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Extended Alexander Matrices of 3-Manifolds II

—Applications—

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§1. Statement of results.

In this paper we study 3-manifolds obtained from S^3 by Dehn surgery along knots. Let p be a positive integer and q be an integer relatively prime to p.

DEFINITION. For a knot $k\subset S^3$, let $L(p, q; k)$ be a 3-manifolds obtained from S^3 by Dehn surgery along k with coefficient p/q.

Note that when k is an unknot $L(p, q; k)$ means just a lens space $L(p, q)$. Clearly $L(p, q; k)$ is a homology lens space and $H_1(L(p, q; k))=\boldsymbol{Z}_p$ is generated by an element corresponding to a meridian of the tubular neighbourhood $N(k)$ of k. We denote this element by t. Then an element of a group ring $\boldsymbol{Z}H_1(L(p, q; k))=\boldsymbol{Z}[\boldsymbol{Z}_p]$ can be represented by a polynomial of t with integer coefficients where $t^p=1$. We give a necessary condition for $L(p, q; k)$ to be a lens space.

THEOREM 1. *Let k be a knot with the Alexander polynomial Δ_k. Suppose that $L(p, q; k)$ is homeomorphic to $L(p, q')$. Let r, r' be integers such that $rq\equiv 1(p)$ and $r'q'\equiv 1(p)$. Then there are $u\in\boldsymbol{Z}[\boldsymbol{Z}_p]$ and $l, s\in\boldsymbol{Z}$ such that $(p, s)=1$ which satisfy the equation*

$$(1+t+\cdots+t^{r-1})\Delta_k(t)\equiv\pm t^l u\bar{u}(1+t^s+\cdots+t^{s(r'-1)}) \bmod (1+t+\cdots+t^{p-1})$$

in $\boldsymbol{Z}[\boldsymbol{Z}_p]$.

As a corollary we can prove:

THEOREM 2. *Let k be a knot with trivial Alexander polynomial. Then $L(p, q; k)$ and $L(p, q')$ can be homeomorphic only if $q\equiv\pm q'(p)$ or $qq'\equiv\pm 1(p)$.*

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In case that k is unknotted, Theorem above yields well known classification of lens spaces which has been proved by Reidemeister [5], Franz [4] and Brody [1].

§2. Extended Alexander matrices.

In [3] the author and Kanno defined extended Alexander matrices of Heegaard splittings of 3-manifolds and studied their fundamental properties (Theorem 1 and Theorem 2 in [3]). We need the following result to prove Theorem 1:

THEOREM 3. *Suppose that there is a homeomorphism* $f: M \to N$. *Let* $\binom{A}{B}$ *and* $\binom{A'}{B'}$ *be EA-matrices of H-splittings of M and N. Then there are* $m, n \in N$ *such that* $\begin{pmatrix} A\oplus E_m \\ B\oplus 0_m \end{pmatrix}^{f_*} \sim \begin{pmatrix} A'\oplus E_n \\ B'\oplus 0_n \end{pmatrix}$.

§3. An EA-matrix of $L(p, q; k)$.

The precise definition of $t \in H_1(L(p, q; k))$ is as follows. Let $\mu \in H_1(S^3 - N(k))$ be a homology class represented by a meridian of $\partial N(k)$ where $N(k)$ denotes a tubular neighbourhood of k. Then t is a image of μ under the homomorphism $H_1(S^3 - N(k)) \to H_1(L(p, q; k)) = \boldsymbol{Z}_p$. Then we have:

LEMMA 1. *There is an EA-matrix of* $L(p, q; k)$ *which has the form*

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1+t+\cdots+t^{p-1} & 0 \\ 0 & C \\ 1+t+\cdots+t^{r-1} & 0 \\ * & D \end{pmatrix}$$

where $rq \equiv 1(p)$, *C and D are square matrices and C is an Alexander matrix of the knot k.*

PROOF. We suppose that a given knot k is in regular position in

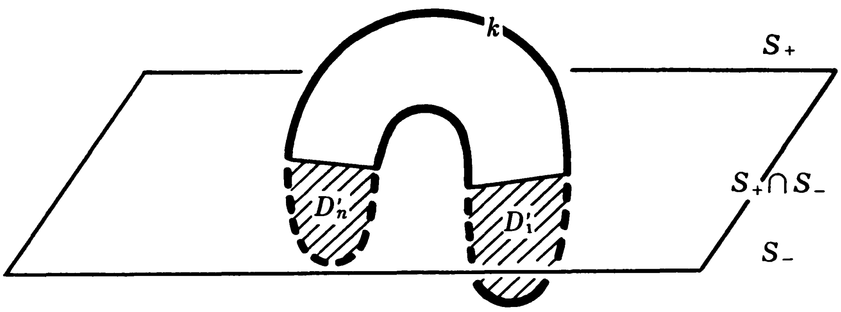

FIGURE 1

$S^3=\boldsymbol{R}^3\cup\infty$. Let $S_+=\boldsymbol{R}^3_+\cup\infty$ and $S_-=\boldsymbol{R}^3_-\cup\infty$ be upper and lower hemispheres such that overpasses and underpasses lie in S_+ and S_- respectively. Let $D'_1, \cdots, D'_n$ be 2-disks obtained as traces of underpasses projected by a projection map $S_-\to S_+\cap S_-$ and let $D_i=D'_i\cap(S^3-\mathring{N}(k))$. Let $N(D_i)$ denote a tubular neighbourhood of D_i in $S_-\cap(S^3-\mathring{N}(k))$.

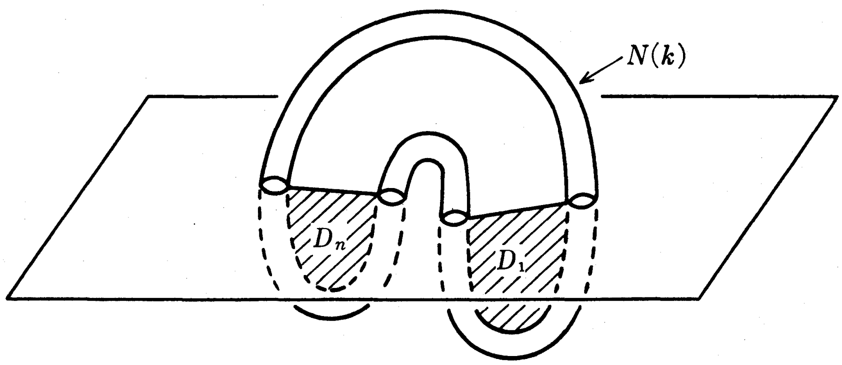

FIGURE 2

Set $T=(S_+\cap(S^3-\mathring{N}(k)))\cup(\cup_{i=1}^n N(D_i))$. Then T is a handle body of genus n and $\pi_1(T)$ is a free group generated by $x_1, \cdots, x_n$ which corresponds to the upperpasses.

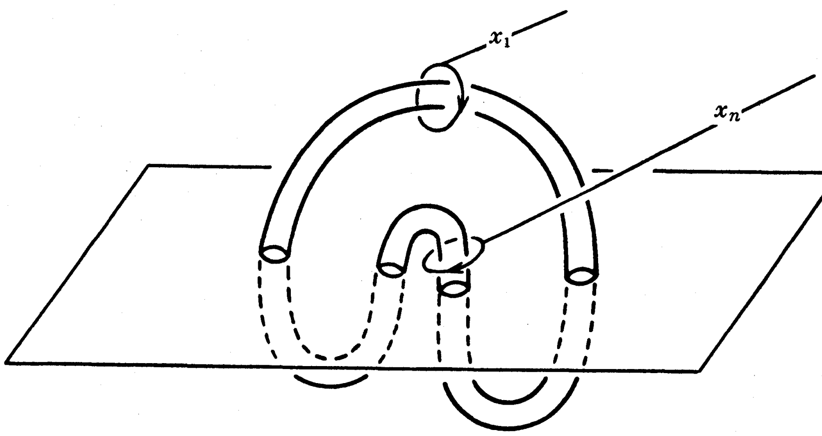

FIGURE 3

Let $T'=L(p, q; k)-\mathring{T}$ then T' is also a handle body of genus n.

Next we investigate meridian disks of T'. Consider 2-disks WD_i $(i=2, \cdots, n)$ wrapping D'_i as in Figure 4 which correspond to relators of Wirtinger presentation of the knot group. We can suppose that $\partial WD_i\subset(S_+\cap S_-)$.

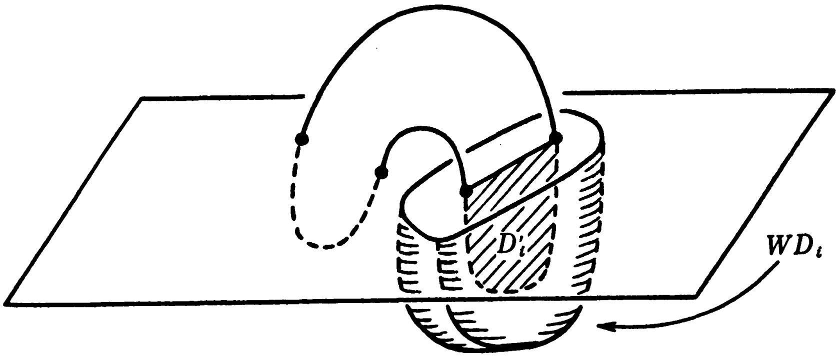

FIGURE 4

Let m and l be a meridian and a preferred longitude of $\partial N(k)$. We can assume that m and l lie on $\partial T \cap \partial N(k)$ and m is mapped to x_1 by the homomorphism $\pi_1(\partial T) \to \pi_1(T)$. Then we can choose simple loops a_1, b_1 on $\partial T \cap \partial N(k)$ such that, as homotopy classes, $a_1 = m^p l^q x$ and $b_1 = m^r l^s y$ where x and y belong to the commutater subgroup generated by l and m and $r, s \in N$ satisfy $ps - qr = -1$ (see Figure 5).

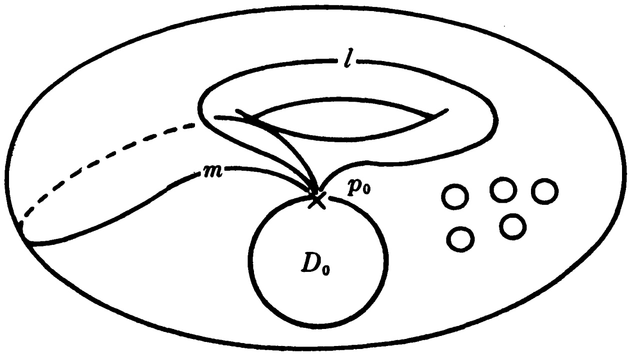

FIGURE 5

By the definition of p/q Dehn surgery, we can assume that $S^1 \times D^2$ is attached to $S^3 - \mathring{N}(k)$ such that a meridian $\{*\} \times \partial D^2$ $(* \in S^1)$ is identified with a_1 and a preferred longitude $S^1 \times \{**\}$ $(\{**\} \in \partial D^2)$ is identified with b_1.

Since $L(p, q; k)$ is obtained from T by attaching 2-handles corresponding to $\{*\} \times D^2$, WD_2, WD_3, $\cdots$, WD_n and a 3-ball, thus $\{\{*\} \times D^2, WD_2, WD_3, \cdots, WD_n\}$ is a system of meridian disks of $T' = L(p, q; k) - \mathring{T}$. Let a_i $(i=2, \cdots, n)$ denote loops on $\partial T = \partial T'$ which are obtained from ∂WD_i $(i=2, \cdots, n)$ by connecting to the base point p_0. Then $a_1, a_2, \cdots, a_n, b_1$ form a part of m-l system. Choose simple loops $b_2, \cdots, b_n$ such that $\{a_i, b_i\}$ should be a m-l system of T' (This is an abuse of the term 'm-l system', but readers should not be confused).

Now for an inclusion map $h\colon \partial T' - D_0 \hookrightarrow T$, let us compute $(\partial h(a_i)/\partial x_j)^\alpha$ and $(\partial h(b_i)/\partial x_j)^\alpha$. First, since $(x_1, \cdots, x_n | h(a_2), \cdots, h(a_n))$ is the Wirtinger presentation of the knot group $\pi_1(S^3 - N(k))$, the matrix $(\partial h(a_i)/\partial x_j)^\alpha_{i=2,\cdots,n}$ is the Alexander matrix reduced from $\boldsymbol{Z}[Z]$ to $\boldsymbol{Z}[Z_p]$. Next let us compute $(\partial h(a_1)/\partial x_j)^\alpha$ and $(\partial h(b_1)/\partial x_j)^\alpha$. Let $\alpha_j = (\partial l^q x/\partial x_j)^\alpha$ and $\beta_j = (\partial l^s y/\partial x_j)^\alpha$. Then we have

$$\left(\frac{\partial h(a_1)}{\partial x_j}\right)^\alpha = \begin{cases} 1+t+\cdots+t^{p-1}+t^p\alpha_1 & (j=1) \\ t^p\alpha_j & (j=2, \cdots, n) \end{cases}$$

and

$$\left(\frac{\partial h(b_1)}{\partial x_j}\right)^\alpha = \begin{cases} 1+t+\cdots+t^{r-1}+t^r\beta_1 & (j=1) \\ t^r\beta_j & (j=2, \cdots, n)\,. \end{cases}$$

Thus an EA-matrix $\binom{A}{B}$ has the form

$$\begin{pmatrix} 1+t+\cdots+t^{p-1}+t^p\alpha_1,\ t^p\alpha_2,\ \cdots,\ t^p\alpha_n \\ \left(\left(\dfrac{\partial h(a_i)}{\partial x_j}\right)^\alpha\right)_{i=2,\cdots,n} \\ 1+t+\cdots+t^{r-1}+t^r\beta_1,\ t^r\beta_2,\ \cdots,\ t^r\beta_n \\ \left(\left(\dfrac{\partial h(b_i)}{\partial x_j}\right)^\alpha\right)_{i=2,\cdots,n} \end{pmatrix}$$

We use the following sublemma the proof of which will be given later.

SUBLEMMA. *α_j and β_j have the forms $\alpha_j = \sum_{i=2}^n c_i(\partial h(a_i)/\partial x_j)^\alpha$ and $\beta_j = \sum_{i=2}^n d_i(\partial h(a_i)/\partial x_j)^\alpha$ where coefficients c_i and d_i are independent of j.*

Then by multiplying to $\binom{A}{B}$ the following matrix

$$\left(\begin{array}{cccc|cccc} 1, & -t^pc_2, & \cdots, & -t^pc_n & & & & \\ 0, & 1 & & 0 & & 0 & & \\ & & \ddots & & & & & \\ & 0 & & 1 & & & & \\ \hline 0, & -t^rd_2, & \cdots, & -t^rd_n & 1 & & & \\ & & & & t^{-p}\bar{c}_2 & 1 & & 0 \\ & & 0 & & \vdots & & \ddots & \\ & & & & t^{-p}\bar{c}_n & 0 & & 1 \end{array}\right)$$

from left, we obtain

$$\begin{pmatrix} 1+t+\cdots+t^{p-1} & 0 \\ & \left(\left(\dfrac{\partial h(a_i)}{\partial x_j}\right)^{\alpha}\right)_{i=2,\cdots,n} \\ 1+t+\cdots+t^{r-1} & 0 \\ * & \end{pmatrix}.$$

As is well known (See Crowell-Fox [2], pp. 122–123), the linear combination of all columns of $((\partial h(a_i)/\partial x_j)^{\alpha})_{i=2,\cdots,n}$ is zero. Thus the above matrix is equivalent to the matrix of the following form

$$\begin{pmatrix} 1+t+\cdots+t^{p-1} & 0 \\ 0 & C \\ 1+t+\cdots+t^{r-1} & 0 \\ * & D \end{pmatrix}$$

where C is also an Alexander matrix. Since $\begin{pmatrix} A \\ B \end{pmatrix}$ is equivalent to the above matrix, we complete the proof except for Sublemma.

PROOF OF SUBLEMMA. Since $l^q x$ and $l^s y$ are presented as products of conjugates of l and l^{-1}, it is sufficient to prove that for a product of conjugates of l or l^{-1}, say z, $(\partial h(z)/\partial x_j)^{\alpha}$ has the form $\sum_{i=2}^{n} m_i(\partial h(a_i)/\partial x_j)^{\alpha}$. As is well known $h(l)$ is mapped to the second commutater subgroup by $k\colon \pi_1(T)\to\pi_1(S^3-N(k))$. l is represented by $l_0 r$ where l_0 belongs to the second commutater group of $\pi_1(T)$ and r belongs to $\ker k$. Since l_0 is negligible when we consider free differential calculus, it follows that $(\partial l/\partial x_j)^{\alpha}=(\partial r/\partial x_j)^{\alpha}$. Note that r is represented by a product of conjugates of $h(a_2), \cdots, h(a_n)$ and their inverses, and

$$\left(\frac{\partial gh(a_i)g^{-1}}{\partial x_j}\right)^{\alpha}=(1-h(a_i))^{\alpha}\left(\frac{\partial g}{\partial x_j}\right)^{\alpha}+g^{\alpha}\left(\frac{\partial h(a_i)}{\partial x_j}\right)=\alpha(g)\left(\frac{\partial h(a_i)}{\partial x_j}\right)^{\alpha}$$

holds. Hence $(\partial r/\partial x_j)^{\alpha}$ has the form $\sum_{i=2}^{n} m_i(\partial h(a_i)/\partial x_j)^{\alpha}$. Furthermore, since z is the product of conjuates of l and l^{-1}, $(\partial z/\partial x_j)^{\alpha}$ also has the form $\sum_{i=2}^{n} m_i(\partial h(a_i)/\partial x_j)^{\alpha}$ as required.

§4. Proofs of theorems.

Now we will prove Theorem 1.

PROOF OF THEOREM 1. For $L(p, q; k)$, by Lemma 1, there is an EA-matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ of the form:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1+t+\cdots+t^{p-1} & 0 \\ 0 & C \\ 1+t+\cdots+t^{r-1} & 0 \\ * & D \end{pmatrix}$$

where $rq \equiv 1(p)$, C is an Alexander matrix and $\det C$ equals to the Alexander polynomial of the knot k. For the lens space $L(p, q')$ there is an EA-matrix $\begin{pmatrix} A' \\ B' \end{pmatrix}$ of the form

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} 1+\tau+\cdots+\tau^{p-1} \\ 1+\tau+\cdots+\tau^{r'-1} \end{pmatrix}$$

where $r'q' \equiv 1(p)$.

Suppose that there is a homeomorphism $f: L(p, q') \to L(p, q: k)$. Then, by Theorem 3, there are matrices G, $\begin{pmatrix} U & 0 \\ W & \pm {}^*U^{-1} \end{pmatrix}$ with $\det G$, $\det U \in \pm H_1(L(p, q: k))$ and $m, n \in N$ such that, for stabilized EA-matrices $\begin{pmatrix} R \\ S \end{pmatrix} = \begin{pmatrix} A \oplus E_m \\ B \oplus 0_m \end{pmatrix}$ and $\begin{pmatrix} R' \\ S' \end{pmatrix} = \begin{pmatrix} A' \oplus E_n \\ B' \oplus 0_n \end{pmatrix}^{f_*}$, $\begin{pmatrix} U & 0 \\ W & \pm {}^*U^{-1} \end{pmatrix} \begin{pmatrix} R \\ S \end{pmatrix}^{G^{-1}} = \begin{pmatrix} R' \\ S' \end{pmatrix}$ holds. This means

(1) $$UR = R'G \quad \text{and}$$

(2) $${}^*UWR \pm S = {}^*US'G\ .$$

Let $f_*: H_1(L(p, q')) \to H_1(L(p, q: k))$ be represented by $f_*(\tau) = t^s$ for some $s \in N$ relatively prime to p and $s < p$. Set $\alpha = 1+t+\cdots+t^{p-1}$, $\beta = 1+t+\cdots+t^{r-1}$, $\beta' = 1+t^s+\cdots+t^{s(r'-1)}$. Then R, S, R' and S' are represented as follows:

$$R = \begin{pmatrix} \alpha & 0 \\ 0 & X \end{pmatrix}, \quad S = \begin{pmatrix} \beta & 0 \\ Z & Y \end{pmatrix}, \quad R' = \begin{pmatrix} \alpha & 0 \\ 0 & E \end{pmatrix}, \quad S' = \begin{pmatrix} \beta' & 0 \\ 0 & 0 \end{pmatrix}$$

where $\det X = \det C$ coincides with the Alexander polynomial $\Delta_k(t)$ up to multiplication of $\pm t^j$ and E denotes a unit matrix. Set $U = \begin{pmatrix} u_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ and $G = \begin{pmatrix} g_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$ where u_{11}, g_{11} are 1×1 matrices.

From (1) we obtain

(3) $$u_{11}\alpha = \alpha g_{11}\ ,$$

(4) $$U_{12}X = \alpha G_{12}\ ,$$

(5) $$U_{21}\alpha = G_{21}\ ,$$

$$(6) \qquad U_{22}X=G_{22}\,.$$

Next we compare the (1, 1)-th entries of the both sides of the equation (2). Since the (1, 1)-th entry of $^*UWR$ is a multiple of α and the (1, 1)-th entry of $^*US'G$ is $\bar{u}_{11}g_{11}\beta'$, we obtain

$$(7) \qquad \pm\beta\equiv\bar{u}_{11}g_{11}\beta' \bmod \alpha\,.$$

Since $\det U$, $\det G\in\pm H_1(L(p, q: k))$, we can set $\det U=\pm t^k$, $\det G=\pm t^l$ for some $k, l\in \boldsymbol{Z}$. Consider the equation $U\begin{pmatrix}1 & 0\\ 0 & X\end{pmatrix}=\begin{pmatrix}u_{11} & U_{12}X\\ U_{21} & U_{22}X\end{pmatrix}$. Then, since $U_{12}X\equiv 0 \bmod \alpha$ by (4),

$$(8) \qquad \pm t^k \det X=\det U \det X\equiv u_{11}\det U_{22}\det X \bmod \alpha\,.$$

Furthermore, since $G_{21}\equiv 0 \bmod \alpha$ by (5), we have

$$(9) \qquad \pm t^l=\det G\equiv g_{11}\det G_{22} \bmod \alpha\,.$$

Thus

$$\begin{aligned} u_{11}(\pm t^l) &\equiv u_{11}g_{11}\det G_{22} \bmod \alpha \quad \text{by (9)}\\ &= u_{11}g_{11}\det U_{22}\det X \quad \text{by (6)}\\ &\equiv g_{11}(\pm t^k\det X) \bmod \alpha \quad \text{by (8)}\,. \end{aligned}$$

This means

$$(10) \qquad g_{11}\det X\equiv\pm t^{l'}u_{11} \bmod \alpha \qquad \text{for some} \quad l'\in\boldsymbol{Z}\,.$$

From (7) and (10), we have

$$\beta\det X\equiv\pm\bar{u}_{11}g_{11}\beta'\det X\equiv\pm t^{l'}u_{11}\bar{u}_{11}\beta' \bmod \alpha\,.$$

Since $\det X=\Delta_k(t)$ we have, by setting $l=l'$ and $u=u_{11}$,

$$(1+t+\cdots+t^{r-1})\Delta_k(t)\equiv\pm t^l u\bar{u}(1+t^s+\cdots+t^{s(r'-1)}) \bmod \alpha$$

as required.

The following lemma is used to prove Theorem 2.

LEMMA 2. *For* $u\in \boldsymbol{Z}[\boldsymbol{Z}_p]$ *and* $q, q', k, s\in\boldsymbol{Z}$ *such that* q, q' and s *are relatively prime to* p, *if it holds that*

$$(11) \qquad u\bar{u}(1+t+\cdots+t^{q-1})\equiv\pm t(1+t^s+\cdots+t^{s(q'-1)}) \bmod (1+t+\cdots+t^{p-1})$$

then $q\equiv\pm q'(p)$ *or* $qq'\equiv\pm 1(p)$.

PROOF. First we assume that $0<q$, $q'<p/2$. For the given identity

(11), multiplying $1-t$ to the both sides we obtain

$$u\bar{u}(1-t^q)=\pm t^k(1-t)(1+t^s+\cdots+t^{s(q'-1)}) . \tag{12}$$

Set $u=a_0+a_1t+\cdots+a_{p-1}t^{p-1}$. Then

$$u\bar{u}=\sum_{i=0}^{p-1}a_i^2+\sum_{i=0}^{p-1}a_ia_{i+1}t+\cdots+\sum_{i=0}^{p-1}a_ia_{i+p-1}t^{p-1}$$

where indices are thought as integer mod p.

Let us express the right hand side of (12) as a linear combination of $1, t, \cdots, t^{p-1}$ over $\boldsymbol{Z}$, then the coefficient of 1 is 0 or ± 1 because t^s is a generater of $\boldsymbol{Z}_p$ and $0<q'<p/2$. Comparing this coefficient with that of the left hand side of (12), we obtain $\sum_{i=0}^{p-1}a_i^2-\sum_{i=0}^{p-1}a_ia_{i-q}=0$ or ± 1. If $\sum_{i=0}^{p-1}a_i^2-\sum_{i=0}^{p-1}a_ia_{i-q}=0$ then $\sum_{i=0}^{p-1}(a_i-a_{i-q})^2=0$, thus $a_i=a_{i-q}$ for any i. Since p and q are coprime, this means that $a_0=a_1=\cdots=a_{p-1}$. Hence $u=a_0(1+t+\cdots+t^{p-1})$ and thus $(1-t)u=0$. Then from (11) we obtain $(1-t)(1+t^s+\cdots+t^{s(q'-1)})=0$. This is a contradiction. Hence $\sum_{i=0}^{p-1}a_i^2-\sum_{i=0}^{p-1}a_ia_{i-q}=\pm 1$. But the case of -1 does not occur because $\sum_{i=0}^{p-1}a_i^2-\sum_{i=0}^{p-1}a_ia_{i-q}=(1/2)\sum_{i=0}^{p-1}(a_i-a_{i-q})^2>0$. Thus $\sum_{i=0}^{p-1}a_i^2-\sum_{i=0}^{p-1}a_ia_{i-q}=1$. This means $\sum_{i=0}^{p-1}(a_i-a_{i-q})^2=2$. Thus there are $n, l\in\boldsymbol{N}$ and $a\in\boldsymbol{Z}$ such that $l<p/2$ and $a_n=a$, $a_{n-q}=a_{n-2q}=\cdots=a_{n-lq}=a\pm 1$, $a_{n-(l+1)q}=\cdots=a_{n-(p-1)q}=a$. Hence

$$\begin{aligned}u\bar{u}&\equiv(t^{n-q}+t^{n-2q}+\cdots+t^{n-lq})\overline{(t^{n-q}+t^{n-2q}+\cdots+t^{n-lq})}\\&\equiv(1+t^q+\cdots+t^{(l-1)q})\overline{(1+t^q+\cdots+t^{(l-1)q})}\\&\equiv t^{-(l-1)q}(1+t^q+\cdots+t^{(l-1)q})^2\\&\quad\bmod (1+t+\cdots+t^{p-1}) .\end{aligned}$$

From (11) and above we have

$$\begin{aligned}&(1+t^q+\cdots+t^{(l-1)q})^2(1+\cdots+t^{q-1})\equiv\pm t^{k'}(1+t^s+\cdots+t^{s(q'-1)})\\&\bmod (1+t+\cdots+t^{p-1}) .\end{aligned}$$

Multiplying $(1-t^q)(1-t)(1-t^s)$ to the both sides of the above equation we obtain

$$(1-t^{lq})^2(1-t^s)=\pm t^{k'}(1-t)(1-t^q)(1-t^{sq'}) . \tag{13}$$

First we see that ql is relatively prime to p. If g.c.d.$(ql, p)=d>1$, then, for $\xi=\exp(2\pi i/d)$, substitute t of (13) by ξ. Then left hand side equals zero while right hand side does not. This is a contradiction. Thus ql is relatively prime to p. Then by applying Franz Independence Lemma ([6], p. 406 and [4]), $\{\overline{ql}, \overline{ql}, \bar{s}\}=\{\bar{1}, \bar{q}, \overline{sq'}\}$ where $\bar{i}$ denotes $i \bmod p$.

Since $\bar{1} \in \{\bar{q}l, \bar{q}l, \bar{s}\}$, $\bar{q}l=\bar{1}$ or $\bar{s}=\bar{1}$.

Case 1: $\bar{s}=\bar{1}$. If $\bar{q}\neq\bar{1}$ then $\bar{q}'=\bar{q}$. If $\bar{q}=\bar{1}$ then $\bar{q}'=\bar{1}$. Therefore if $\bar{s}=\bar{1}$ then $\bar{q}'=\bar{q}$.

Case 2: $\bar{q}l=\bar{1}$. If $\bar{q}=\bar{1}$ then $\bar{q}'=\bar{1}$. If $\bar{q}\neq\bar{1}$ then $\bar{s}\bar{q}'=\bar{1}$ and $\bar{q}=\bar{s}$. Thus in this case $\bar{q}\bar{q}'=\bar{1}$. Therefore if $\bar{q}l=\bar{1}$ then $\bar{q}\bar{q}'=\bar{1}$.

Concluding these we have $q\equiv q'(p)$ or $qq'\equiv 1(p)$. Recall that we assumed that $0<q, q'<p/2$. Without the assumption we have $q\equiv \pm q'(p)$ or $qq'\equiv \pm 1(p)$ completing the proof of Lemma 2.

Since t^s is a generator of Z_p, Lemma 2 can be restated as follows:

LEMMA 2'. *For $u\in Z[Z_p]$ and $q, q', k, s\in Z$ such that q, q' and s are relatively prime to p, if it holds that*

(11') $$(1+t+\cdots+t^{q-1})\equiv \pm t^k u\bar{u}(1+t^s+\cdots+t^{s(q'-1)}) \bmod (1+t+\cdots+t^{p-1})$$

then $q\equiv \pm q'(p)$ or $qq'\equiv \pm 1(p)$.

Now Theorem 2 is an immediate cosequence of Theorem 1 and Lemma 2'.

References

[1] E. J. BRODY, The topological classification of the lens spaces, Ann. of Math. (2), **71** (1960), 163–184.

[2] R. H. CROWELL and R. H. FOX, Introduction to Knot Theory, GTM No. 57, Springer, Berlin-Heidelberg-New York, 1977.

[3] S. FUKUHARA and J. KANNO, Extended Alexander matrices of 3-manifolds I, Tokyo J. Math., **8** (1985), 107–120.

[4] W. FRANZ, Über die Torsion einer Überdeckung, J. Reine Angew. Math., **173** (1935), 245–254.

[5] K. REIDEMEISTER, Homotopieringe und Linseräume, Abh. Math. Sem. Univ. Hamburg, **11** (1935), 102–109.

[6] J. W. MILNOR, Whitehead torsion, Bull. Amer. Math. Soc., **72** (1966), 385–426.

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