2\geqq\delta>0$, and we get easily

$$|\nabla^\alpha_{(y;\tilde{\eta})}T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\zeta})|\leqq C_\alpha(1+|\tilde{\eta}|)\ ,\quad |\alpha|\geqq 1\ . \tag{7.71}$$

Then we get

$$\begin{aligned}(\widetilde{\nabla^l A^{2,2}})_{2,(1)}&(r^{-1},t,x;\hat{\xi},z;\hat{\zeta})\\ &=\iint_{(1/2)C^{-1}\leqq|\tilde{\eta}|\leqq 2C}(L_T^{(3)*})^m[c_{2,3}^{l,T'}(x;r\hat{\xi},y;r\tilde{\eta},z;t\hat{\zeta})r^n]e^{-irT(x;\hat{\xi},y;\tilde{\eta},z;\hat{\zeta})}đyđ\tilde{\eta}\ .\end{aligned}$$

Put as follows:

$$(7.72)\qquad \begin{cases} \tilde{r}=r|\tilde{\eta}|\,, & \tilde{t}_1=|\tilde{\eta}|\,, \quad \tilde{t}_3=t\,, \quad \text{if } |\tilde{\eta}|\leqq K \\ \tilde{r}=r\,, & \tilde{t}_2=1/|\tilde{\eta}|\,, \quad \tilde{t}_3=t\,, \quad \text{if } |\tilde{\eta}|\geqq K^{-1}\,. \end{cases}$$

Then, amplitude function $\tilde{c}_{2,2}$ in (7.69) is smooth on $\Delta_{3,(2,1)}$ and $\Delta_{3,(1,2)}$ for each case in (7.72) respectively. Therefore, we get

$$(7.73)\qquad (\widetilde{\nabla^l A^{2,2}})_{2,(1)}=O(r^{-N}) \quad \text{for any} \quad N\geqq 0\,.$$

By the same computations as above, combining (7.71), we have

$$(7.74)\qquad D^{\alpha}_{(r,t,x;\hat{\xi},z;\hat{\zeta})}(\widetilde{\nabla^l A^{2,2}})_{2,(1)}=O(r^{-N}) \quad \text{for any} \quad N\geqq 0\,.$$

On $\Delta_{2,(1,2)}$: Use the coordinate on $\Delta_{2,(1,2)}$. Then, we have

$$\begin{aligned}&(\widetilde{\nabla^l A^{2,2}})_{2,(1,2)}(r^{-1}, t; x; \hat{\xi}, z; \hat{\zeta})\\ &\qquad =Os-\iint c^{l,T'}_{2,2}(x; r\hat{\xi}, y; \eta, z; (r/t)\hat{\zeta})e^{-iT(x;r\hat{\xi},y;\eta,z;(r/t)\hat{\zeta})}dy d\eta\,.\end{aligned}$$

Set $\eta=r\tilde{\eta}$, and we get

$$\begin{aligned}(7.75)\qquad &(\widetilde{\nabla^l A^{2,2}})_{2,(1,2)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta})\\ &\qquad =\iint_{(1/2)C^{-1}\leqq|\tilde{\eta}|\leqq 2C} r^n c^{l,T'}_{2,2}(x; r\hat{\xi}, y; r\tilde{\eta}, z; (r/t)\hat{\zeta})e^{-irT(x;\hat{\xi},y;\tilde{\eta},z;\hat{\zeta})}dy d\tilde{\eta}\,,\end{aligned}$$

where

$$c^{l,T'}_{2,2}(x; r\hat{\xi}, y; r\tilde{\eta}, z; (r/t)\hat{\zeta})=\tilde{c}_{2,2}(x; r\hat{\xi}, y; r\tilde{\eta}, z; (r/t)\hat{\zeta})r^{|l|}(T')^l(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta})\,.$$

Put as follows:

$$(7.76)\qquad \begin{cases} \tilde{r}=r|\tilde{\eta}|\,, & \tilde{t}_1=|\tilde{\eta}|\,, \quad \tilde{t}_3=t\,, \quad \text{if } |\tilde{\eta}|\leqq K\,, \\ \tilde{r}=r\,, & \tilde{t}_2=1/|\tilde{\eta}|\,, \quad \tilde{t}_3=t|\tilde{\eta}|\,, \quad \text{if } |\tilde{\eta}|\geqq K^{-1}\,, \quad |\tilde{\eta}|t\leqq K\,, \\ \tilde{r}=r\,, & \tilde{t}_2=1/|\tilde{\eta}|t\,, \quad \tilde{t}_3=t\,, \quad \text{if } |\tilde{\eta}|\geqq K^{-1}\,, \quad |\tilde{\eta}|t\geqq K^{-1}\,. \end{cases}$$

Then, the following functions ρ, ρ', ρ'' are smooth on $\Delta_{3,(2,1,3)}$, $\Delta_{3,(1,2,3)}$ and $\Delta_{3,(1,3,2)}$, respectively:

$$(7.77)\qquad \begin{cases} \rho(\tilde{r}^{-1}, \tilde{t}_1, \tilde{t}_3, x; \hat{\xi}, y; \hat{\tilde{\eta}}, z; \hat{\zeta})=\tilde{c}_{2,2}(x; (\tilde{r}/\tilde{t}_1)\hat{\xi}, y; \tilde{r}\hat{\tilde{\eta}}, z; (\tilde{r}/\tilde{t}_1\tilde{t}_3)\hat{\zeta}) \\ \qquad\qquad \text{for} \quad \Delta_{3,(2,1,3)}\,, \\ \rho'(\tilde{r}^{-1}, \tilde{t}_2, \tilde{t}_3, x; \hat{\xi}, y; \hat{\tilde{\eta}}, z; \hat{\zeta})=\tilde{c}_{2,2}(x; \tilde{r}\hat{\xi}, y; (\tilde{r}/\tilde{t}_2)\hat{\tilde{\eta}}, z; (\tilde{r}/\tilde{t}_3\tilde{t}_2)\hat{\zeta}) \\ \qquad\qquad \text{for} \quad \Delta_{3,(1,2,3)}\,, \\ \rho''(\tilde{r}^{-1}, \tilde{t}_2, \tilde{t}_3, x; \hat{\xi}, y; \hat{\tilde{\eta}}, z; \hat{\zeta})=\tilde{c}_{2,2}(x; \tilde{r}\hat{\xi}, y; (\tilde{r}/\tilde{t}_2\tilde{t}_3)\hat{\tilde{\eta}}, z; (\tilde{r}/\tilde{t}_3)\hat{\zeta}) \\ \qquad\qquad \text{for} \quad \Delta_{3,(1,3,2)}\,. \end{cases}$$

Also, we have

$$(7.78)\qquad \begin{cases} \partial_r = |\tilde{\eta}|\partial_{\tilde{r}}\ , & \partial_t = \partial_{\tilde{t}_3} \quad \text{on} \quad \Delta_{3,(2,1,3)}\ , \\ \partial_r = \partial_{\tilde{r}}\ , & \partial_t = |\tilde{\eta}|\partial_{\tilde{t}_3} \quad \text{on} \quad \Delta_{3,(1,2,3)}\ , \\ \partial_r = \partial_{\tilde{r}}\ , & \partial_t = -(\tilde{t}_2/t)\partial_{\tilde{t}_2} + \partial_{\tilde{t}_3} \quad \text{on} \quad \Delta_{3,(1,3,2)}\ . \end{cases}$$

Remark that in the case $\Delta_{3,(1,3,2)}$, $(1/t) \leqq K|\tilde{\eta}| \leqq 2KC$. Use $L_T^{(3)}$ in (7.70) and Lax technique. So, we get

$$(7.79)\qquad (\widetilde{\nabla^l A^{2,2}})_{2,(1,2)} = O(r^{-N}) \quad \text{for any} \quad N \geqq 0\ ,$$

and

$$(7.80)\qquad D^{\alpha}_{(r,t,x;\hat{\xi},z;\hat{\zeta})}(\widetilde{\nabla^l A^{2,2}})_{2,(1,2)} = O(r^{-N}) \quad \text{for and} \quad N \geqq 0\ .$$

On $\Delta_{2,(2,1)}$: Use the coordinate in $\Delta_{2,(2,1)}$. Then we have

$$(\widetilde{\nabla^l A^{2,2}})_{2,(2,1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta}) = Os - \iint c_{2,2}^{l;T'}(x; (r/t)\hat{\xi}, y; \eta, z; r\hat{\zeta}) e^{-iT(x;(r/t)\hat{\xi},y;\eta,z;r\hat{\zeta})} đyđ\eta\ .$$

Set by $(r/t)\tilde{\eta} = \eta$, and we get

$$(7.81)\qquad (\widetilde{\nabla^l A^{2,2}})_{2,(2,1)}(r^{-1}, t, x; \hat{\xi}, z; \hat{\zeta}) = \iint_{(1/2)C^{-1} \leqq |\tilde{\eta}| \leqq 2C} (r/t)^n c_{2,2}^{l;T'}(x; (r/t)\hat{\xi}, y; (r/t)\tilde{\eta}, z; r\hat{\zeta}) \times e^{-i(r/t)T(x;\hat{\xi},y;\tilde{\eta},z;\hat{\zeta})} đyđ\tilde{\eta}\ ,$$

where

$$(7.82)\qquad c_{2,2}^{l;T'}(x; (r/t)\hat{\xi}, y; (r/t)\tilde{\eta}, z; r\hat{\zeta}) = \tilde{c}_{2,2}(x; (r/t)\hat{\xi}, y; (r/t)\tilde{\eta}, z; r\hat{\zeta})(T')^l(x; \hat{\xi}, y; \tilde{\eta}, z; \hat{\zeta})(r/t)^{|l|}\ .$$

Put as follows:

$$(7.83)\qquad \begin{cases} \tilde{r} = r\ , \quad \tilde{t}_1 = t\ , \quad \tilde{t}_2 = r|\tilde{\eta}|/t\ , & \text{if} \quad r|\tilde{\eta}|/t \leqq K\ , \\ \tilde{r} = r\ , \quad \tilde{t}_1 = t\ , \quad \tilde{t}_2 = 1/|\tilde{\eta}|\ , & \text{if} \quad r|\tilde{\eta}|/t \geqq K^{-1}\ , \quad |\tilde{\eta}| \geqq K^{-1}\ , \\ \tilde{r} = (r/t)|\tilde{\eta}|\ , \quad \tilde{t}_1 = t\ , \quad \tilde{t}_3 = |\tilde{\eta}|/t\ , & \text{if} \quad r|\tilde{\eta}|/t \geqq K^{-1}\ , \quad |\tilde{\eta}| \leqq K\ , \quad |\tilde{\eta}|/t \leqq K\ , \\ \tilde{r} = r\ , \quad \tilde{t}_1 = |\tilde{\eta}|\ , \quad \tilde{t}_2 = t/|\tilde{\eta}|\ , & \text{if} \quad r|\tilde{\eta}|/t \geqq K^{-1}\ , \quad |\tilde{\eta}| \leqq K\ , \quad |\tilde{\eta}|/t \geqq K^{-1}\ . \end{cases}$$

Also, the following functions ρ, ρ', ρ'', ρ''' are smooth on $\Delta_{3,(3,1)}$, $\Delta_{3,(3,1,2)}$, $\Delta_{3,(2,3,1)}$ and $\Delta_{3,(3,2,1)}$ for each case of (7.83) respectively.

(7.84)

$$\begin{cases} \rho(\tilde{r}^{-1}, \tilde{t}_1, \tilde{t}_2, x; \hat{\xi}, y; \hat{\eta}, z; \hat{\zeta}) = \tilde{c}_{2,2}(x; (\tilde{r}/\tilde{t}_1)\hat{\xi}, y; \tilde{t}_2\hat{\eta}, z; \tilde{r}\hat{\zeta}) & \text{on } \Delta_{3,(3,1)}, \\ \rho'(\tilde{r}^{-1}, \tilde{t}_1, \tilde{t}_2, x; \hat{\xi}, y; \hat{\eta}, z; \hat{\zeta}) = \tilde{c}_{2,2}(x; (\tilde{r}/\tilde{t}_1)\hat{\xi}, y; (\tilde{r}/\tilde{t}_1\tilde{t}_2)\hat{\eta}, z; \tilde{r}\hat{\zeta}) & \text{on } \Delta_{3,(3,1,2)}, \\ \rho''(\tilde{r}^{-1}, \tilde{t}_1, \tilde{t}_2, x; \hat{\xi}, y; \hat{\eta}, z; \hat{\zeta}) = \tilde{c}_{2,2}(x; (\tilde{r}/\tilde{t}_1\tilde{t}_3)\hat{\xi}, y; \tilde{r}\hat{\eta}, z; (\tilde{r}/\tilde{t}_3)\hat{\zeta}) & \text{on } \Delta_{3,(2,3,1)}, \\ \rho'''(\tilde{r}^{-1}, \tilde{t}_1, \tilde{t}_2, x; \hat{\xi}, y; \hat{\eta}, z; \hat{\zeta}) = \tilde{c}_{2,2}(x; (\tilde{r}/\tilde{t}_1\tilde{t}_2)\hat{\xi}, y; (\tilde{r}/\tilde{t}_2)\hat{\eta}, z; \tilde{r}\hat{\zeta}) & \text{on } \Delta_{3,(3,2,1)}. \end{cases}$$

Remark that

$$(7.85) \quad \begin{cases} \partial_r = \partial_{\tilde{r}} + (|\hat{\eta}|/t)\partial_{\tilde{t}_2}, & \partial_t = \partial_{\tilde{t}_1} - (\tilde{t}_2/t)\partial_{\tilde{t}_2} & \text{for } \Delta_{3,(3,1)}, \\ \partial_r = \partial_{\tilde{r}}, & \partial_t = \partial_{\tilde{t}_1} & \text{for } \Delta_{3,(3,1,2)}, \\ \partial_r = (|\hat{\eta}|/t)\partial_{\tilde{r}}, & \partial_t = \partial_{\tilde{t}_1} - (\tilde{t}_3/t)\partial_{\tilde{t}_3} & \text{for } \Delta_{3,(2,3,1)}, \\ \partial_r = \partial_{\tilde{r}}, & \partial_t = -(\tilde{t}_2/|\hat{\eta}|)\partial_{\tilde{t}_2} & \text{for } \Delta_{3,(3,2,1)}. \end{cases}$$

Use $L_T^{(3)}$ in (7.70) and Lax technique. So, we get

$$(7.86) \qquad (\widetilde{\nabla^l A^{2,2}})_{2,(2,1)} = O(r^{-N}t^N) \quad \text{for any } N \geqq 0,$$

and

$$(7.87) \qquad D^{\alpha}_{(r,t,x;\hat{\xi},z;\hat{\zeta})}(\widetilde{\nabla^l A^{2,2}})_{2,(2,1)} = O(r^{-N}t^N) \quad \text{for any } N \geqq 0.$$

By the same reasoning as in the last paragraph of Lem. $A^{2,1}$, we can see the differentiability of $A^{(2,2)}$ with respect to (a, τ_1, τ_2). □

By 7.1-7.5, we obtain Proposition 6.1, completely. □

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Present Address:
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND TECHNOLOGY
KEIO UNIVERSITY
HIYOSHI, KOHOKU-KU, YOKOHAMA, 223

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND TECHNOLOGY
SCIENCE UNIVERSITY OF TOKYO
NODA, CHIBA, 278

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE AND TECHNOLOGY
KEIO UNIVERSITY
HIYOSHI, KOHOKU-KU, YOKOHAMA, 223

AND

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
TOKYO METROPOLITAN UNIVERSITY
FUKAZAWA, SETAGAYA-KU, TOKYO, 158