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On Classification of Parahermitian Symmetric Spaces

Soji KANEYUKI

Sophia University and University of Massachusetts

Introduction

The purpose of this paper is to give a method of classifying parahermitian symmetric spaces with semisimple automorphism groups up to local paraholomorphic equivalence (see Definition 1.1 for local paraholomorphic equivalence). The outline of the classification was given in our previous paper [1]. In §1 we reduce the problem to the classification of parahermitian symmetric coset spaces of semisimple Lie groups. Proposition 1.6 is the main result in §1. In §2 we consider the reduction of the problem to the case where the groups are simple. In §3 we give the main theorem (Theorem 3.4) which establishes a one-to-one correspondence between parahermitian symmetric spaces with simple automorphism groups and a certain class of simple graded Lie algebras which was worked out by Kobayashi-Nagano [2]. The explicit infinitesimal forms of these spaces are given in the previous paper [1].

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NOTATIONS. The Lie algebra of a Lie group G is denoted by the corresponding German small letter or Lie G. G^0 denotes the identity component of a Lie group G. ϕ_{*p} denotes the differential of a map ϕ at p. $T_p(M)$ denotes the tangent space to M at p. id denotes the identity mapping. The Lie group homomorphism and the corresponding Lie algebra homomorphism are denoted by the same letter unless otherwise stated.

§1. Some properties of the automorphism groups.

Let (M, I) be an almost paracomplex manifold, and let $\tilde{M}$ be a covering manifold of M and $\pi: \tilde{M} \to M$ be the natural projection.

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LEMMA 1.1. *There exists an almost paracomplex structure* $\tilde{I}$ *on* $\tilde{M}$ *such that* π *is an almost paraholomorphic map. Furthermore,* $\tilde{I}$ *is integrable, if and only if* I *is integrable.*

PROOF. Take a point $\tilde{p}\in\tilde{M}$ and put $\pi(\tilde{p})=p$. Consider a tangent vector $\tilde{X}\in T_{\tilde{p}}(\tilde{M})$ (=the tangent space of $\tilde{M}$ at $\tilde{p}$). Let us define a linear endomorphism $\tilde{I}_{\tilde{p}}$ of $T_{\tilde{p}}(\tilde{M})$ by putting

$$\pi_{*\tilde{p}}(\tilde{I}_{\tilde{p}}\tilde{X})=I_p\pi_{*\tilde{p}}\tilde{X}\ .$$

Then it is easily seen that the assignment $\tilde{I}:\tilde{p}\to\tilde{I}_{\tilde{p}}$ is smooth and that $\tilde{I}$ is an almost paracomplex structure on $\tilde{M}$. For an arbitrary smooth vector field X on M, one can find a smooth vector field $\tilde{X}$ on $\tilde{M}$ which is π-related to X. Such a vector field $\tilde{X}$ is called a projectable vector field. Take two projectable vector fields $\tilde{X}$, $\tilde{Y}$ on $\tilde{M}$, and denote by $\tilde{T}$ (resp. T) the torsion tensor field of $\tilde{I}$ (resp. I). Then we have $\pi_*\tilde{T}(\tilde{X},\tilde{Y})=T(\pi_*\tilde{X},\pi_*\tilde{Y})$. Hence, if I is integrable, then $\tilde{T}$ vanishes for any two projectable vector fields. But, this implies that $\tilde{T}$ is identically zero, since any vector field on $\tilde{M}$ can be expressed as a linear combination of projectable vector fields over the ring of smooth functions on $\tilde{M}$. This proves that $\tilde{I}$ is integrable. The converse implication is also easily derived.

LEMMA 1.2. *Let* (M, I, g) *be a (almost) parahermitian (resp. parakähler) manifold with metric* g, $\tilde{M}$ *be a covering manifold of* M, *and* $\pi:\tilde{M}\to M$ *be the natural projection. Put* $\tilde{g}=\pi^*g$. *Then* $(\tilde{M},\tilde{I},\tilde{g})$ *is a (almost) parahermitian (resp. parakähler) manifold, where* $\tilde{I}$ *is the (almost) paracomplex structure given in Lemma* 1.1.

The proof is easy and so omitted. For the definitions see [1].

LEMMA 1.3. *Let* (M, I, g) *be a parahermitian symmetric space and* $\tilde{M}$ *be the universal covering manifold of* M *endowed with the paracomplex structure* $\tilde{I}$ *and with the parahermitian metric* $\tilde{g}$ *given in Lemmas* 1.1 *and* 1.2. *Then* $(\tilde{M},\tilde{I},\tilde{g})$ *is a parahermitian symmetric space.*

PROOF. $(\tilde{M},\tilde{g})$ is a simply connected affine symmetric space with respect to the Levi-Civita connection. Therefore one can define the symmetry $s_{\tilde{p}}$ at each point $\tilde{p}\in\tilde{M}$ satisfying $\pi\cdot s_{\tilde{p}}=s_p\cdot\pi$, where s_p is the symmetry at $p\in M$ (cf. Kobayashi-Nomizu [3]). It is easy to see that $s_{\tilde{p}}\in\mathrm{Aut}(\tilde{M},\tilde{I},\tilde{g})$.

We need the following definition.

DEFINITION 1.1. Let (M, I, g) and (M', I', g') be two parahermitian symmetric spaces. Then (M, I, g) and (M', I', g') are said to be *paraholomorphically equivalent*, if there exists a diffeomorphism ϕ of M onto M' *such that*

(i) $\phi \cdot s_p = s_{\phi(p)} \cdot \phi$ for $p \in M$,

(ii) $\phi_{*p} \cdot I_p = I'_{\phi(p)} \cdot \phi_{*p}$ for $p \in M$,

where s_p and $s_{\phi(p)}$ are the symmetries of M and M' at p and $\phi(p)$ respectively. (M, I, g) and (M', I', g') are said to be *locally paraholomorphically equivalent*, if the universal covering manifolds (endowed with the structures given in Lemma 1.3) of M and M' are paraholomorphically equivalent.

Let (M, g) be a pseudo-Riemannian symmetric space with pseudo-Riemannian metric g, $I^0(M, g)$ be the identity component of the isometry group, and L be the isotropy subgroup of $I^0(M, g)$ at a point $o \in M$. We denote by σ the involutive automorphism of $I^0(M, g)$ which is naturally induced by the symmetry s_o at o. We denote also by σ the corresponding automorphism of the Lie algebra $\mathfrak{i}$ of $I^0(M, g)$. Then we have the eigenspace decomposition of $\mathfrak{i}$ by σ:

$$\mathfrak{i} = \mathfrak{l} + \mathfrak{m} ,$$

where $\mathfrak{l} = \mathrm{Lie}\, L$ and $\mathfrak{m}$ is the -1-eigenspace of σ.

LEMMA 1.4. *Let* $X \in \mathfrak{m}$. *For a fixed* $t_0 \in \boldsymbol{R}$, *put* $p = \exp t_0 X \cdot o$, $p' = (\exp(t_0/2)X) \cdot o$. *Then we have* $\exp t_0 X = s_{p'} \circ s_o$, *where* s_q *denotes the symmetry at* $q \in M$.

PROOF. The proof is quite analogous to that in the Riemannian case. Note that the linear isotropy representation of L is faithful.

PROPOSITION 1.5. *Let* (M, I, g) *be a parahermitian symmetric space. Let us denote by* $\mathrm{Aut}^0(M, I, g)$ *the identity component of the automorphism group* $\mathrm{Aut}(M, I, g)$. *If either one of* $I^0(M, g)$ *and* $\mathrm{Aut}^0(M, I, g)$ *is semisimple, then we have* $I^0(M, g) = \mathrm{Aut}^0(M, I, g)$.

PROOF. Suppose that $I^0(M, g)$ is semisimple. Take $X \in \mathfrak{m}$. Then for an arbitrary fixed $t \in \boldsymbol{R}$, $\exp tX$ can be written as a product of two symmetries (Lemma 1.4). But symmetries are in $\mathrm{Aut}(M, I, g)$. So we have $\exp tX \in \mathrm{Aut}(M, I, g)$, which implies $\mathfrak{m} \subset \mathfrak{a}$ (=the Lie algebra of $\mathrm{Aut}(M, I, g)$). Since $\mathfrak{i}$ is semisimple, one has $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{l}$, and so $\mathfrak{l} \subset \mathfrak{a}$. Therefore we have $\mathfrak{i} = \mathfrak{a}$. The proof in the case that $\mathrm{Aut}^0(M, I, g)$ is semisimple is quite similar to that of the next Proposition 1.6, so we can omit it.

The next proposition is an analogue to a well-known fact for a Hermitian symmetric space.

PROPOSITION 1.6. *Let (M, I, g) be a parahermitian symmetric space. Let G be a connected semisimple Lie subgroup of $\mathrm{Aut}(M, I, g)$, and H be the isotropy subgroup of G at a point $o \in M$. Suppose that G/H is a symmetric coset space and that M is represented in the form G/H as a symmetric coset space. Then we have $G = \mathrm{Aut}^0(M, I, g)$.*

PROOF. Let σ be the involutive automorphism of G which induces the given symmetric space structure on G/H. It follows from the assumption that the symmetry of G/H at o induced from σ coincides with the symmetry s_o of M at o. Hence, if we denote by $\tilde{\sigma}$ the involutive automorphism of $A = \mathrm{Aut}^0(M, I, g)$ which is induced by s_o, then the restriction of $\tilde{\sigma}$ to G coincides with σ. Let $\mathfrak{a}$ and $\mathfrak{g}$ be the Lie algebras of A and G, respectively. Let us denote by the same notations the involutive automorphisms of $\mathfrak{a}$ and $\mathfrak{g}$ induced by $\tilde{\sigma}$ and σ, respectively. Then $\mathfrak{a}$ and $\mathfrak{g}$ can be decomposed in the form

$$(1.1) \qquad \mathfrak{a} = \mathfrak{b} + \mathfrak{m}\,, \qquad \mathfrak{g} = \mathfrak{h} + \mathfrak{m}\,,$$

where $\mathfrak{h} = \mathrm{Lie}\,H$, $\mathfrak{b}$ is the isotropy subalgebra of $\mathfrak{a}$ at o, and $\mathfrak{m}$ is the common -1-eigenspace of $\tilde{\sigma}$ and σ. Here we have

$$(1.2) \qquad \mathfrak{b} \supset \mathfrak{h}\,, \qquad [\mathfrak{b}, \mathfrak{m}] \subset \mathfrak{m}\,, \qquad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}\,.$$

Also we have (cf. Nomizu [4])

$$(1.3) \qquad [\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}\,,$$

since G/H is an effective symmetric coset space with G semisimple. Our aim is to prove $\mathfrak{b} = \mathfrak{h}$. Take $X \in \mathfrak{b}$. In view of (1.3) we have $[X, \mathfrak{h}] \subset \mathfrak{h}$. Hence $\mathrm{ad}\,X$ leaves $\mathfrak{g}$ invariant (cf. (1.2)). Therefore $\mathrm{ad}_{\mathfrak{g}}\,X$ is a derivation of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, there exists an element $X' \in \mathfrak{g}$ such that

$$(1.4) \qquad [X, Y] = [X', Y]\,, \qquad Y \in \mathfrak{g}\,.$$

We write X' in the form $X' = X_1' + X_2'$, $X_1' \in \mathfrak{h}$, $X_2' \in \mathfrak{m}$. Then we have $[X, \mathfrak{m}] = [X', \mathfrak{m}] = [X_1', \mathfrak{m}] + [X_2', \mathfrak{m}]$. So, from (1.2) and (1.3) it follows that $[X_2', \mathfrak{m}] = 0$. The same argument shows that $[X_2', \mathfrak{h}] = 0$. Therefore $X_2' = 0$, since $\mathfrak{g}$ is is semisimple. So we have $X' \in \mathfrak{h}$ and consequently $X - X' \in \mathfrak{b}$. On the other hand, from (1.4) we have $[X - X', \mathfrak{m}] = 0$. Since the representation $\mathfrak{b} \to \mathrm{ad}_{\mathfrak{m}}\,\mathfrak{b}$ is faithful, we get $X = X' \in \mathfrak{h}$. Thus we proved $\mathfrak{g} = \mathfrak{a}$.

Let G be a connected Lie group and H be a closed subgroup, σ be an involutive automorphism. Suppose that $(G/H, I, g)$ is a parahermitian symmetric coset space (cf. [1]). Then the quadruple (G, H, σ, I) is called a *parahermitian symmetric quadruple*. The quadruple is called *effective* *semisimple*, if the pair (G, H) is effective with G semisimple. For a parahermitian symmetric coset space $(G/H, I, g)$ with G semisimple, the metric g is always supposed to be the one induced from the Killing form of the Lie algebra $\mathfrak{g}$ of G (cf. Proposition 3.3 [1]). We consider the following two sets:

$\widehat{PHSS}$ the set of all parahermitian symmetric spaces (M, I, g) with a base point $o \in M$, having the semisimple $\mathrm{Aut}(M, I, g)$,

$\widehat{PHSQ}$ the set of all effective semisimple parahermitian symmetric quadruples (G, H, σ, I).

PROPOSITION 1.7. *There exists a natural bijection* Φ_1: $\widehat{PHSS} \to \widehat{PHSQ}$.

PROOF. We preserve the notations in the proof of Proposition 1.6. Take $(M, I, g) \in \widehat{PHSS}$. Let B be the isotropy subgroup of $A = \mathrm{Aut}^0(M, I, g)$ at $o \in M$. A being transitive on M, M can be expressed as $M = A/B$. Let us define Φ_1 by putting

$$\Phi_1((M, I, g)) = (A, B, \tilde{\sigma}, I) . \tag{1.5}$$

Then Φ_1 is obviously injective. The surjectivity of Φ_1 follows from Proposition 1.6.

§ 2. Semisimple graded Lie algebras.

DEFINITION 2.1. Let $\mathfrak{g}$ be a real semisimple Lie algebra of non-compact type, and Z be an element of $\mathfrak{g}$. Then the pair $(\mathfrak{g}, Z)$ is called an *effective semisimple graded Lie algebra*, if the following conditions are satisfied:

(i) $\mathfrak{g}$ is written as

$$\mathfrak{g} = \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 , \tag{2.1}$$

where each $\mathfrak{g}^\lambda$ is the λ-eigenspace of $\mathrm{ad}\, Z$,

(ii) the pair $(\mathfrak{g}, \mathfrak{g}^0)$ is effective.

Two effective semisimple graded Lie algebras $(\mathfrak{g}, Z)$ and $(\mathfrak{g}', Z')$ are said to be *isomorphic*, if there exists an isomorphism f of $\mathfrak{g}$ onto $\mathfrak{g}'$ such that $f(Z) = Z'$. The following lemma is a simplification of Proposition 4.1 in [1].

LEMMA 2.1. *There exists a one-to-one correspondence between the following two objects*:

(i) *an effective semisimple symmetric triple* $(\mathfrak{g}, \mathfrak{h}, \sigma)$ *satisfying* (C_2) (cf. [1]);

(ii) *an effective semisimple graded Lie algebra* $(\mathfrak{g}, Z)$.

PROOF. The proof is essentially the same as in Proposition 4.1 in [1]. Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be an effective semisimple symmetric triple satisfying (C_2). Then $\mathfrak{g}$ is obviously of noncompact type. By Lemmas 2.6 and 3.1 in [1], there exists a unique element $Z^0 \in \mathfrak{g}$ such that $(\mathfrak{g}, Z^0)$ is an effective semisimple graded Lie algebra with $\mathfrak{h}=\mathfrak{g}^0$.

Now let $(\mathfrak{g}, Z)$ be an effective semisimple graded Lie algebra, and let us consider the decomposition (2.1). Then Z is in the center $\mathfrak{z}(\mathfrak{g}^0)$ of $\mathfrak{g}^0$. Let us define a linear endomorphism σ of $\mathfrak{g}$ by putting $\sigma|_{\mathfrak{g}^0}=\mathrm{id}$ and $\sigma|_{\mathfrak{g}^1+\mathfrak{g}^{-1}}=-\mathrm{id}$. Then σ is an involutive automorphism of $\mathfrak{g}$. Let $\tilde{\tau}$ be a Cartan involution of $\mathfrak{g}$ satisfying $\sigma\tilde{\tau}=\tilde{\tau}\sigma$, and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the corresponding Cartan decomposition, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1-eigenspaces of $\tilde{\tau}$, respectively. Since $\tilde{\tau}$ leaves $\mathfrak{g}^0$ (and consequently $\mathfrak{z}(\mathfrak{g}^0)$) invariant, we have $\mathfrak{z}(\mathfrak{g}^0)=\mathfrak{z}(\mathfrak{g}^0)\cap\mathfrak{k}+\mathfrak{z}(\mathfrak{g}^0)\cap\mathfrak{p}$. We can write $Z=Z'+Z''$, where $Z'\in\mathfrak{z}(\mathfrak{g}^0)\cap\mathfrak{k}$, $Z''\in z(\mathfrak{g}^0)\cap\mathfrak{p}$. Then, by the same argument as in the proof of Proposition 3.3 in [1], we get $Z'=0$. Therefore $\tilde{\tau}(Z)=-Z$ and so $\tilde{\tau}(\mathfrak{g}^\lambda)=\mathfrak{g}^{-\lambda}$. In particular, we have $\dim\mathfrak{g}^{-1}=\dim\mathfrak{g}^1$. Set $\mathfrak{m}^\pm=\mathfrak{g}^{\pm1}$, $\mathfrak{m}=\mathfrak{m}^++\mathfrak{m}^-$ and $I_o=\mathrm{ad}_{\mathfrak{m}}Z$. Furthermore, let $\langle\ ,\ \rangle$ be the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{m}$. Then $(\mathfrak{g}, \mathfrak{h}, \sigma)$ satisfies (C_2).

Let us denote by $\widehat{SGLA}$ the set of all effective semisimple graded Lie algebras $(\mathfrak{g}, Z)$. The following lemma is well-known (cf. Proposition 3.2 in [1]).

LEMMA 2.2. *Let* $(\mathfrak{g}, Z)\in\widehat{SGLA}$, and let $\mathfrak{g}=\mathfrak{g}_1+\cdots+\mathfrak{g}_s$ *be the decomposition into simple ideals. Let us write* $Z=Z_1+\cdots+Z_s$, $Z_i\in\mathfrak{g}_i$. *Then each* Z_i is *non-zero, and* $(\mathfrak{g}, Z)$ *is the direct sum of the simple (that is,* $\mathfrak{g}_i$ *simple) graded Lie algebras* $(\mathfrak{g}_i, Z_i)$, $1\leqq i\leqq s$.

Now let $(G, H, \sigma, I)\in\widehat{PHSQ}$. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. Then $(\mathfrak{g}, \mathfrak{h}, \sigma)$ satisfies (C_2) (cf. [1]). By Lemma 2.1, there corresponds to it an effective semisimple graded Lie algebra $(\mathfrak{g}, Z)$. Let us define the map $\Phi_2\colon \widehat{PHSQ}\to\widehat{SGLA}$ by putting

$$\Phi_2((G, H, \sigma, I))=(\mathfrak{g}, Z)\ . \tag{2.2}$$

Suppose that $(M, I, g)\in\widehat{PHSS}$ is simply connected, and let $\Phi_1((M, I, g))=(A, B, \tilde{\sigma}, I)$ (cf. (1.5)). Let $\mathfrak{a}$ and $\mathfrak{b}$ be Lie algebras of A and B, respectively. Then, by Lemmas 2.1 and 2.2, $\Phi_2((A, B, \tilde{\sigma}, I))=(\mathfrak{a}, Z)$ can be decomponsed into the direct sum of simple graded Lie algebras:

$$(\mathfrak{a}, Z)=(\mathfrak{a}_1, Z_1)\oplus\cdots\oplus(\mathfrak{a}_s, Z_s)\ . \tag{2.3}$$

$\tilde{\sigma}$ leaves each $\mathfrak{a}_i$ stable (cf. [1]). Put $\tilde{\sigma}_i=\sigma|_{\mathfrak{a}_i}$ and $\mathfrak{b}_i=\mathfrak{b}\cap\mathfrak{a}_i$. Then it is known ([1] and Lemma 2.2) that the symmetric triple $(\mathfrak{a}, \mathfrak{b}, \sigma)$ is the direct sum of the symmetric triples $(\mathfrak{a}_i, \mathfrak{b}_i, \tilde{\sigma}_i)$ which satisfy (C_2).

PROPOSITION 2.3. *Suppose that $(M, I, g) \in \widehat{PHSS}$ is simply connected. Then M is paraholomorphically equivalent to the direct product of simply connected parahermitian symmetric spaces with simple automorphism groups.*

PROOF. We preserve the notations above. Let $\tilde{A}$ be the universal covering group of A, and $\tilde{B}$ be the analytic subgroup of $\tilde{A}$ generated by $\mathfrak{b}$, the Lie algebra of B. Then M is written as the coset space $\tilde{A}/\tilde{B}$. Let $\tilde{A}_i$ and $\tilde{B}_i$ be the analytic subgroups of $\tilde{A}$ generated by $\mathfrak{a}_i$ and $\mathfrak{b}_i$, respectively. Then we have

$$\tilde{A}=\tilde{A}_1\times\cdots\times\tilde{A}_s\ ,$$
$$\tilde{B}=\tilde{B}_1\times\cdots\times\tilde{B}_s\ .$$

By putting $M_i=\tilde{A}_i/\tilde{B}_i$, M can be written as

$$M=M_1\times\cdots\times M_s\ .$$

Since $(\mathfrak{a}_i, \mathfrak{b}_i, \tilde{\sigma}_i)$ satisfies (C_2), and since $\tilde{B}_i$ is connected, the coset space $\tilde{A}_i/\tilde{B}_i$ endowed with the metric induced from the Killing form of $\mathfrak{a}_i$ satisfies the condition (C_1) in [1]. Hence $M_i=\tilde{A}_i/\tilde{B}_i$ is a parahermitian symmetric space (cf. [1]). It is easy to see that the natural diffeomorphism of M to $M_1\times\cdots\times M_s$ gives a paraholomorphic equivalence.

PROPOSITION 2.4. *Let $(\mathfrak{g}, Z) \in \widehat{SGLA}$. Let G^* be the adjoint group corresponding to the Lie algebra $\mathfrak{g}$, and let $C^*(Z)$ be the centralizer of Z in G^*. Then $G^*/C^*(Z)$ is an effective parahermitian symmetric coset space.*

PROOF. Let G^C be the simply connected Lie group corresponding to the Lie algebra $\mathfrak{g}^C$, the complexification of $\mathfrak{g}$, and let G be the analytic subgroup of G^C generated by $\mathfrak{g}$. By Lemma 2.1, to $(\mathfrak{g}, Z)$ there corresponds the effective symmetric triple $(\mathfrak{g}, \mathfrak{h}, \sigma)$ such that $\mathfrak{h}$ is the centralizer $\mathfrak{c}(Z)$ of Z. The involutive automorphism σ here can be extended to the involutive automorphism of G, which is denoted by the same letter σ. It is known [1] that

$$C(Z)=G_\sigma\ , \tag{2.4}$$

where $C(Z)$ denotes the centralizer of Z in G, and G_σ denotes the σ-fixed set in G. Let π be the covering homomorphism of G onto G^*, and σ^*

be the involutive automorphism of G^* induced by σ. Then we have

$$\sigma^* \cdot \pi = \pi \cdot \sigma . \tag{2.5}$$

Therefore it follows that

$$C^*(Z) = \pi(C(Z)) = \pi(G_\sigma) \subset G_{\sigma^*} ,$$

where G_{σ^*} denotes the σ^*-fixed set in G^*. On the other hand (2.5) implies that the Lie algebra of G_σ is isomorphic to that of G_{σ^*} under π. Hence we have

$$(G_{\sigma^*})^0 = \pi((G_\sigma)^0) = \pi(C^0(Z)) \subset \pi(C(Z)) = C^*(Z) .$$

Thus we have proved $(G_{\sigma^*})^0 \subset C^*(Z) \subset G_{\sigma^*}$. Therefore, by the same arguments as in the proof of Theorem 3.7 [1], we conclude that $G^*/C^*(Z)$ is an effective parahermitian symmetric coset space.

§ 3. Classification theorem.

LEMMA 3.1. *Let (M, I, g), $(M', I', g') \in \widehat{PHSS}$, and let $\Phi_2\Phi_1((M, I, g)) = (\mathfrak{g}, Z)$ and $\Phi_2\Phi_1((M', I', g')) = (\mathfrak{g}', Z')$. If (M, I, g) and (M', I', g') are locally paraholomorphically equivalent, then $(\mathfrak{g}, Z)$ and $(\mathfrak{g}', Z')$ are isomorphic.*

PROOF. Without loss of generality, we can assume that M and M' are simply connected and that there exists a diffeomorphism ϕ of M onto M' satisfying the conditions (i), (ii) in Definition 1.1. Let o and o' be the base points of M and M' respectively. Let $\Phi_1((M, I, g)) = (A, B, \sigma, I)$ and $\Phi_1((M', I', g')) = (A', B', \sigma', I')$. Let us denote by G and G' the universal covering groups of A and A' respectively, and extend σ and σ' to the involutive automorphisms of G and G', which are denoted by the same letters. We represent M and M' as $M = A/B = G/H$ and $M' = A'/B' = G'/H'$, where H and H' are the (connected) isotropy subgroups of G and G' at o and o', respectively. Consider the eigenspace decompositions of the Lie algebras $\mathfrak{g} = \mathrm{Lie}\, G$ and $\mathfrak{g}' = \mathrm{Lie}\, G'$ by σ and σ':

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} , \qquad \mathfrak{g}' = \mathfrak{h}' + \mathfrak{m}' , \tag{3.1}$$

where $\mathfrak{h} = \mathrm{Lie}\, H$ and $\mathfrak{h}' = \mathrm{Lie}\, H'$. Since A and A' are semisimple, we have (cf. (1.3))

$$[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h} , \qquad [\mathfrak{m}', \mathfrak{m}'] = \mathfrak{h}' , \tag{3.2}$$

From this it follows that A (resp. A') is generated by $\exp \mathfrak{m}$ (resp. $\exp \mathfrak{m}'$) (cf. [5]). Therefore, by Lemma 1.4, every element of A (resp. A') is expressed as a product of symmetries of M (resp. M'). By using this and the condition (i) of Definition 1.1, we can conclude $\phi A\phi^{-1}=A'$. Making the composite of ϕ with an element of A' if necessary, we can assume further $\phi(o)=o'$. Let us consider the isomorphism $\tilde{\phi}: A\rightarrow A'$ defined by $\tilde{\phi}(a)=\phi\cdot a\cdot\phi^{-1}$, $a\in A$. By extending $\tilde{\phi}$ to an isomorphism of G onto G' (denoted by the same $\tilde{\phi}$), we have the following commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\tilde{\phi}} & G' \\ \pi\downarrow & & \downarrow\pi' \\ G/H & \xrightarrow{\phi} & G'/H' \end{array} \tag{3.3}$$

where π and π' are the natural projections. Let us identify the tangent spaces $T_o(G/H)$ and $T_{o'}(G'/H')$ with $\mathfrak{m}$ and $\mathfrak{m}'$, respectively. Then it is easy to see that

$$\tilde{\phi}|_{\mathfrak{m}}=\phi_{*_o}\ . \tag{3.4}$$

Since ϕ is paraholomorphic, we have

$$\phi_{*_o}\cdot I_o=I'_{o'}\cdot\phi_{*_o}\ . \tag{3.5}$$

From the equalities $I_o=\mathrm{ad}_{\mathfrak{m}} Z$ and $I'_{o'}=\mathrm{ad}_{\mathfrak{m}'}Z'$, and from (3.4) and (3.5) it follows that

$$\tilde{\phi}\cdot\mathrm{ad}\, Z=\mathrm{ad}\, Z'\cdot\tilde{\phi}\ , \tag{3.6}$$

which implies that $\tilde{\phi}(Z)=Z'$.

LEMMA 3.2. *Let (M, I, g) and (M', I', g') be parahermitian symmetric spaces with semisimple automorphism groups. Let $\Phi_2\Phi_1(M, I, g))=(\mathfrak{g}, Z)$ and $\Phi_2\Phi_1((M', I', g'))=(\mathfrak{g}', Z')$. If $(\mathfrak{g}, Z)$ and $(\mathfrak{g}', Z')$ are isomorphic, then* (M, I, g) and (M', I', g') *are locally paraholomorphically equivalent.*

PROOF. M and M' can be assumed to be simply connected. Let G and G' be the simply connected Lie groups corresponding to $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. By the assumption, there exists a graded isomorphism $\tilde{\phi}$ of $\mathfrak{g}$ onto $\mathfrak{g}'$ which can be extended to the isomorphism of G onto G', which is denoted by the same $\tilde{\phi}$. Consider the subalgebras $\mathfrak{h}=\mathfrak{g}^0\subset\mathfrak{g}$ and $\mathfrak{h}'=\mathfrak{g}'^0\subset\mathfrak{g}'$. Let us denote by H and H' the analytic subgroups of G and G' generated by $\mathfrak{h}$ and $\mathfrak{h}'$, respectively. Then M and M' can be written as $M=G/H$ and $M'=G'/H'$. On the other hand we have $\tilde{\phi}(\mathfrak{h})=\mathfrak{h}'$, which implies $\tilde{\phi}(H)=H'$. So $\tilde{\phi}$ induces a diffeomorphism ϕ of G/H onto G'/H' for which the diagram (3.3) is valid. $\tilde{\phi}$ being a graded isomorphism, the relation (3.6) is valid. So (3.5) is valid. Therefore, from the invariance of I and I' under G and G', it follows that $\phi_{*_p}\cdot I_p = I'_{\phi(p)}\cdot\phi_{*_p}$ for each $p\in M$.

Let $o\in M$ and $o'\in M'$ be the base points, and let σ and σ' be the involutive automorphisms of G and G', respectively. Then, considering the gradations of $\mathfrak{g}$ and $\mathfrak{g}'$, we have $\tilde{\phi}\cdot\sigma=\sigma'\cdot\tilde{\phi}$, from which it follows that ϕ satisfies the condition (i) in Definition 1.1.

Thus we have the following classification theorem.

THEOREM 3.4. *Let PHSS be the set of local paraholomorphic equivalence classes of all parahermitian symmetric spaces with semisimple automorphism groups, and let SGLA be the set of all isomorphism classes of noncompact semisimple graded Lie algebras* (*in the sense of Definition* 2.1). *Then there exists a bijection Φ of PHSS onto SGLA.*

PROOF. Take $(M, I, g)\in\widehat{PHSS}$ and consider the equivalence class $[(M, I, g)]\in PHSS$. Let $\Phi_2\Phi_1((M, I, g))=(\mathfrak{g}, Z)$, and let $[(\mathfrak{g}, Z)]$ be the corresponding isomorphism class in $SGLA$. We define Φ by putting

$$\Phi([(M, I, g)])=[(\mathfrak{g}, Z)]\ .$$

Then Φ is well-defined by Lemma 3.1, and is one-to-one by Lemma 3.2. Propositions 1.6 and 2.4 imply that Φ is surjective.

The set $SGLA$ was determined by Kobayashi-Nagano [2] and we get the table of classification of parahermitian symmetric spaces with simple automorphism groups, which was already given in [1].

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Present Address:
DEPARTMENT OF MATHEMATICS
SOPHIA UNIVERSITY
KIOICHO, CHIYODA-KU, TOKYO 102