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Local Topological Properties of Differentiable Mappings II

[Dedicated to Professor Morio Obata on his sixtieth Birthday]

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Introduction

In the preceding paper [2], it was shown that *almost every* C^∞ map-germ: $(R^n, 0)\to(R^p, 0)$, $n\leqq p$, has rather good topological structures. In particular it was shown that they are topologically equivalent to the cones of topologically stable mappings of S^{n-1} into S^{p-1}, where the cone of a mapping $f: X\to Y$ is the mapping $Cf: X\times[0, 1)/X\times\{0\}\to Y\times[0, 1)/Y\times\{0\}$ defined by $Cf(x, t)=(f(x), t)$. Here *almost every* is used in the rather strong sense that the complement of the set of these map-germs should have infinite codimension in the space of all C^∞ map-germs.

This paper has two purposes. One is to show similar generic properties for the remaining case $n>p$. The other is to show, as an application of these generic properties, that for almost every mapping into the plane $f: (R^n, 0)\to(R^2, 0)$ a Poincare-Hopf type equality, in some cases the Morse inequalities as well, holds between the Betti numbers of the set $f^{-1}(0)\cap S_\varepsilon^{n-1}$ and the indices of the singular points of f appearing around the origin, where $S_\varepsilon^{n-1}=\{x\in R^n \mid \|x\|=\varepsilon\}$ and ε is supposed to be small. The index of a singular point of a mapping into the plane will be defined later in this section.

Let us explain these properties more precisely. $J^r(n, p)$ is the set of the r-jets of all C^∞ map-germs: $(R^n, 0)\to(R^p, 0)$. For a positive number $\varepsilon>0$, we set

$$D_\varepsilon^m=\{x\in R^m \mid \|x\|\leqq\varepsilon\}\ ,$$
$$S_\varepsilon^{m-1}=\{x\in R^m \mid \|x\|=\varepsilon\}\ .$$

THEOREM 1. *For each positive integer r, there exists a closed*

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semi-algebraic subset $\Sigma_r(n, p)$ *of* $J^r(n, p)$ *such that*

(1) codim. $\Sigma_r(n, p) \to \infty$ *as* $r \to \infty$,

(2) *if a* C^∞ *mapping* $f: R^n \to R^p$ *represents an element of* $J^r(n, p) - \Sigma_r(n, p)$, *then for any sufficiently small positive numbers* ε *and* δ, *the upper bound of* ε *depending on* f *and the upper bound of* δ *depending on* ε *and* f, *the following properties hold.*

(a) $D_\varepsilon^n \cap f^{-1}(S_\delta^{p-1})$ *is a* C^∞ *manifold, in general with boundary, and its differentiable structure is independent of* ε *and* δ.

(b) *The restricted mapping* $f: D_\varepsilon^n \cap f^{-1}(S_\delta^{p-1}) \to S_\delta^{p-1}$ *is topologically stable* (C^∞ *stable if* (n, p) *is a nice pair of dimensions in J. Mather's sense) and its topological type is independent of* ε *and* δ.

REMARK. Moreover we can prove that (c) *the topological type of* $f: D_\varepsilon^n \cap f^{-1}(S_\delta^{p-1}) \to S_\delta^{p-1}$ *determines the topological type of the germ of* f *at the origin of* R^n. The proof of this property is very similar to the proof of the corresponding property in the case $n \leqq p$ given in [2], and we will not give it here.

REMARK. Combining with A. du Plessis's work [1], we can say that *the germ at the origin of such* f *is topologically r-determined.*

Now we explain our Poincare-Hopf equality and the Morse inequalities. From Theorem 1, if the jet $j^r f(0)$ of a C^∞ mapping $f: R^n \to R^p$ belongs to $J^r(n, p) - \Sigma_r(n, p)$, then for sufficiently small ε and δ, the restricted mapping $f: D_\varepsilon^n \cap f^{-1}(S_\delta^1) \to S_\delta^1$ is C^∞ stable. In other words, defining a function $\theta: R^2 - \{0\} \to (R \bmod. 2\pi)$ by

$$x + iy = \sqrt{x^2+y^2}\, e^{i\theta(x,y)}, \quad (x, y) \in R^2 - \{0\},$$

the composed mapping $\theta \circ f: D_\varepsilon^n \cap f^{-1}(S_\delta^1) \to (R \bmod. 2\pi)$ can be regarded as a Morse function. Although it is not a Morse function in the strict sense (it's values are not in R but in R mod. 2π), we can define the indices of critical points of $\theta \circ f: D_\varepsilon^n \cap f^{-1}(S_\delta^1) \to (R \bmod. 2\pi)$ as usual. Now we set

$m_i(f)$ = the number of critical points having index i of the Morse function $\theta \circ f: D_\varepsilon^n \cap f^{-1}(S_\delta^1) \to R \bmod. 2\pi$,

$b_i(M)$ = the i-th Betti number of a manifold M,

$\chi(M) = \Sigma(-1)^i b_i(M)$ the Euler characteristic number of M.

Then we have

THEOREM 2. *If* $f: R^n \to R^2$ *represents an element of* $J^r(n, p) - \Sigma_r(n, p)$, *then*

(i) *the number* $m_i(f)$ *and* $b_j(f^{-1}(0) \cap S_\varepsilon^{n-1})$ *are independent of* ε *and*

δ provided that ε and δ are sufficiently small,

(ii) *we have the following Poincare-Hopf type equality;*

$$\sum_{i=0}^{n-1}(-1)^i m_i(f)+\chi(f^{-1}(0)\cap S_\varepsilon^{n-1})=\chi(S^{n-1}),$$

and moreover

(iii) *if 0 is an isolated point of $f^1(0)$,* i.e. *if $0\notin\overline{f^{-1}(0)-\{0\}}$, and if $n\geqq 3$, then we have the following Morse inequalities;*

$$m_0(f)\geqq b_0(S^{n-1})$$
$$m_1(f)-m_0(f)\geqq b_1(S^{n-1})-b_0(S^{n-1})$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\sum_{i=0}^{k}(-1)^{k-i}m_i(f)\geqq\sum_{i=0}^{k}(-1)^{k-i}b_i(S^{n-1}),\quad k<n-1$$
$$\sum_{i=0}^{n-1}(-1)^i m_i(f)=\chi(S^{n-1}).$$

REMARK. We will see that the numbers $m_i(f)$ and $b_j(f^{-1}(0)\cap S_\varepsilon^{n-1})$ are not only independent of ε and δ, but also they are determined only by the singularities appearing around the origin. In particular we will see

(iv) *a point p of $D_\varepsilon^n\cap f^{-1}(S_\delta^1)$ is a critical potnt of $\theta\circ f$: $D_\varepsilon^n\cap f^{-1}(S_\delta^1)\to$ (R mod. 2π) if and only if it is a singular point of f: $R^n\to R^2$, and*

(v) *the index of a critical point p of $\theta\circ f$: $D_\varepsilon^n\cap f^{-1}(S_\delta^1)\to$ (R mod. 2π) and the index of a critical point q of $\theta\circ f$: $D_\varepsilon^n\cap f^{-1}(S_\delta^1)\to R$ mod. 2π agree if and only if the singular points p and q of f: $R^n\to R^2$ are C^∞ equivalent under diffeomorphisms which preserve the orientation of the target space R^2*: there exist diffeomorphic germs $h_1(R^n, p)\to(R^n, q)$ and h_2: $(R^2, f(p))\to(R^2, f(q))$ such that the equality $f\circ h_1=h_2\circ f$ holds around p and such that h_2 preserve the orientation of R^2.

Contents

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§ 1. Transversality theorem.

In this chapter we recall a transversality theorem which was proved in the preceding paper [2]. This theorem and J. Mather's canonical stratification of the jet spaces play major roles in this paper.

NOTATIONS. The notations used here are about the same as R. Thom's [7] and J. Mather's [4, 5]. $j^r f(x)$ denotes the r-jet of a smooth mapping f at a point x. $J^r(n, p)$ is the space of the r-jets of smooth map-germs: $(R^n, 0) \to (R^p, 0)$, and $J^r(R^n, R^p)$ is the r-jet bundle of the r-jets of smooth mappings of R^n into R^n. ${}_mJ^r(R^n, R^p)$ is the m-fold r-jet bundle of smooth mapping of R^n into R^p: ${}_mJ^r(R^n, R^p) = \{(j^r g_1(q_1), \cdots, j^r g_m(q_m)) \in (J^r(R^n, R^p))^m \mid (q_1, \cdots, q_m) \in (R^n)^{(m)}\}$, where for a set X, $X^{(m)} = \{(q_1, \cdots, q_m) \in X^m \mid q_i \neq q_j \text{ if } i \neq j\}$. For a mapping $f: R^n \to R^p$, ${}_mj^r f: (R^n)^{(m)} \to {}_mJ^r(R^n, R^p)$ denotes the m-fold r-jet extension of f defined by

$$ {}_mj^r f(q_1, \cdots, q_m) = (j^r f(q_1), \cdots, j^r f(q_m)) \, . $$

For integers r and s with $s > r > 0$, $\pi_r^s: J^s(n, p) \to J^r(n, p)$ denotes the canonical projection defined by $\pi_r^s(j^s f(0)) = j^r f(0)$. $\pi_1: (R^n)^m \to R^n$ denotes the projection to the first factor: $\pi_1(q_1, \cdots, q_m) = q_1$. For positive integers l and m with $l \leqq m$, we set

$$ \Delta_l = \{(j^r g_1(q_1), \cdots, j^r g_m(q_m)) \in {}_mJ^r(R^n, R^p) \mid g_1(q_1) = \cdots = g_l(q_l)\} \, . $$

Then our transversality theorem can be stated as follows.

THEOREM 3 (Transversality). *Let W be a semi-algebraic subset of $J^r(n, p)$ and let X be a semi-algebraic submanifold of ${}_mJ^k(R^n, R^p)$. Then there exists a closed semi-algebraic subset Σ_W of $(\pi_r^{r+m(k+1)})^1(W)$ having codimension $\geqq 1$ such that for any mapping $f: R^n \to R^p$ with $j^{r+m(k+1)} f(0) \in (\pi_r^{r+m(k+1)})^{-1}(W) - \Sigma_W$, there exists a neighborhood U of the origin of R^n such that*

(1) *${}_mj^k f$ is transversal to X at every point of*

$$ (U - \{0\})^{(m)} = \{(q_1, \cdots, q_m) \in (U - \{0\})^m \mid q_i \neq q_j \ \text{if} \ i \neq j\} \, , $$

(2) *if* codim. $X = mn$, *then* ${}_mj^k f((U - \{0\})^{(m)}) \cap X = \varnothing$.

Moreover given a polynomial function $\mu: (J^k(R^n, R^p))^m \to R$ whose restriction on X, $\mu | X$, has no critical points and such that $\mu((\{0\} \times \{0\} \times J^k(n, p))^m) = 0$, where we regard $J^k(R^n, R^p) = R^n \times R^p \times J^k(n, p)$, then Σ_W and

U can be chosen so that

(3) ${}_mj^kf(U-\{0\})^{(m)}$ *is transversal to* $X\cap\mu^{-1}(\varepsilon)$ *for all* $\varepsilon\in R$.

This was proved in [2].

§ 2. J. Mather's canonical stratification of jet bundles.

In this section we recall J. Mather's canonical stratification of jet bundles. Let $J^k(n, p)$ be the space of the k-jets of smooth map-germs: $(R^n, 0)\to(R^p, 0)$ and let $J^k(N, P)$ be the jet bundle of k-jets of smooth mappings of a manifold N into another P. Let $L^k(n)$ be the group of the k-jets of diffeomorphic germs: $(R^n, 0)\to(R^n, 0)$. Then $L^k(n)\times L^k(p)$ acts on $J^k(n, p)$ as a Lie transformation group: the action is defined by $(j^kh_1(0), j^kh_2(0))j^kf(0)=j^k(h_2\circ f\circ h_1^{-1})(0)$. Now let A be a subset of $J^k(n, p)$ which is invariant under the action of $L^k(n)\times L^k(p)$. Then for manifolds N and P with dim. $N=n$ and dim. $P=p$, there is a unique subbundle $A_{N,P}$ of the bundle $J^k(N, P)$ with fibre A which is invariant under the action on $J^k(N, P)$ of the group of pairs of diffeomorphisms of N and P. We call $A_{N,P}$ the *subset of* $J^k(N, P)$ *corresponding to* A. For a stratification $\mathscr{S}$ of an $L^k(n)\times L^k(p)$-invariant subset A of $J^k(n, p)$ whose strata are also $L^k(n)\times L^k(p)$-invariant, we set

$$\mathscr{S}_{N,P}=\{X_{N,P}\,|\,X\in\mathscr{S}\}\ .$$

We call $\mathscr{S}_{N,P}$ the stratification of $A_{N,P}$ corresponding to $\mathscr{S}$.

THEOREM 2.1 (J. Mather [4, 5], see also [3]). *For any pair* (n, p) *of positive integers, there exist a positive integer* $k=k(n, p)$, *and* $L^k(n)\times L^k(p)$ *invariant closed semi-algebraic subset* $\Sigma=\Sigma(n, p)$ *of* $J^k(n, p)$ *and a Whitney stratification* $\mathscr{S}=\mathscr{S}(n, p)$ *of* $J^k(n, p)$ *satisfying the following conditions*:

(a) *Strata of* $\mathscr{S}$ *are all* $L^k(n)\times L^k(p)$ *invariant and they are semi-algebraic subsets of* $J^k(n, p)$.

(b) codim. $\Sigma(n, p)>n$ *and* $\Sigma(n, p)$ *is a stratified subset of* $J^k(n, p)$; i.e. *if* $X\cap\Sigma(n, p)\neq\emptyset$ *and* $X\in\mathscr{S}$, *then* $X\subset\Sigma(n, p)$.

(c) *Let* N *and* P *be manifolds with* dim. $N=n$ *and* dim. $P=p$. *Let* $\mathscr{S}_{N,P}$ *be the stratification of* $J^k(N, P)$ *corresponding to* $\mathscr{S}$. *If a proper smooth mapping* $f\colon N\to P$ *is multi-transversal to* $\mathscr{S}_{N,P}$, *then* f *is topologically stable.* (C^∞ *stable if the pair* (n, p) *is a nice pair in J. Mather's sense.*).

Where we say that a mapping $f\colon N\to P$ is *multi-transversal to* $\mathscr{S}_{N,P}$ if for a sufficiently large integer m ($m=p+1$ is large enough), ${}_mj^kf\colon N^{(m)}\to$

${}_mJ^k(N, P)$ is transversal to every manifold of the form $(X_1\times\cdots\times X_m)\cap\Delta_l$, $l\leqq m$ and $X_i\in\mathcal{S}_{N,P}$, where

$$N^{(m)}=\{(x_1, \cdots, x_m)\in N^m\,|\,x_i\neq x_j \text{ if } i\neq j\}\ ,$$
$$\Delta_l=\{(j^kg_1(q_1), \cdots, j^kg_m(q_m))\in {}_mJ^k(N, P)\,|\,g_1(q_1)=\cdots=g_l(q_l)\}\ .$$

DEFINITION. We call $\mathcal{S}(n, p)$ and $\mathcal{S}_{N,P}$ the *canonical stratifications* of the jet spaces $J^k(n, p)$ and $J^k(N, P)$ respectively.

In the case where N is a compact manifold with boundary, as a corollary of the proof of Theorem 2.1, we have

COROLLARY 2.2. *Let N be a compact manifold with boundary. Let $f\colon N\to P$ be a smooth mapping such that*

(1) *the restricted mapping $f\colon\partial N\to P$ is a submersion,*

(2) *$f(N-\partial N)\colon(N-\partial N)\to P$ is multi-transversal to the canonical stratification $\mathcal{S}_{(N-\partial N),P}$.*

Then f is topologically stable. (C^∞ stable if (n, p) is a nice pair.).

§ 3. A stratification of $J^k(R^n, R^p-\{0\})$.

Let $k=k(n-1, p-1)$, $\Sigma=\Sigma(n-1, p-1)$ and $\mathcal{S}=\mathcal{S}(n-1, p-1)$ be the integer, the closed semi-algebraic subset of $J^k(n-1, p-1)$ and the canonical stratification of $J^k(n-1, p-1)$ given by Theorem 2.1 respectively. Set

$$Q=\{j^kf(x)\in J^k(R^n, R^p-\{0\})\,|\,\mathrm{grad}(f_1^2+\cdots+f_p^2)(x)\neq 0\}\ ,$$

where $f(x)=(f_1(x), \cdots, f_p(x))$. Then

$$C=J^k(R^n, R^p-\{0\})-Q$$

is a closed semi-algebraic subset of $J^k(R^n, R^p-\{0\})$ having codimension n.

The purpose of this section is to construct a stratification induced in a way from the canonical stratification $\mathcal{S}(n-1, p-1)$ of $J^k(n-1, p-1)$. For a stratum X of $\mathcal{S}(n-1, p-1)$, we define a subset $X(Q)$ of Q as follows. Take a jet $j^kf(x_0)\in Q$ and let $f\colon R^n\to R^p$ be a smooth representative of $j^kf(x_0)$. Since $j^kf(x_0)\in Q$, there is a neighbourhood U of x_0 such that $U\cap f^{-1}(S_\delta^{p-1})$ is a smooth hypersurface of U for every $\delta>0$, where S_δ^{p-1} is the $(p-1)$-sphere centered at the origin of R^p and with radius δ. Consider the restricted mapping $f\colon U\cap f^{-1}(S_\delta^{p-1})\to S_\delta^{p-1}$. Now we define that $j^kf(x_0)\in X(Q)$ if and only if $j^k(f\,|\,U\cap f^{-1}(S_\delta^{p-1}))(x_0)$ is contained in $X(U\cap f^{-1}(S_\delta^{p-1}), S_\delta^{p-1})$, where $X(U\cap f^{-1}(S_\delta^{p-1}), S_\delta^{p-1})$ is the subset of $J^k(U\cap f^{-1}(S_\delta^{p-1}), S_\delta^{p-1})$ corresponding to $X\subset J^k(n-1, p-1)$, which was defined in § 2.

PROPOSITION 3.1. (1) *For each straum X of $\mathscr{S}(n-1, p-1)$, $X(Q)$ is a semi-algebraic submanifold of Q.*

(2) *$\mathscr{S}(Q)=\{X(Q) \mid X \in \mathscr{S}(n-1, p-1)\}$ is a Whitney stratification of Q.*

Before prove this we state its corollary whose proof will be given after the proof of the proposition.

COROLLARY 3.2. *Let $f: V \to R^p$ be a smooth mapping of an open subset V of R^n into R^p. Suppose that*

(a) *$j^k f(V-f^{-1}(0)) \subset Q$,*

(b) *for a positive number δ and for any integer m with $m \leqq p+1$, ${}_m j^k f$; $(V-f^{-1}(0)) \to {}_m J^k(R^n, R^p-\{0\})$ is transversal to the submanifolds of ${}_m J^k(R^n, R^p-\{0\})$ of the form*

$$\Delta_m \cap (X_1(Q) \times \cdots \times X_m(Q)) \cap \mu_1^{-1}(\delta^2)\ ,$$

where $X_i(Q) \in \mathscr{S}(Q)$, $\Delta_m = \{(j^k g_1(x_1), \cdots, j^k g_m(x_m) \in {}_m J^k(R^n, R^p) \mid g_1(x_1) = g_2(x_2) = \cdots = g_m(x_m)\}$ and $\mu_1(j^k g_1(x_1), \cdots, j^k g_m(x_m)) = \|g_1(x_1)\|^2$.

Then the following properties hold.

(1) *$f^{-1}(S_\delta^{p-1})$ is a smooth hypersurface of V.*

(2) *The restricted mapping $f: f^{-1}(S_\delta^{p-1}) \to S_\delta^{p-1}$ is multi-transversal to the canonical stratification $\mathscr{S}(f^{-1}(S_\delta^{p-1}), S_\delta^{p-1})$ of $J^k(f^{-1}(S_\delta^{p-1}), S_\delta^{p-1})$ corresponding to $\mathscr{S}(n-1, p-1)$.*

(3) *If $f: f^{-1}(S_\delta^{p-1}) \to S_\delta^{p-1}$ is proper, then it is topologically stable.*

PROOF OF PROPOSITION 3.1. We prove the proposition by showing that Q is covered by a finite number of semi-algebraic open subsets $Q_1, \cdots, Q_l$ such that for each i, $i=1, \cdots, l$, there is a rational submersion $g_i: Q_i \to J^k(R^{n-1}, R^{p-1})$ such that for each stratum X of $\mathscr{S}(n-1, p-1)$ we have $Q_i \cap X(Q) = g_i^{-1}(X(R^{n-1}, R^{p-1}))$, where $X(R^{n-1}, R^{p-1})$ is the subset of $J^k(R^{n-1}, R^{p-1})$ corresponding to X. Here we call a mapping $q=(q_1, \cdots, q_m)$ of an open subset of a euclidean space into R^m a rational mapping if each component q_j is a rational function, i.e. $q_j = p_j/r_j$ for some polynomials p_j and r_j.

Now take a jet $j^k f(x_0) \in Q$, then we have $f(x_0) = (f_1(x_0), \cdots, f_p(x_0)) \neq 0$ and grad.$(f_1^2 + \cdots + f_p^2)(x_0) \neq 0$. Then operating linear transformations of R^n and R^p if necessary, we may suppose that $f_1(x_0) \neq 0, \cdots, f_p(x_0) \neq 0$ and $\partial/\partial x_1(f_1^2 + \cdots + f_p^2)(x_0) \neq 0$. Hence to prove the proposition, it is enough to prove that for the set

$$Q_{1,1} = \{j^k f(x) \in Q \mid f_1(x) \neq 0,\ \cdots,\ f_p(x) \neq 0,\ \partial/\partial x_1(f_1^2 + \cdots + f_p^2)(x) \neq 0.\}$$

there is a rational submersion $\tilde{\pi}: Q_{1,1} \to J^k(R^{n-1}, R^{p-1})$ such that for each $X \in \mathcal{S}(n-1, p-1)$ we have

$$Q_{1,1} \cap X(Q) = \tilde{\pi}^{-1}(X(R^{n-1}, R^{p-1})) .$$

We define $\tilde{\pi}$ as follows. Let $j^k f(x^0) \in Q_{1,1}$ and let $\delta = \|f(x^0)\|$. Since $\partial/\partial x_1(f_1^2 + \cdots + f_p^2)(x^0) \neq 0$, from the implicit function theorem, there is a neighbourhood U of x^0 and a smooth function $h(x_2, \cdots, x_n)$ defined in an open subset W of R^{n-1} such that we have

$$U \cap f^{-1}(S_\delta^{p-1}) = \{(h(x_2, \cdots, x_n), x_2, \cdots, x_n) \mid (x_2, \cdots, x_n) \in W\} .$$

Define $\tilde{h}: W \to U$ by $\tilde{h}(x_2, \cdots, x_n) = (h(x_2, \cdots, x_n), x_2, \cdots, x_n)$ and $\pi: R^p \to R^{p-1}$ by $(y_1, \cdots, y_p) = (y_2, \cdots, y_p)$. Then we define $\tilde{\pi}$ by $\tilde{\pi}(j^k f(x^0)) = j^k(\pi \circ f \circ \tilde{h})(x_2^0, \cdots, x_n^0)$.

Now, to prove the proposition it is enough to prove

LEMMA. (1) *$\tilde{\pi}: Q_{1,1} \to J^k(R^{n-1}, R^{p-1})$ is a rational submersion.*

(2) *For each stratum X of $\mathcal{S}(n-1, p-1)$, we have*

$$Q_{1,1} \cap X(Q) = \tilde{\pi}^{-1}(X(R^{n-1}, R^{p-1})) .$$

PROOF OF LEMMA. (1) Let $j^k f(x^0) \in Q_{1,1}$ and let U, W, h and h be those ones constructed just before lemma. Since $(f_1^2 + \cdots + f_p^2) h(x_2, \cdots, x_n) = \delta^2 = \text{constant}$, we have

$$\begin{aligned} 0 &= \partial/\partial x_i(f_1^2 + \cdots + f_p^2) \circ h \\ &= \sum_{k=1}^p 2f_k \circ \tilde{h}((\partial f_k/\partial x_1) \circ \tilde{h} \cdot \partial h/\partial x_i + \partial f_k/\partial x_i \circ h) . \end{aligned}$$

Hence

$$\begin{aligned} \partial h/\partial x_i &= -\sum_{k=1}^p (f_k \circ \tilde{h})((\partial f_k/\partial x_i) \circ \tilde{h}) \Big/ \sum_{k=1}^p (f_k \circ \tilde{h})(\partial f_k/\partial x_1 \circ h) \\ &= -(\partial/\partial x_i(f_1^2 + \cdots + f_p^2)) \circ \tilde{h} / (\partial/\partial x_1(f_1^2 + \cdots + f_p^2)) \circ \tilde{h} . \end{aligned}$$

Therefore $j^k h(x_2, \cdots, x_n)$ is given by a rational function of the variables $j^k f(x_1, x_2, \cdots, x_n)$.

Now for the point $x^0 = (x_1^0, \cdots, x_n^0)$ we set $x^{0\prime} = (x_2^0, \cdots, x_n^0)$. Then for $i, j \geqq 2$, we have

$$\begin{aligned} &\partial^2(f_i \circ \tilde{h})/\partial x_j(x^{0\prime}) = \partial f_i/\partial x_j(h(x^{0\prime})) + (\partial f_i/\partial x_1)(\tilde{h}(x^{0\prime}))(\partial h/\partial x_j(x^{0\prime})) , \\ &\partial^2(f_i \circ h)/\partial x_j x_l(x^{0\prime}) \\ &\quad = \partial^2 f_i/\partial x_j \partial x_l(h(x^{0\prime})) + \partial^2 f_i/\partial x_1^2(h(x^{0\prime}))(\partial h/\partial x_j(x^{0\prime}))\partial h/\partial x_l(x^{0\prime}) \\ &\qquad + \partial^2 f_i/\partial x_l x_1(h(x^{0\prime}))\partial h/\partial x_j(x^{0\prime}) + \partial f_i/\partial x_1(h(x^{0\prime}))\partial^2 h/\partial x_j x_l(x^{0\prime}) \\ &\quad = \partial^2 f_i/\partial x_j x_l(x^0) + \partial^2 f_i/x_1^2(x^0)\partial h/\partial x_j(x^{0\prime})\partial h/\partial x_l(x^{0\prime}) \\ &\qquad + \partial^2 f_i/\partial x_1 \partial x_l(x^0)\partial h/\partial x_j(x^{0\prime}) + \partial f_i/\partial x_1(x^0)\partial^2 h/x_j x_l(x^{0\prime}) . \end{aligned}$$

In general $\partial^m(f_i \circ \tilde{h})/\partial x^{\omega}(x^{0\prime})$ is a polynomial of the variables $(j^m f_i(x^0), j^m h(x^{0\prime}))$ which contains the term $\partial^m f_i/\partial x^{\omega}(x^0)$. Thus $\tilde{\pi}\colon Q_{1,1} \to J^k(R^{n-1}, R^{p-1})$: $j^k f(x^0) \mapsto j^k(\pi \circ f \circ \tilde{h})(x^{0\prime}) = (j^k(f_2 \circ \tilde{h})(x^{0\prime}), \cdots, j^k(f_p \circ \tilde{h})(x^{0\prime}))$, is a rational submersion.

(2) $j^k f(x^0) \in Q_{1,1} \cap X(Q) \Leftrightarrow j^k f(x^0) \in Q_{1,1}$ and $j^k(f \mid U \cap f^{-1}(S_\delta^{p-1}))(x^0) \in X(U \cap f^{-1}(S_\delta^{p-1}), S_\delta^p) \Leftrightarrow j^k f(x^0) \in Q_{1,1}$ and $j^k(\pi \circ f \circ \tilde{h})(x^{0\prime}) \in X(R^{n-1}, R^{n-1}) \Leftrightarrow j^k f(x^0) \in \pi^{-1}(X(R^{n-1}, R^{p-1}))$. Q.E.D. of lemma and hence of Proposition 3.1.

PROOF OF COROLLARY 3.2. Let $f\colon V \to R^p$ be a smooth mapping, V being an open subset of R^n. Suppose that

(a) $j^k f(V - f^{-1}(0)) \subset Q$, and

(b) for any integer m, ${}_m j^k f\colon (V - f^{-1}(0))^{(m)} \to {}_m J^k(R^n, R^p - \{0\})$ is transversal to the submanifolds of the form

$$\Delta_m \cap (X_1(Q) \times \cdots \times X_m(Q)) \cap \mu_1^{-1}(\delta^2)\,, \quad \text{where} \quad X_i(Q) \in \mathscr{S}(Q)\,.$$

Then to prove the corollary we have to prove that

(1) *$f^{-1}(S_\delta^{p-1})$ is a smooth hypersurface of V,*

(2) *$f \mid f^{-1}(S_\delta^{p-1})\colon f^{-1}(S_\delta^{p-1}) \to S_\delta^{p-1}$ is multi-transversal to the canonical stratification $\mathscr{S}(f^{-1}(S_\delta^{p-1}), S_\delta^{p-1})$ of $J^k(f^{-1}(S_\delta^{p-1}), S_\delta^{p-1})$ corresponding to $\mathscr{S}(n-1, p-1)$, and*

(3) *if $f\colon f^{-1}(S_\delta^{p-1}) \to S_\delta^{p-1}$ is proper, then it is topologically stable.*

(3) is trivial from (2) and Theorem 2.1.

Proof of (1). Since $j^k f(V - f^{-1}(0)) \subset Q$, we have $\mathrm{grad.}(f_1^2 + \cdots + f_p^2)(x) \neq 0$ at any point $x \in V - f^{-1}(0)$. Hence $f^{-1}(S_\delta^{p-1}) = (f_1^2 + \cdots + f_p^2)^{-1}(\delta^2)$ is a smooth hypersurface of $V - f^{-1}(0)$.

Proof of (2). Let $f\colon V \to R^p$ be a smooth mapping satisfying (a) and (b). First we show that

(c) *$j^k(f \mid f^{-1}(S_\delta^{p-1}))\colon f^{-1}(S_\delta^{p-1}) \to J^k(f^{-1}(S_\delta^{p-1}), S_\delta^{p-1})$ is transveral to the stratification $\mathscr{S}(f^{-1}(S_\delta^{p-1}), S_\delta^{p-1}))$ of $J^k(f^{-1}(S_\delta^{p-1}), S_\delta^{p-1})$ corresponding to $\mathscr{S}(n-1, p-1)$.*

Take any point x^0 of $f^{-1}(S_\delta^{p-1})$. We will show that $j^k(f \mid f^{-1}(S_\delta^{p-1}))$ is transversal to $\mathscr{S}(f^{-1}(S_\delta^{p-1}), S_\delta^{p-1})$ at $x^0 = (x_1^0, \cdots, x_n^0)$. We may assume that $j^k f(x^0) \in Q_{1,1}$, where $Q_{1,1}$ is the set constructed in the proof of Proposition 3.1, i.e. $Q_{1,1} = \{j^k g(x) \in Q \mid g_1(x) \neq 0, \cdots, g_p(x) \neq 0, \ \partial/\partial x_1(g_1^2 + \cdots + g_p^2)(x) \neq 0\}$. Now consider the following diagram:

$$\begin{array}{ccc} R^{n-1} \supset W & \xrightarrow{\ \tilde{h}\ } & f^{-1}(S_\delta^{p-1}) \\ \Big\downarrow {\scriptstyle \pi \circ f \circ \tilde{h}} & & \Big\downarrow {\scriptstyle f} \\ R^{p-1} & \xleftarrow{\ \pi\ } & S_\delta^{p-1} \end{array} ,$$

,where W, $\tilde{h}$ and π are the ones constructed in the proof of Proposition 3.1. Then we see that

(d) *$j^k(f|f^{-1}(S_\delta^{p-1}))$ is transversal to $\mathscr{S}(f^{-1}(S^{p-1}), S_\delta^{p-1})$ at $x^0=(x_1^0, \cdots, x_n^0)$ if and only if $j^k(\pi\circ f\circ \tilde{h})$ is transversal to $\mathscr{S}(R^{n-1}, R^{p-1})$ at $x^{0\prime}=(x_2^0, \cdots, x_n^0)$.*

Now consider the following diagram.

$$\begin{array}{ccccc} W & \xrightarrow{\tilde{h}} & f^{-1}(S_\delta^{p-1}) & \subset & V=\bigcup_{\delta'} f^{-1}(S_{\delta'}^{p-1}) \\ \Big\downarrow{\scriptstyle j^k(\pi\circ f\circ \tilde{h})} & & \Big\downarrow{\scriptstyle j^k f} & & \Big\downarrow{\scriptstyle j^k f} \\ J^k(R^{n-1}, R^{p-1}) & \xleftarrow{\tilde{\pi}} & Q_{1,1}\cap \mu^{-1}(\delta^2) & \subset & Q_{1,1} \end{array},$$

where $\mu\colon J^k(R^n, R^p)\to R$ is defined by $\mu(j^kg(x))=\|g(x)\|^2$. Since, from (b), j^kf is transversal to $X(Q)\cap\mu^{-1}(\delta^2)$ for every stratum X of $\mathscr{S}(n-1, p-1)$, $j^kf|f^{-1}(S_\delta^{p-1})\colon f^{-1}(S_\delta^{p-1})\to Q_{1,1}\cap\mu^{-1}(\delta^2)$ is transversal to $X(Q)\cap\mu^{-1}(\delta^2)$ in $Q_{1,1}\cap\mu^{-1}(\delta^2)$. Hence noticing that the restriction of $\tilde{\pi}$ to $Q_{1,1}\cap\mu^{-1}(\delta^2)$ is also a submersion into $J^k(R^{n-1}, R^{p-1})$, we see, from the commutativity of the above diagram, that $j^k(\pi\circ f\circ\tilde{h})$ is transversal to $X(R^{n-1}, R^{p-1})$. Therefore from (d), we see that $j^k(f|f^{-1}(S_\delta^{p-1}))$ is transveral to $\mathscr{S}(f^{-1}(S_\delta^{p-1}), S_\delta^{p-1})$. This completes the proof of (c).

Now, since ${}_mj^kf\colon (V-\{0\})^{(m)}\to {}_mJ^k(R^n, R^p-\{0\})$ is transversal to $\Delta_m\cap(X_1(Q)\times\cdots\times X_m(Q))\cap\mu_1^{-1}(\delta^2)$ for any integer $m\leqq p+1$ and any strata $X_1(Q), \cdots, X_m(Q)$ of $\mathscr{S}(Q)$, we see that the images of $(j^kf)^{-1}(X_1(Q))\cap f^{-1}(S_\delta^{p-1})$, and $(j^kf)^{-1}(X_2(Q)), \cdots, (j^kf)^{-1}(X_m(Q))$ under f meet transversally in R^p, which means $f((j^kf)^{-1}(X_1(Q)))\cap S_\delta^{p-1}$ and $f((j^kf)^{-1}(X_2(Q)))\cap S_\delta^{p-1}, \cdots, f((j^kf)^{-1}(X_m(Q))\cap S_\delta^{p-1}$ meet transversally in S_δ^{p-1}. Hence the images $f((j^kf)^{-1}(X_1(Q))\cap f^{-1}(S_\delta^{p-1})), \cdots, f((j^kf)^{-1}(X_m(Q))\cap f^{-1}(S_\delta^{p-1}))$ meet transversally in S_δ^{p-1}.

Therefore, since $(j^kf)^{-1}(X(Q))\cap f^{-1}(S_\delta^{p-1})=(j^k(f|f^{-1}(S_\delta^{p-1}))^{-1}(X(f^{-1}(S_\delta^{p-1}), S_\delta^{p-1})$ for any stratum X of $\mathscr{S}(n-1, p-1)$, we see that

(f) $f((j^k(f|f^{-1}(S_\delta^{p-1}))^{-1}(X_1(f^{-1}(S_\delta^{p-1}), S_\delta^{p-1})), \cdots, f((j^k(f|f^{-1}(S_\delta^{p-1}))^{-1}(X_m\times(f^{-1}(S_\delta^{p-1}), S_\delta^{p-1}))$ *meet transversally.*

Therefore ${}_mj^k(f|f^{-1}(S_\delta^p))$ is transversal to

$$\Delta_m\cap(X_1(f^{-1}(S_\delta^{p-1}), S_\delta^{p-1})\cdots X_m(f^{-1}(S_\delta^{p-1}), S_\delta^{p-1}))\ .$$

Thus $f|f^{-1}(S_\delta^{p-1})\colon f^{-1}(S_\delta^{p-1})\to S_\delta^{p-1}$ is mutitransversal to $\mathscr{S}(f^{-1}(S_\delta^{p-1}), S_\delta^{p-1})$.

Q.E.D. of Corollary 3.2.

§4. Proof of Theorem 1.

First we state Theorem 1 in a slightly different form. For positive integers s and r with $s>r$, let $\pi_r^s\colon J^s(n, p)\to J^r(n, p)$ denote the canonical projection defined by $\pi_r^s(j^s f(0))=j^r f(0)$.

THEOREM 1′. *Suppose $n>p$. Then for any semi-algebraic subset W of $J^r(n, p)$, there exists an integer s, greater than r and depending only on r, n and p, and there exists a closed semi-algebraic subset Σ_W of $(\pi_r^s)^{-1}(W)$ with* $\dim.\Sigma_W<\dim.(\pi_r^s)^{-1}(W)$ *such that every smooth mapping $f\colon R^n\to R^p$ with $j^s f(0)\in(\pi_r^s)^{-1}(W)-\Sigma_W$ satisfies either the following* I) (i)–(iv) *or* II) (v)–(vi).

Case I) *If the origin 0 is not an isolated point of $f^{-1}(0)$*, i.e. $0\in\overline{f^{-1}(0)-\{0\}}$, *then there exist a positive number ε_0 and a strictly increasing smooth function $\delta\colon[0, \varepsilon_0]\to[0, \infty)$ with $\delta(0)=0$ such that for every ε and δ with $0<\varepsilon\leqq\varepsilon_0$ and $0<\delta<\delta(\varepsilon)$ the following properties* (i)–(iv) *hold*

(i) *$f^{-1}(0)\cap S_\varepsilon^{n-1}$ is an $(n-p-1)$-dimensional manifold and it is diffeomorphic to $f^{-1}(0)\cap S_{\varepsilon_0}^{n-1}$.*

(ii) *$D_\varepsilon^n\cap f^{-1}(S_\delta^{p-1})$ is a smooth manifold with boundary and it is diffeomorphic to $D_{\varepsilon_0}^n\cap f^{-1}(S_{\delta(\varepsilon_0)}^{p-1})$.*

(iii) *$\partial(D_\varepsilon^n\cap f^{-1}(D_\delta^{p-1}))$ is homeomorphic to S_ε^{n-1}.*

(iv) *The restricted mapping $f\colon D_\varepsilon^n\cap f^{-1}(S_\delta^{p-1})\to S_\delta^{p-1}$ is topologically stable (C^∞ stable if (n, p) is a nice pair) and its topological type is independent of ε and δ.*

Case II) *If 0 is an isolated point of $f^{-1}(0)$*, i.e. $0\notin\overline{f^{-1}(0)-\{0\}}$, *then there exists a positive number ε_0 such that for every ε with $0<\varepsilon\leqq\varepsilon_0$ the following properties* (v) *and* (vi) *hold.*

(v) *$f^{-1}(S_\varepsilon^{p-1})$ is diffeomorphic to S^{n-1}.*

(vi) *The restricted mapping $f\colon f^{-1}(S_\varepsilon^{p-1})\to S_\varepsilon^{p-1}$ is topologically stable (C^∞ stable if (n, p) is a nice pair) and its topological type is independent of ε.*

This theorem implies the following corollary and hence Theorem 1 stated in the introduction.

COROLLARY. *For any positive integer r, there exists a closed semi-algebraic subset Σ_r of $J^r(n, p)$ such that* $\text{codim}.\Sigma_r\to\infty$ *as $r\to\infty$ and such that every smooth mapping $f\colon R^n\to R^p$ with $j^r f(0)\in(J^r(n, p)-\Sigma_r)$ satisfies either* I) (i)–(iv) *or* II) (v)–(vi) *above.*

PROOF OF COROLLARY. Set $W_1=J^1(n, p)$. Then from Theorem 1′, there exist an integer s_1 and a closed semi-algebraic subset Σ_{W_1} of $(\pi_1^{s_1})^{-1}(W_1)=J^{s_1}(n, p)$ satisfying the conditions in Theorem 1′. Now set $W_2=\Sigma_{W_1}$. Then again from Theorem 1′, there exist an integer s_2 and a closed semi-algebraic subset Σ_{W_2} of $(\pi_{s_1}^{s_2})^{-1}(W_2)$ satisfying the conditions in

Theorem 1′. Thus we obtain inductively increasing integers s_i and closed semialgebraic subsets Σ_{W_i} in $J^{s_i}(n, p)$. Set $\Sigma_r = \cap_{s_i \leq r}(\pi^r_{s_i})^{-1}(\Sigma_{W_i})$. Then Σ_r is the desired one.

Construction of Σ_W.

Let (n, p) be a pair of positive integers with $n > p$ and let W be a semi-algebraic subset of $J^k(n, p)$. Let $k = k(n-1, p-1)$, $\Sigma = \Sigma(n-1, p-1)$ and $\mathscr{S}(n-1, p-1)$ be the integer, the closed semi-algebraic subset of $J^k(n-1, p-1)$ and the canonical stratification given in J. Mather's theorem stated in § 2 respectively. Let Q and $\mathscr{S}(Q)$ be the semi-algebraic open subset of $J^k(R^n, R^n - \{0\})$ and its stratification constructed in § 3. Let $V = \{j^k f(x) \in J^k(R^n, R^p) \mid f(x) = 0\}$. Then from Theorem 3 stated in § 1, we have

LEMMA 4.1. *There exists a closed semi-algebraic subset Σ_W of $(\pi^s_r)^{-1}(W)$, where $s = r + (p+1)(k+1)$, with* dim. $\Sigma_W <$ dim.$(\pi^s_r)^{-1}(W)$ *such that for any smooth mapping $f: R^n \to R^p$ with $j^s f(0) \in (\pi^s_r)^{-1}(W) - \Sigma_W$, there exists a neighbourhood U of the origin of R^n satisfying the following conditions* (1)-(4).

(1) $j^k f(U - f^{-1}(0)) \subset Q$. *(Note that Q is semi-algebraic and* codim.$(J^k(R^n, R^p - \{0\}) - Q) = n$.).

(2) *$j^k(f \mid U - \{0\})$ is transversal to $V \cap \mu^{-1}_{1,R^n}(\varepsilon)$ for every $\varepsilon > 0$, where $\mu_{1,R^n}: J^k(R^n, R^p) \to R$ is defined by $\mu_{1,R^n}(j^k f(x)) = \|x\|^2$.*

(3) *For any positive number δ and for any positive integer m with $m \leqq p+1$, ${}_m j^k f: (U - \{0\})^{(m)} \to {}_m J^k(R^n, R^p - \{0\})$ is transversal to the submanifolds of ${}_m J^k(R^n, R^p - \{0\}$ of the form*

$$\Delta_m \cap (X_1(Q) \times \cdots \times X_m(Q)) \cap \mu^{-1}_{1,R^p}(\delta)\ ,$$

where $X_1(Q), \cdots, X_m(Q) \in \mathscr{S}(Q)$ and $\mu_{1,R^p}: {}_m J^k(R^n, R^p) \to R$ is defined by $\mu_{1,R^p}(j^k f_1(x_1), \cdots, j^k f_m(x_m)) = \|f_1(x_1)\|^2$.

(4) *For any stratum X of $\mathscr{S}(Q)$ and for any positive number ε, $j^k f: U - f^{-1}(0) \to J^k(R^n, R^p - \{0\})$ is transversal to $X \cap \mu^{-1}_{1,R^n}(\varepsilon)$, where $\mu_{1,R^n}: J^k(R^n, R^p) \to R$ is defined by $\mu_{1,R^n}(j^k f(x)) = \|x\|^2$.*

PROOF OF CASE I).

PROOF OF (i). Let ε_0 be a so small number that $S^{n-1}_{\varepsilon_0} \subset U$. Let $\mu_{R^n}: R^n \to R$ be the canonical metric function on R^n defined by $\mu_{R^n}(x_1, \cdots, x_n) = x_1^2 + \cdots + x_n^2$. Let f be a smooth mapping with $j^s f(0) \in (\pi^s_r)^{-1}(W) - \Sigma_W$. We define a mapping $f \times \mu_{R^n}: R^n \to R^p \times R$ by $(f \times \mu_{R^n})(x) = (f(x), \mu_{R^n}(x))$. Then from (2) in Lemma 4.1, we see that

(5) *$f \times \mu_{R^n}$ has no singular points on $f^{-1}(0) \cap (U - \{0\})$.* Hence for any positive number ε with $\varepsilon < \varepsilon_0$ we see that

(6) *$f^{-1}(0)\cap S_\varepsilon^{n-1}$ and $f^{-1}(0)\cap S_{\varepsilon_0}^{n-1}$ are diffeomorphic,* where the one parameter group of diffeomorphisms generated by the gradient vector field of the function $\mu_{R^n}\colon f^{-1}(0)\cap(U-\{0\})\to R$ gives a diffeomorphism between $f^{-1}(0)\cap S_\varepsilon^{n-1}$ and $f^{-1}(0)\cap S_{\varepsilon_0}^{n-1}$. This proves (i).

PROOF OF (ii). From (5) we see that

(7) *there exists a tubular neighbourhood N of $f^{-1}(0)\cap(U-\{0\})$ in $U-\{0\}$ such that the restricted mapping $f\times\mu_{R^n}\colon N\to R^p\times R$ is a submersion.* Hence and since ε_0 is so small that $D_{\varepsilon_0}^n\subset U$,

(8) *There is a strictly increasing smooth function $\delta\colon[0,\varepsilon_0]\to[0,\infty)$ with $\delta(0)=0$ such that for every ε and δ with $0<\varepsilon\leqq\varepsilon_0$ and $0<\delta<2\delta(\varepsilon)$ we have $S_\varepsilon^{n-1}\cap f^{-1}(D_\delta^{p-1})\subset N$ and hence S_ε^{n-1} and $f^{-1}(S_\delta^{p-1})$ meet transversally in R^n.*

On the other hand, from (3) in Lemma 4.1., we see that

(9) *$\mu_{R^p}\circ f\colon U-f^{-1}(0)\to R$ has no critical points,* where $\mu_{R^p}\colon R^p\to R$ is defined by $\mu_{R^p}(y)=\|y\|^2$. And from (7), we see that

(10) *$\mu_{R^n}\times(\mu_{R^p}\circ f)\colon N-f^{-1}(0)\to R\times R$ has no singular points.*

From (8), (9) and (10) we see that if $0<\varepsilon\leqq\varepsilon_0$ and $0<\delta<\delta(\varepsilon)$, then $D_\varepsilon^n\cap f^{-1}(S_\delta^{p-1})=\mu_{R^n}^{-1}([0,\varepsilon^2])\cap\mu_{R^p}f)^{-1}(\delta^2)$ is a differentiable manifold with boundary and it is diffeomorphic to $D_{\varepsilon_0}^n\cap f^{-1}(S_{\delta(\varepsilon_0)}^{p-1})=\mu_{R^n}^{-1}([0,\varepsilon_0^2])\cap(\mu_{R^p}\circ f)^{-1}(\delta(\varepsilon_0)^2)$. This completes the proof of (ii).

PROOF OF (iii). Consider the gradient vector field of $(\mu_{R^p}\circ f)$. Define a map $h\colon\partial(D_\varepsilon^n\cap f^{-1}(D_\delta^p))\to S_\varepsilon^{n-1}$ as follows: For a point x of $\partial(D_\varepsilon^n\cap f^{-1}(D_\delta^p))$, let $h(x)$ be the point where the integral curve of grad.$(\mu_{R^p}\circ f)$ passing through x meets S_ε^{n-1}. Then h is a homeomorphism between $\partial(D_\varepsilon^n\cap f^{-1}(D_\delta^p)$ and S_ε^{n-1}. This proves (iii).

PROOF OF (iv). Let $0<\varepsilon<\varepsilon_0$ and $0<\delta<\delta(\varepsilon)$. Then from (1) and (3) and from Corollary 3.2, we see that the restricted mapping $f\colon f^{-1}(S_\delta^{p-1})\cap U\to S_\delta^{p-1}$ is multi-transversal to the canonical stratification $\mathscr{S}(f^{-1}(S_\delta^{p-1})\cap U, S_\delta^{p-1})$ of $J^k(f^{-1}(S_\delta^{p-1})\cap U, S_\delta^{p-1})$. Hence $f\colon D_\varepsilon^n\cap f^{-1}(S_\delta^{p-1})\to S_\delta^{p-1}$ is multi-transversal to the canonical stratification $\mathscr{S}(D_\varepsilon^n\cap f^{-1}(S_\delta^{p-1}), S_\delta^{p-1})$ of $J^k(f^{-1}(S_\delta^{p-1})\cap D_\varepsilon^n, S_\delta^{p-1})$. On the other hand, from (6) we see that $f\colon\partial(D_\varepsilon^n\cap f^{-1}(S_\delta^{p-1}))\to S_\delta^{p-1}$ is a submersion. Therefore from Corollary 2.2, the restricted mapping $f\colon D_\varepsilon^n\cap f^{-1}(S_\delta^{p-1})\to S_\delta^{p-1}$ is topologically stable, and moreover it is C^∞ stable if (n, p) is a nice pair.

Now let's prove that for any two pairs (ε_i,δ_i), $i=1, 2$, with $0<\varepsilon_i<\varepsilon_0$ and $0<\delta_i<\delta(\varepsilon_i)$, $f\colon D_{\varepsilon_1}^n\cap f^{-1}(S_{\delta_1}^{p-1})\to S_{\delta_1}^{p-1}$ and $f\colon D_{\varepsilon_2}^n\cap f^{-1}(S_{\delta_2}^{p-1})\to S_{\delta_2}^{p-1}$ are topologically equivalent. (C^∞ equivalent if (n, p) is a nice pair.). It is enough to prove it for the case where ε_1 and ε_2 are sufficiently close to each other and so are δ_1 and δ_2. In this case, let $h_2: S_{\delta_1}^{p-1} \to S_{\delta_2}^{p-1}$ be the diffeomorphism defined by $h_2(y) = (\delta_2/\delta_1)y$. From (ii), there exists a diffeomorphism $h_1: D_{\varepsilon_1}^n \cap f^{-1}(S_{\delta_1}^{p-1}) \to D_{\varepsilon_2}^n \cap f^{-1}(S_{\delta_2}^{p-1})$. Then from the proof of (ii), we see that we may choose h_1 so that $h_2^{-1} \circ f \circ h_1: D_{\varepsilon_1}^n \cap f^{-1}(S_{\delta_1}^{p-1}) \to S_{\delta_1}^{p-1}$ is sufficiently close to $f: D_{\varepsilon_1}^n \cap f^{-1}(S_{\delta_1}^{p-1}) \to S_{\delta_1}^{p-1}$ in the Whitney topology. Since $f: D_{\varepsilon_1}^n \cap f^{-1}(S_{\delta_1}^{p-1}) \to S_{\delta_1}^{p-1}$ is topologically stable (C^∞ stable if (n, p) is a nice pair), $h_2^{-1} \circ f \circ h_1$ is topologically equivalent to $f: D_{\varepsilon_1}^n \cap f^{-1}(S_{\delta_1}^{p-1}) \to S_{\delta_1}^{p-1}$. Therefore $f: D_{\varepsilon_2}^n \cap f^{-1}(S_{\delta_2}^{p-1}) \to S_{\delta}^{p-1}$ is topologically equivalent (C^∞ equivalent if (n, p) is a nice pair) to $f: D_{\varepsilon_1}^n \cap f^{-1}(S_{\delta_1}^{p-1}) \to S_{\delta_1}^{p-1}$. Q.E.D. of Case I.

THE PROOF OF CASE II. The proof of (v) can be found in [2]. (vi) can be proved in the same way as (iv). Q.E.D. of the proof of Theorem 1.

§5. Proof of the Poincare-Hopf equality (Theorem 2).

Let W be a semi-algebraic subset of $J^r(n, p)$. Let Σ_W be the corresponding closed semi-algebraic subset of $(\pi_r^s)^{-1}(W)$ constructed in the proof of Theorem 1′. We will prove that if a C^∞ mapping $f: R^n \to R^2$ represents an element of $(\pi_r^s)^{-1}(W) - \Sigma_W$, then f has properties (i), (ii) and (iii) in Theorem 2. By an argument similar to the proof of corollary in §4, we see that this implies Theorem 2.

Now let $f: R^n \to R^2$ be a C^∞ mapping with $j^s f(0) \in (\pi_r^s)(W) - \Sigma_W$. Let U be a neighbourhood of the origin of R^n satisfying conditions (1)–(4) in Lemma 4.1.

5.1. Proof of (i).

Let ε_0 and $\delta: [0, \delta_0] \to [0, \infty)$ be the positive number and the strictly increasing function respectively given in (8) in the proof of Theorem 1 in 4. In the case where 0 is an isolated point of $f^{-1}(0)$, we may choose the function δ so small that $f^{-1}(S_{\delta(\varepsilon)}^1)$ is contained in D^n. Let ε and δ be any positive numbers with $0 < \varepsilon < \varepsilon_0$ and $0 < \delta < \delta(\varepsilon)$. Then we see that

(a) *if* $0 \in \overline{f^{-1}(0) - \{0\}}$, *then* $f^{-1}(S_\delta^1) \cap D_\varepsilon^n$ *is a* C^∞ *manifold with boundary, and* $\partial(D_\varepsilon^n \cap f^{-1}(D_\delta^2))$ *is homeomorphic to* S_ε^{n-1}, *and if* $0 \notin \overline{f^{-1}(0) - \{0\}}$, *then* $f^{-1}(S_\delta^1) \cap D_\varepsilon^n = f^{-1}(S_\delta^1)$ *is diffeomorphic to* S_ε^{n-1},

(b) *the restricted mapping* $f: D_\varepsilon^n \cap f^{-1}(S_\delta^1) \to S_\delta^1$ *is a* C^∞ *stable mapping and* $S_\varepsilon^{n-1} \cap f^{-1}(D_\delta^2)$ *contains no singular points of* f.

From (4) in Lemma 4.1, we see that

(c) $f: U - \{0\} \to R^2$ *has only* C^∞ *stable singularities of codimension less than* n *which, in this case where the dimension of the target space is* 2, *are "fold type" singularities*; a point p of R^n is a fold singularity of $f: R^n \to R^2$ if there exist coordinate systems $(\xi_1, \cdots, \xi_n)$ around p and

(η_1, η_2) around $f(p)$ such that $\eta_1 \circ f = \xi_1$ and $\eta_2 \circ f = \xi_2^2 \pm \xi_3^2 \pm \cdots \pm \xi_n^2$.

Since fold singularities are of codimension $n-1$, letting $S(f)$ be the set of singular points of f, we see that

(d) *$S(f) \cap U = \{0\}$ or $S(f) \cap U$ is the union of a finite number of smooth curves, say $s_i(t)$, $0 \leqq t < 1$ and $s_i(0) = 0$, $i = 1, \cdots, k$, which meet S_ε^{n-1} and $f^{-1}(S_\delta^1)$ transversally* (see (2) and (4) in Lemma 4.1).

Now parametrize S_δ^1 by angle θ, $S_\delta^1 = \{\delta e^{i\theta}\}$, as we did so in the introduction. Since $f: D_\varepsilon^n \cap f^{-1}(S_\delta^1) \to S_\delta^1$ is C^∞ stable, we may regard the composed mapping $\theta \circ f: D_\varepsilon^n \cap f^{-1}(S_\delta^1) \to R$ mod. 2π as a Morse function, though $\theta \circ f$ is not a function in the strict sense that its values should be in R. Then we see that

(e) *if $p \in S(f) \cap D_\varepsilon^n f^{-1}(S_\delta^1)$ and $q \in S(f) \cap D_{\varepsilon'}^n \cap f^{-1}(S_{\delta'}^1)$ are in the same curve $s_i(t)$ given in* (d), *then the index of the critical point p of $\theta \circ (f \mid D_\varepsilon^n \cap f^{-1}(S_\delta^1))$ and the index of the critical point q of $\theta \circ (f \mid D_{\varepsilon'}^n \cap f^{-1}(S_{\delta'}^1))$ are the same.*

Proof of (e). Since $\theta \circ (f \mid D_\varepsilon^n \cap f^{-1}(S_\delta^1))$ is a Morse function for every ε and δ, it does not bifurcate as ε and δ vary. This proves (e).

Now from (i) of Theorem 1′, we see that

(f) *the Betti numbers $b_i(S_\varepsilon^{n-1} \cap f^{-1}(0))$ are independent of ε provided that $0 < \varepsilon \leqq \varepsilon_0$.*

This completes the proof of (i).

5.2. Proof of Remark stated below Theorem 2 in introduction.

Let f and ε and δ be as in 5.1. Then

(g) *the set of critical points of $\theta \circ (f \mid D_\varepsilon^n \cap f^{-1}(S_\delta^1))$ is equal to $S(f) \cap (D_\varepsilon^n \cap f^{-1}(S_\delta^1))$.*

Proof of (*g*). If $p \in f^{-1}(S_\delta^1)$ is not a singular point of $f: R^n \to R^2$, then p is not a singular point of $f: f^{-1}(S_\delta^1) \to S_\delta^1$. Therefore it is not a critical point of $\theta \circ f: f^{-1}(S_\delta^1) \to R$ mod. 2π.

On the other hand, since $\mu_{R^2} \circ f$ has no critical points in $U - f^{-1}(0)$, if a point p is not a critical point of $\theta \circ f: f^{-1}(S_\delta^1) \to R$ mod. 2π, then p is not a singular point of f; precisely, from (9) in § 4, $\mu_{R^2} \circ f: U - f^{-1}(0) \to R$ has no critical points, hence there exists a coordinate system $(\xi_1, \cdots, \xi_n)$ around p with $\xi_1 = \mu_{R^2} \circ f$. On the other hand, there exists a coordinate system (η_1, η_2) around $f(p)$ with $\eta_1 = \mu_{R^2}$. Since p is not a critical point of $\theta \circ f: f^{-1}(S_\delta^1) \to R$ mod. 2π and since $\eta_1 = \mu_{R^2}$, we see that p is not a critical point of $\eta_2 \circ (f \mid f^{-1}(S_\delta^1))$. From this and from the fact that $\eta_1 \circ f = \xi_1$, we see that p is not a singular point of f. Q.E.D. of (*g*).

Let $p_i \in S(f) \cap D_{\varepsilon_i}^n \cap f^{-1}(S_{\delta_i}^1)$, $i = 1, 2$, with $0 < \delta_i < 2\delta(\varepsilon_i)$. Then

(h) *the indices of the critical points* p_i *of* $\theta\circ f\colon D^n_{\varepsilon_i}\cap f^{-1}(S^1_{\delta_i})\to R$ mod. 2π *are equal to each other if and only if the singular points* p_1 *and* p_2 *are* C^∞ *equivalent under target-orientation-preserving diffeomorphisms,* i.e., there exist diffeomorphic germs $h_1\colon (R^n, p_1)\to(R^n, p_2)$ and $h_2\colon (R^2, f(p_1))\to(R^2, f(p_2))$, h_2 preserving the orientation of R^2, such that $f\circ h_1=h_2\circ f$.

PROOF OF (h). Let the pair (μ_{R^2}, θ) be the so-called polar coordinate system on $R^2-\{0\}$. Let λ_i be the indices of the critical points p_i. Then from the Morse lemma there exists a local coordinate system $(\bar\xi_2, \cdots, \bar\xi_n)$ around p_1 in $f^{-1}(S^1_{\delta_1})$ such that

$$\theta\circ f\,|\,D^n_{\varepsilon_1}\cap f^{-1}(S^1_{\delta_1})=\theta\circ f(p_1)-\bar\xi_2^2-\cdots-\bar\xi^2_{\lambda_1+1}+\bar\xi^2_{\lambda_1+2}+\cdots+\bar\xi^2_n\,.$$

From the Morse lemma for functions with parameters, we see that $\bar\xi_2, \cdots, \bar\xi_n$ can be extended to functions $\xi_2, \cdots, \xi_n$ defined in a neighbourhood of p_1 in D^n such that

$$\theta\circ f=\theta(f(p_1))-\xi_2^2-\cdots-\xi^2_{\lambda_1+1}+\xi^2_{\lambda_1+2}+\cdots+\xi^2_n\,.$$

Let $\xi_1=\mu_{R^2}\circ f-\delta_1$. Then $(\xi_1, \cdots, \xi_n)$ is a local coordinate system around p_1 under which f is of the form

$$(*)\qquad \begin{aligned}&\mu_{R^2}\circ f=\xi_1+\delta_1\\ &\theta\circ f=-\xi_2^2-\cdots-\xi^2_{\lambda_1+1}+\xi^2_{\lambda_1+2}+\cdots+\xi^2_n+\theta(f(p_1))\,.\end{aligned}$$

For p_2 also, with the same argument, there exists a local coordinate system $(\xi'_1, \cdots, \xi'_n)$ around p_2 such that

$$(**)\qquad \begin{aligned}&\mu_{R^2}f=\xi'_1+\delta_2\\ &\theta\circ f=-\xi'^2_2-\cdots-\xi'^2_{\lambda_2+1}+\xi'^2_{\lambda_2+2}+\cdots+\xi'^2_n+\theta(f(p_2))\,.\end{aligned}$$

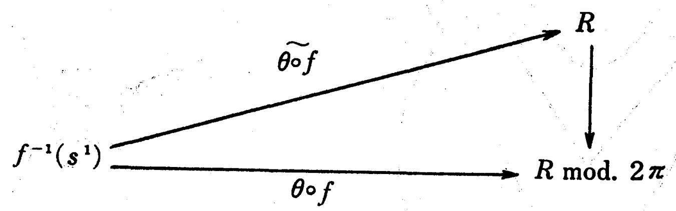
Now it is clear that $\lambda_1=\lambda_2$ if and only if the singularities (*) and (**) are C^∞ equivalent under target-orientation-preserving diffeomorphisms.

Q.E.D. of (h).

(g) and (h) complete the proof of Remark stated below Theorem 2 in the introduction.

5.3. Proof of the Morse inequalities (iii).

Before we prove the Euler-Poincare equality (ii), we prove (iii) whose proof is much shorter. Let $f\colon R^n\to R^2$ be the mapping under consideration. Suppose that 0 is an isolated point of $f^{-1}(0)$. Suppose also that $n\geqq 3$. Let δ be a sufficiently small positive number. Then from (a) and (b) in 5.1, we see that $f^{-1}(S^1_\delta)$ is diffeomorphic to S^{n-1} and the restricted function $\theta \circ f\colon f^{-1}(S^1_\delta) \to R \bmod. 2\pi$ is a Morse function. Since $f^{-1}(S^1_\delta)$ is homeomorphic to S^{n-1} and $n \geqq 3$, we see that the fundamental group of $f^{-1}(S^1_\delta)$ is trivial. Hence we can lift $\theta \circ f\colon f^{-1}(S^1_\delta) \to R \bmod. 2\pi$ to a function $\tilde{\theta} \circ f\colon f^{-1}(S^1_\delta) \to R$



where we regard R as the universal covering space of $R \bmod. 2\pi$. Then $\widetilde{\theta \circ f}$ is a Morse function in the usual sense. A point p of $f^{-1}(S^1_\delta)$ is a critical point of $\widetilde{\theta \circ f}$ with index i if and only if it is a critical point of $\theta \circ (f \mid f^{-1}(S^1_\delta))$ with index i. Therefore the number of critical points of $\widetilde{\theta \circ f}$ with index i, which we denote by $\mu_i(\theta \circ f)$, is equal to the number of critical points of $\theta \circ (f \mid f^{-1}(S^1_\delta))$ with index i, which we denote by $m_i(f)$.

Now from the ordinary Morse inequality for the function $\widetilde{\theta \circ f}$, we have

$$m_0(f) = \mu_0(\widetilde{\theta \circ f}) \geqq b_0(f^{-1}(S^1_\delta)) = b_0(S^{n-1})$$

$$m_1(f) - m_0(f) = \mu_1(\widetilde{\theta \circ f}) - \mu_0(\widetilde{\theta \circ f}) \geqq b_1(f^{-1}(S^1_\delta)) - b_0(f^{-1}(S^1_\delta)) = b_1(S^{n-1}) - b_0(S^{n-1})$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum (-1)^i m_i(f) = \sum (1)^i \mu_i(\widetilde{\theta \circ f}) = \chi(f^{-1}(S^1_\delta)) = \chi(S^{n-1})\ .$$

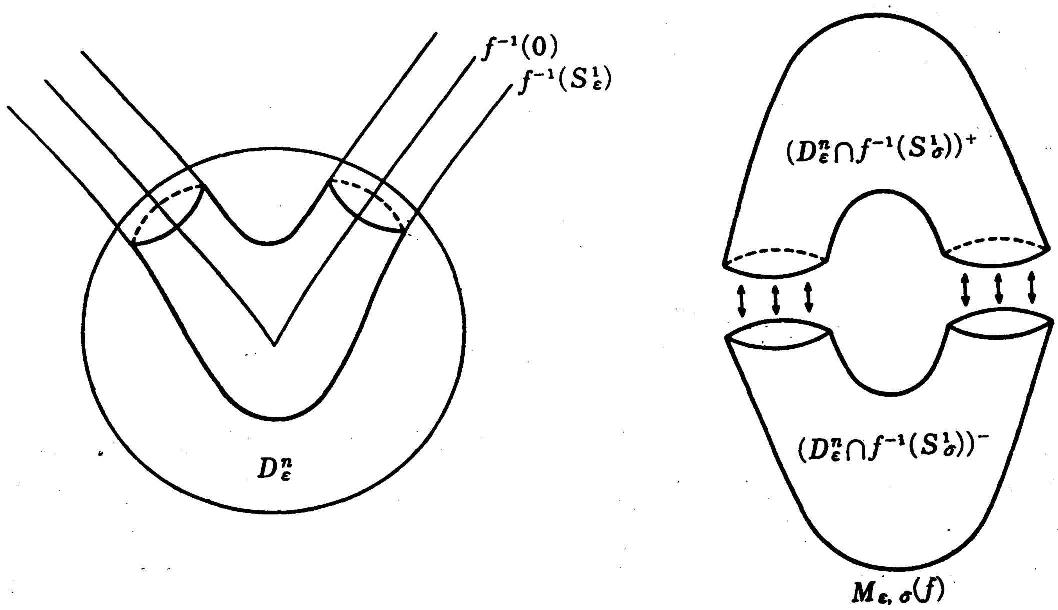
This completes the proof of (iii).

5.4. Proof of the Poincare-Hopf equality (ii).

Let $M_{\varepsilon,\delta}(f)$ be the manifold obtained from two copies of $D^n_\varepsilon \cap f^{-1}(S^1_\delta)$ by identifying their boundaries by the identity mapping of the boundary $S^{n-1}_\varepsilon \cap f^{-1}(S^1_\delta)$ (See the figure below): Namely let $(D^n_\varepsilon \cap f^{-1}(S^1_\delta))^+$ and $(D^n_\varepsilon \cap f^{-1}(S^1_\delta))^-$ be the two copies of $D^n_\varepsilon \cap f^{-1}(S^1_\delta)$ and for a point x of $D^n_\varepsilon \cap f^{-1}(S^1_\delta)$ let x^+ and x^- denote the corresponding points of $(D^n_\varepsilon \cap f^{-1}(S^1_\delta))^+$ and $(D^n_\varepsilon \cap f^{-1}(S^1_\delta))^-$ respectively. Then $M_{\varepsilon,\delta}(f)$ is defined as the quotient space

$$M_{\varepsilon,\delta}(f) = (D^n_\varepsilon \cap f^{-1}(S^1_\delta))^+ \cup (D^n_\varepsilon \cap f^{-1}(S^1_\delta))^- / \sim$$

where $x^+ \sim y^-$ if and only if $x = y$ and $x = y \in S^n_\varepsilon \cap f^{-1}(S^1_\delta)$. Then $M_{\varepsilon,\delta}(f)$ has a unique smooth structure compatible with those of $D^n_\varepsilon \cap f^{-1}(S^1_\delta))^+$ and $(D^n_\varepsilon \cap f^{-1}(S^1_\delta))^-$.



FIGURE

Now we define a function $F: M_{\varepsilon,\delta}(f) \to S^1$ by

$$F(x^+) = f(x) \quad \text{and} \quad F(x^-) = f(x)\ .$$

Then the composed function $\theta \circ F: M_{\varepsilon,\delta}(f) \to (R \bmod. 2\pi)$ is a Morse function in the sense that it has no degenerate critical points. And it is obvious that

(j) *x^+ (resp. x^-) is a critical point of $\theta \circ F$ if and only if the corresponding point x of $D^n_\varepsilon \cap f^{-1}(S^1_\delta)$ is a critical point of $\theta \circ (f \mid D^n_\varepsilon f^{-1}(S^1_\delta))$ and the index of x^+ (resp. x^-) is equal to the index of x.*

Hence we have

(k) $\mu_i(\theta \circ F) = 2\mu_i(\theta \circ f \mid D^n_\varepsilon \cap f^{-1}(S^1_\delta))$ $(=2m_i(f))$,

where $\mu_i(g)$ denotes the number of critical points of a function g with index i.

Now consider the gradient vector field of $\theta \circ F$ with respect to any Riemannian metric of $M_{\varepsilon,\delta}(f)$. Then

(l) $p \in M_{\varepsilon,\delta}(f)$ is a singular point of the gradient vector field grad. $\theta \circ F$ if and only if p is a critical point of $\theta \circ F$. Moreover p is a critical point of $\theta \circ F$ with index i, then the index of p as a singular point of grad. $\theta \circ F$ is $(-1)^i$.

LEMMA 5.1. (Poincare-Hopf, see [6]). *Let ξ be a smooth vector field on a closed manifold M whose singular points are isolated. Then the sum of the indices of the singular points of ξ is equal to the Euler characteristic number $\chi(M)$ of M.*

From Lemma 5.1 and (l) we have

(m) $\chi(M_{\epsilon,\delta}(f))=\sum(-1)^i\mu_i(\theta\circ F)=\sum(-1)^i 2m_i(f)$.

Hence to prove (ii) it remains to prove

LEMMA 5.2. (n) $\chi(M_{\epsilon,\delta}(f))=-2\chi(S_\epsilon^{n-1}\cap f^{-1}(0))+2\chi(S^{n-1})$.

PROOF OF LEMMA 5.2. First, from the property (iii) given in Theorem 1′, note that

(o) $(D_\epsilon^n\cap f^{-1}(S_\delta^1))\cup(S_\epsilon^{n-1}\cap f^{-1}(D_\delta^2))$ *is homeomorphic to* S^{n-1}.

(p) *Therefore* $D_\epsilon^n\cap f^{-1}(S_\delta^1)$ *is homeomorphic to* $S_\epsilon^{n-1}-f^{-1}(\mathring{D}_\delta^2)\cap S_\epsilon^{n-1}$, *and from* (7) *in* § 4, $f^{-1}(D_\delta^2)\cap S_\epsilon^{n-1}$ *is homeomorphic to* $(S_\epsilon^{n-1}\cap f^{-1}(0))\times D^2$.

Now apply the Mayer-Vietoris exact sequence

$$\longrightarrow H_{i-1}(X_1\cup X_2)\xrightarrow{\delta_*}H_i(X_1\cap X_2)\longrightarrow H_i(X_1)\oplus H_i(X_2)\longrightarrow H_i(X_1\cup X_2)$$
$$\xrightarrow{\delta_*}H_{i-1}(X_1\cap X_2)\longrightarrow\cdots$$

to the pair of $X_1=D_\epsilon^n\cap f^{-1}(S_\delta^1)$ and $X_2=S_\epsilon^{n-1}\cap f^{-1}(D_\delta^2)$. Since $X_1\cup X_2$ is homeomorphic to S_ϵ^{n-1}, $X_1\cap X_2$ is homeomorphic to $(S_\epsilon^{n-1}\cap f^{-1}(0))\times S^1$ and $X_2=S_\epsilon^{n-1}\cap f^{-1}(D_\delta^2)$ is homeomorphic to $(S_\epsilon^{n-1}\cap f^{-1}(0))\times D^2$, we have

(q) $b_0(D_\epsilon^n\cap f^{-1}(S_\delta^1))=1$,
$b_i(D_\epsilon^n\cap f^{-1}(S_\delta^1))=b_{i-1}(S_\epsilon^{n-1}\cap f^{-1}(0))$, $1\leqq i\leqq n-3$,
$b_{n-2}(D_\epsilon^n\cap f^{-1}(S_\delta^1))=b_{n-3}(S_\epsilon^{n-1}\cap f^{-1}(0))-1$,
$b_{n-1}(D_\epsilon^n\cap f^{-1}(S_\delta^1))=0$.

Applying again the Mayer-Vietoris exact sequence to the pair of $X_1=(D_\epsilon^n\cap f^{-1}(S_\delta^1))^+$ and $X_2=(D_\epsilon^n\cap f^{-1}(S_\delta^1))^-$, where note that $X_1\cup X_2=M_{\epsilon,\delta}(f)$ and $X_1\cap X_2$ is homeomorphic to $(S_\epsilon^{n-1}\cap f^{-1}(0))\times S^1$, we have

(r) $b_0(M_{\epsilon,\delta}(f))=1$,
$b_1(M_{\epsilon,\delta}(f))=b_0(S_\epsilon^{n-1}\cap f^{-1}(0))=1$,
$b_i(M_{\epsilon,\delta}(f))=2b_{i-1}(S_\epsilon^{n-1}\cap f^{-1}(0))$, $2\leqq i\leqq n-3$,
$b_{n-2}(M_{\epsilon,\delta}(f))=2b_{n-3}(S_\epsilon^{n-1}\cap f^{-1}(0))-1$,
$b_{n-1}(M_{\epsilon,\delta}(f))=1$.

From (r), we have

$$\begin{aligned}\chi(M_{\epsilon,\delta}(f))&=\sum_{i=0}^{n-1}(-1)^i b_i(M_{\epsilon,\delta}(f))\\&=1-1+\sum_{i=2}^{n-2}(-1)^i b_i(M_{\epsilon,\delta}(f))+(-1)^{n-1}\\&=(-1)^{n-1}+\sum_{i=2}^{n-2}2(-1)^i b_{i-1}(S_\epsilon^{n-1}\cap f^{-1}(0))+(-1)^{n-3}\\&=-2\chi(S_\epsilon^{n-1}\cap f^{-1}(0))+2(-1)^{n-1}+2\\&=-2\chi(S_\epsilon^{n-1}\cap f^{-1}(0))+2\chi(S^{n-1})\ .\end{aligned}$$

Q.E.D. of Lemma 5.2.

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