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Homogeneity Theorems on Perfect Codes in Hamming Schemes and Generalized Hamming Schemes

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Introduction

Let F be a finite set of q elements, where $q>1$, q is not necessarily assumed to be a prime power, and let X be the set of all d-tuples over F. We may assume $F=\{0, 1, \cdots, q-1\}$ without loss of generality, and we regard X as an additive group. For $\boldsymbol{x}=(x_i)\in X$, $\boldsymbol{y}=(y_i)\in X$, we define the Hamming distance on X by $\partial(x, y)=|\{i \mid x_i\neq y_i\}|$, and distance relations R_i by $R_i=\{(\boldsymbol{x}, \boldsymbol{y})\in X\times X \mid \partial(\boldsymbol{x}, \boldsymbol{y})=i\}$ for $i=0, 1, \cdots, d$. Then $(X, \{R_i\}_{i=0}^{d})$ is a symmetric association scheme, which is called a Hamming scheme, and is denoted by $H(d, q)$. A perfect e-error-correcting code in X (or a perfect e-code in $H(d, q)$) is a subset C of X such that for every $\boldsymbol{x}\in X$ there exists exactly one $\boldsymbol{c}\in C$ satisfying $\partial(\boldsymbol{x}, \boldsymbol{c})\leqq e$.

The classification of perfect e-codes in $H(d, q)$ is completed for the case $e\geqq 3$ by Tietäväinen, van Lint, Bannai, Reuvers, Best, Hong, and many others (see [4] for details). For the case $e=2$, the known perfect 2-codes have the following parameters (see [6, chapter V]):

(1) $d=1, 2$ (trivial codes)

(2) $d=5, q=2$ (binary repetition code)

(3) $d=11, q=3$ (ternary Golay code)

and they are unique up to isomorphism. Tietäväinen-van Lint [5, 10] showed that there exists no unknown perfect 2-code in $H(d, q)$, provided q is a prime power. But if q is not a prime power, the (non)existence problem remains open. We know two necessary conditions for the existence of a perfect e-code in $H(d, q)$ with q arbitrary.

The first is called the sphere packing condition. Let $S_e(\boldsymbol{c})$ denote the sphere of radius e with center $\boldsymbol{c}\in X$, i.e., $S_e(\boldsymbol{c})=\{\boldsymbol{x}\in X \mid \partial(\boldsymbol{x}, \boldsymbol{c})\leqq e\}$. Then a subset C of X is a perfect e-code in $H(d, q)$ if and only if $\{S_e(\boldsymbol{c}) \mid \boldsymbol{c}\in C\}$ is a partition of X. Thus the following condition is necessary for the

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existence of a perfect e-code in $H(d, q)$, and is called the sphere packing condition:

$$\frac{q^d}{|S_e(c)|} \text{ is an integer .}$$

The second is the generalized Lloyd's theorem: If there exists a perfect e-code in $H(d, q)$, then the following polynomial (which we shall call the Lloyd polynomial)

$$\sum_{i=0}^{e} K_i(d, q; x)$$

has e distinct integral zeros among $\{1, 2, \cdots, d\}$, where $K_i(d, q; x)$ is the Krawtchouk polynomial defined by

$$(1) \qquad K_i(d, q; x) = \sum_{j=0}^{i} (-1)^j (q-1)^{i-j} \binom{x}{j} \binom{d-x}{i-j} .$$

The main purpose of this paper is to prove the following theorem.

THEOREM 1. *Let u be the minimum zero of the Lloyd polynomial, and let k be an integer less than u. Define the subgroup Y of X by $Y = \{\boldsymbol{x} = (x_i) \in X \mid x_i = 0$ for $1 \leqq i \leqq k\}$. If there exists a perfect e-code C in $H(d, q)$, then C is distributed evenly to the cosets of X by Y, i.e., $|C \cap (\boldsymbol{a} + Y)| = |C \cap (\boldsymbol{b} + Y)|$ for any $\boldsymbol{a}, \boldsymbol{b} \in X$. In particular,*

$$\frac{q^{d-u+1}}{|S_e(c)|}$$

is an integer.

To show this theorem, we shall slightly modify the method used in [8] by K. Nomura.

By the inequality (see [7])

$$u \geqq \frac{(d-e+1)(q-1)+e}{q-1+e}$$

we have $u > 1$. Therefore the condition in our theorem is really stronger than the sphere packing condition. Moreover, in the case where $e=1$ and 2, we can calculate u explicitly:

$$u=\begin{cases}\dfrac{d(q-1)+1}{q} & \text{if} \quad e=1\\ \dfrac{2d(q-1)+4-q-\sqrt{q^2+4(q-1)(d-2)}}{2q} & \text{if} \quad e=2\,.\end{cases}$$

The above result is extended to association schemes of bilinear forms. Now we assume that q is a prime power and let X be the set of all $d\times n$ matrices over the finite field $GF(q)$ of q elements, where $d\leqq n$. Define a distance ∂ on X by $\partial(\boldsymbol{x}, \boldsymbol{y})=\operatorname{rank}(\boldsymbol{x}-\boldsymbol{y})$, for $\boldsymbol{x}, \boldsymbol{y}\in X$, and define the distance relations $R_i=\{(\boldsymbol{x}, \boldsymbol{y})\in X\times X|\partial(\boldsymbol{x}, \boldsymbol{y})=i\}$. Then $(X, \{R_i\}_{i=0}^d)$ is a symmetric association scheme, which is called an association scheme of bilinear forms, or the generalized Hamming scheme with parameters (d, n, q) (see e.g. [2], p. 306). The perfect e-code is defined similarly as in the Hamming schemes. The sphere packing condition for this case is stated as follows:

$$\frac{q^{dn}}{|S_e(\boldsymbol{c})|} \quad \text{is an integer}\,.$$

To state the generalized Lloyd's theorem for perfect e-codes in generalized Hamming schemes, we recall the definition of the hypergeometric series.

$${}_3\varphi_2\left(\begin{matrix}a_1, a_2, a_3,\\ b_1, b_2,\end{matrix}; q, x\right)=\sum_{t=0}^{\infty}\frac{(a_1; q)_t(a_2; q)_t(a_3; q)_t x^t}{(b_1; q)_t(b_2; q)_t(q; q)_t}\,,$$

where

$$(a: q)_t=\begin{cases}(1-a)\cdots(1-aq^{t-1}) & (t=1, 2, \cdots)\,,\\ 1 & (t=0)\,.\end{cases}$$

Then the generalized Lloyd's theorem insists that, if there exists a perfect e-code in the generalized Hamming scheme with parameters (d, n, q), then the following function in x (which is a polynomial of degree e in q^{-x})

$$\sum_{i=0}^{e} K_i^{Aff}(x, q^{-d}, n; q) \tag{2}$$

has e distinct integral zeros among $\{1, \cdots, d\}$, where $K_i^{Aff}(x, q^{-d}, n; q)$ is the affine q-Krawtchouk polynomial defined in [9] by

$$K_i^{Aff}(x, q^{-d}, n; q)=\frac{(q^n; q^{-1})_i(q^d; q^{-1})_i}{(q; q)_i}q^{i(i-1)/2}(-1)^i$$

$$\times {}_3\varphi_2\begin{pmatrix} q^{-i}, q^{-x}, 0 \\ q^{-n}, q^{-d} \end{pmatrix}; q, q\Big) .$$

Then we have the following Theorem 2, which we shall prove in Section 2. The method is essentially the same as that of Theorem 1, but the computation is more complicated.

THEOREM 2. *Let u be the minimum zero of the function* (2), *and let k be an integer less than u. Define the subgroup Y of X by $Y=\{\boldsymbol{x}=(x_{ij})\in X|x_{ij}=0$ for $1\leqq i\leqq k, 1\leqq j\leqq n\}$. If there exists a perfect e-code C in the generalized Hamming scheme with parameters (d, n, q), then C is distributed evenly to the cosets of X by Y, i.e., $|C\cap(\boldsymbol{a}+Y)|=|C\cap(\boldsymbol{b}+Y)|$ for any $\boldsymbol{a}, \boldsymbol{b}\in X$. In particular,*

$$\frac{q^{(d-u+1)n}}{|S_e(\boldsymbol{c})|} \quad \textit{is an integer} .$$

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§ 1. Proof of Theorem 1.

For each coset $\boldsymbol{a}+Y$ of X by Y, we can choose $\boldsymbol{a}\in X$ such that $\boldsymbol{a}=(a_1, \cdots, a_k, 0, \cdots, 0)$, so that the set of representatives of the cosets of X by Y is identified with F^k. Thus we write as follows: $\{\boldsymbol{x}=(x_i)\in X|x_i=a_i$ for $i=1, \cdots, k\}=\boldsymbol{a}+Y$ for $\boldsymbol{a}=(a_i)\in F^k$.

Let C be a perfect e-code in $H(d, q)$. We shall consider the sphere covering of each coset. For $\boldsymbol{a}\in F^k$ and $\boldsymbol{c}\in C$, we shall calculate the cardinarity $|S_e(\boldsymbol{c})\cap(\boldsymbol{a}+Y)|$. The Hamming distance on F^k will be denoted by ∂', and for each $\boldsymbol{c}\in X=F^d$, the first k-tuple $(c_1, \cdots, c_k)\in F^k$ of $\boldsymbol{c}$ will be denoted by $\boldsymbol{c}'$. Put $j=\partial'(\boldsymbol{a}, \boldsymbol{c}')$. Then it is easily seen that

$$|S_e(\boldsymbol{c})\cap(\boldsymbol{a}+Y)|=\sum_{i=j}^{e}\binom{d-k}{i-j}(q-1)^{i-j} .$$

Since $\{S_e(\boldsymbol{c})|\boldsymbol{c}\in C\}$ is a partition of X, we have

$$\boldsymbol{a}+Y=\bigcup_{\boldsymbol{c}\in C}\{S_e(\boldsymbol{c})\cap(\boldsymbol{a}+Y)\} \quad \text{(disjoint union)} ,$$

and

$$|\boldsymbol{a}+Y|=\sum_{\boldsymbol{c}\in C}|S_e(\boldsymbol{c})\cap(\boldsymbol{a}+Y)|$$

$$=\sum_{j=0}^{e}\sum_{\substack{c\in C\\ \partial'(a,c')=j}}|S_e(\boldsymbol{c})\cap(\boldsymbol{a}+Y)|$$

$$=\sum_{j=0}^{e}\sum_{\substack{c\in C\\ \partial'(a,c')=j}}\sum_{i=j}^{e}\binom{d-k}{i-j}(q-1)^{i-j}$$

$$=\sum_{j=0}^{e}\sum_{i=j}^{e}\binom{d-k}{i-j}(q-1)^{i-j}\sum_{\substack{b\in F^k\\ \partial'(a,b)=j}}|C\cap(\boldsymbol{b}+Y)| \ .$$

Since ∂' is the Hamming distance on $H(k, q)$, we can rewrite the above equation by using the adjacency matrices A_j ($j=0, 1, \cdots, k$) of the Hamming scheme $H(k, q)$, whose rows and columns are labelled by $\boldsymbol{b}\in F^k$:

$$\sum_{j=0}^{e}\sum_{i=j}^{e}\binom{d-k}{i-j}(q-1)^{i-j}A_j\xi=|Y|\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}, \tag{3}$$

where ξ is the column vector whose $\boldsymbol{b}$-entry ($\boldsymbol{b}\in F^k$) is $|C\cap(\boldsymbol{b}+Y)|$.

We shall prove that the matrix

$$A=\sum_{j=0}^{e}\sum_{i=j}^{e}\binom{d-k}{i-j}(q-1)^{i-j}A_j \tag{4}$$

is nonsingular. To show this, we shall calculate the eigenvalues of A. The next lemma is well-known.

Lemma 1. *Let k and n be positive integers. Then*

$$\binom{x}{n}=\sum_{m=0}^{n}\binom{k}{n-m}\binom{x-k}{m} \ .$$

Lemma 2. *Let $K_i(d, q; x)$ be the Krawtchouk polynomials defined by (1). Then we have the following identity:*

$$\sum_{j=0}^{e}\sum_{i=j}^{e}\binom{d-k}{i-j}(q-1)^{i-j}K_j(k, q; x)=\sum_{i=0}^{e}K_i(d, q; x) \ .$$

Proof. We have

$$\sum_{j=0}^{e}\sum_{i=j}^{e}\binom{d-k}{i-j}(q-1)^{i-j}K_j(k, q; x)$$

$$=\sum_{j=0}^{e}\sum_{i=j}^{e}\binom{d-k}{i-j}(q-1)^{i-j}\sum_{m=0}^{j}(-1)^m(q-1)^{j-m}\binom{x}{m}\binom{k-x}{j-m}$$

$$=\sum_{i,j,m=0}^{e}(-1)^m(q-1)^{i-m}\binom{x}{m}\binom{d-k}{i-j}\binom{k-x}{j-m}$$

$$=\sum_{i,j,t=0}^{e}(-1)^{i+t-e}(q-1)^{e-t}\binom{x}{i+t-e}\binom{d-k}{i-j}\binom{k-x}{j-i+e-t}$$

$$=\sum_{i,t=0}^{e}(-1)^{i+t-e}(q-1)^{e-t}\binom{x}{i+t-e}\sum_{j=0}^{e}\binom{d-k}{i-j}\binom{k-x}{j-i+e-t}$$

$$=\sum_{i,t=0}^{e}(-1)^{i+t-e}(q-1)^{e-t}\binom{x}{i+t-e}\binom{d-x}{e-t}\qquad\text{(by Lemma 1)}$$

$$=\sum_{i,j=0}^{e}(-1)^{j}(q-1)^{i-j}\binom{x}{j}\binom{d-x}{i-j}$$

$$=\sum_{i=0}^{e}\sum_{j=0}^{i}(-1)^{j}(q-1)^{i-j}\binom{x}{j}\binom{d-x}{i-j}$$

$$=\sum_{i=0}^{e}K_i(d,\, q;\, x)\ .$$

LEMMA 3. *The eigenvalues of A are given by the values of the Lloyd polynomial at $x=0, 1, \cdots, k$.*

PROOF. Since A_j is the j-th adjacency matrix of the Hamming scheme $H(k, q)$, the eigenvalues of A_j are $K_j(k, q; x)$, $(x=0, 1, \cdots, k)$ (see [2], 3.2). Thus the assertion is obvious by (4) and Lemma 2.

Since u is the minimum zero of the Lloyd polynomial, the assumption $k<u$ implies that the matrix A is nonsingular. Since the row sum of A_j is the j-th valency $K_j(k, q; 0)$, Lemma 2 shows that every row sum of A is equal to

$$\sum_{j=0}^{e}\sum_{i=j}^{e}\binom{d-k}{i-j}(q-1)^{i-j}K_j(k,\, q\colon 0)=\sum_{i=0}^{e}K_i(d,\, q;\, 0)=|S_e(\boldsymbol{c})|\ .$$

Hence the column vector ${}^t(1, \cdots, 1)$ is an eigenvector of A belonging to the eigenvalue $|S_e(\boldsymbol{c})|$. So the equation (3) has a solution

$$\xi=\frac{|Y|}{|S_e(\boldsymbol{c})|}\,{}^t(1, \cdots, 1)\ ,$$

and this is unique by the nonsingularity of A. Thus we have

$$|C\cap(\boldsymbol{b}+Y)|=\frac{|Y|}{|S_e(\boldsymbol{c})|}\quad\text{for any}\quad \boldsymbol{b}\in F^k\ ,$$

this implies that C is distributed evenly to the cosets of X by Y. Putting $k=u-1$, we have

$$|C\cap(\boldsymbol{b}+Y)|=\frac{q^{d-u+1}}{|S_e(\boldsymbol{c})|}\ .$$

Since $|C\cap(\boldsymbol{b}+Y)|$ is an integer, the theorem is proved.

§ 2. Proof of Theorem 2.

For each $k\times n$ matrix $\boldsymbol{a}=(a_{ij})$, $\boldsymbol{a}+Y$ denotes the "coset" of X by Y:

$$\{\boldsymbol{x}=(x_{ij})\in X|x_{ij}=a_{ij}\quad \text{for}\quad 1\leqq i\leqq k,\ 1\leqq j\leqq n\}\ .$$

For each $d\times n$ matrix $\boldsymbol{c}$, the submatrix consisting of upper k rows of $\boldsymbol{c}$ is denoted by $\boldsymbol{c}'$.

Assume that there exists a perfect e-code C in the generalized Hamming scheme with parameters (d, n, q). For $k\times n$ matrix $\boldsymbol{a}$ and $\boldsymbol{c}\in C$, we shall calculate the cardinarity $|S_e(\boldsymbol{c})\cap(\boldsymbol{a}+Y)|$. Put $j=\operatorname{rank}(\boldsymbol{a}-\boldsymbol{c}')$, then

$$\begin{aligned}|S_e(\boldsymbol{c})\cap(\boldsymbol{a}+Y)|&=|\{\boldsymbol{y}\in\boldsymbol{a}+Y|\operatorname{rank}(\boldsymbol{c}-\boldsymbol{y})\leqq e\}|\\&=\sum_{i=j}^{e}|\{\boldsymbol{y}\in\boldsymbol{a}+Y|\operatorname{rank}(\boldsymbol{c}-\boldsymbol{y})=i\}|\ .\end{aligned}$$

It is easily seen that

$$|\{\boldsymbol{y}\in\boldsymbol{a}+Y|\operatorname{rank}(\boldsymbol{c}-\boldsymbol{y})=i\}|=\begin{bmatrix}d-k\\i-j\end{bmatrix}q^{j(d-k)}(q^{n-j}-1)\cdots(q^{n-j}-q^{i-j-1})\ ,$$

where $\begin{bmatrix}n\\m\end{bmatrix}$ denotes the Gaussian polynomial which is equal to the number of m-dimensional subspaces of a n-dimensional vector space over $GF(q)$, namely,

$$\begin{bmatrix}n\\m\end{bmatrix}=\frac{(q^n;\,q^{-1})_m}{(q;\,q)_m}=\frac{(q^n-1)\cdots(q^n-q^{m-1})}{(q^m-1)\cdots(q^m-q^{m-1})}\ .$$

Hence

$$\begin{aligned}|\boldsymbol{a}+Y|&=\sum_{\boldsymbol{c}\in C}|S_e(\boldsymbol{c})\cap(\boldsymbol{a}+Y)|\\&=\sum_{j=0}^{e}\sum_{\substack{\boldsymbol{c}\in C\\ \operatorname{rank}(\boldsymbol{a}-\boldsymbol{c}')=j}}|S_e(\boldsymbol{c})\cap(\boldsymbol{a}+Y)|\\&=\sum_{j=0}^{e}\sum_{\substack{\boldsymbol{c}\in C\\ \operatorname{rank}(\boldsymbol{a}-\boldsymbol{c}')=j}}\sum_{i=j}^{e}\begin{bmatrix}d-k\\i-j\end{bmatrix}q^{j(d-k)}(q^{n-j}-1)\cdots(q^{n-j}-q^{i-j-1})\ .\end{aligned}$$

$$=\sum_{j=0}^{e}\sum_{i=j}^{e}\begin{bmatrix} d-k \\ i-j \end{bmatrix} q^{j(d-k)}(q^{n-j}-1)\cdots(q^{n-j}-q^{i-j-1}) \sum_{\substack{b:k\times n \text{ matrix} \\ \operatorname{rank}(a-b)=j}} |C\cap(b+Y)| .$$

We can rewrite the above equation by using the adjacency matrices A_j $(j=0, 1, \cdots, k)$ of the generalized Hamming scheme with parameters (k, n, q), whose rows and columns are labelled by $k\times n$ matrix b:

$$(5) \qquad \sum_{j=0}^{e}\sum_{i=j}^{e}\begin{bmatrix} d-k \\ i-j \end{bmatrix} q^{j(d-k)}(q^{n-j}-1)\cdots(q^{n-j}-q^{i-j-1})A_j\xi=|Y|\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

where ξ is the column vector whose b-entry (b: $k\times n$ matrix) is $|C\cap(b+Y)|$.

We shall prove that the matrix

$$(6) \qquad A=\sum_{j=0}^{e}\sum_{i=j}^{e}\begin{bmatrix} d-k \\ i-j \end{bmatrix} q^{j(d-k)}(q^{n-j}-1)\cdots(q^{n-j}-q^{i-j-1})A_j$$

is nonsingular. To show this, we shall calculate the eigenvalues of A. As an analogue of Lemma 1, we use next lemma.

LEMMA 4 ([1, (3.3.10)]). *Let k be a nonnegative integer less than d, then*

$$\sum_{j=0}^{i}\begin{bmatrix} d-k \\ i-j \end{bmatrix}\begin{bmatrix} k \\ j \end{bmatrix} q^{j(d-k-i+j)}=\begin{bmatrix} d \\ i \end{bmatrix}.$$

LEMMA 5. *We have the following identity.*

$$\sum_{j=0}^{e}\sum_{i=j}^{e}\begin{bmatrix} d-k \\ i-j \end{bmatrix} q^{j(d-k)}(q^{n-j}-1)\cdots(q^{n-j}-q^{i-j-1})K_j^{Aff}(x, q^{-k}, n; q)$$
$$=\sum_{i=0}^{e} K_i^{Aff}(x, q^{-d}, n; q) .$$

PROOF. The left hand side is

$$(7) \qquad \sum_{i=0}^{e}\sum_{j=0}^{i}\sum_{t=0}^{j}\begin{bmatrix} d-k \\ i-j \end{bmatrix} q^{j(d-k)}(q^{n-j}-1)\cdots(q^{n-j}-q^{i-j-1})$$
$$\times\frac{(q^n; q^{-1})_j(q^k; q^{-1})_j}{(q; q)_j}q^{j(j-1)/2}(-1)^j\times\frac{(q^{-j}; q)_t(q^{-x}; q)_t q^t}{(q^{-k}; q)_t(q^{-n}; q)_t(q; q)_t} .$$

By the identity

$$(q^{n-j}-1)\cdots(q^{n-j}-q^{i-j-1})(q^n; q^{-1})_j(-1)^j q^{j(j-1)/2}=\frac{(q^n-1)\cdots(q^n-q^{i-1})}{q^{j(i-j)}} ,$$

this is equal to

$$\sum_{i=0}^{e}\sum_{t=0}^{i}\sum_{j=t}^{i}\begin{bmatrix} d-k \\ i-j \end{bmatrix}(q^n-1)\cdots(q^n-q^{i-1})\frac{(q^k-q^t)\cdots(q^k-q^{j-1})}{(q^j-q^t)\cdots(q^j-q^{j-1})}$$

$$\times q^{j(d-k)-j(i-j)+tk-jt}\frac{(q^{-x};q)_t q^t}{(q^{-n};q)_t(q;q)_t}$$

$$=\sum_{i=0}^{e}\sum_{t=0}^{i}\frac{(q^{-x};q)_t q^t}{(q^{-n};q)_t(q;q)_t}(q^n-1)\cdots(q^n-q^{i-1})q^{t(d-i)}$$

$$\times\sum_{j=t}^{i}\begin{bmatrix} d-k \\ i-j \end{bmatrix}\begin{bmatrix} k-t \\ j-t \end{bmatrix}q^{(j-t)(d-k-i+j)}.$$

By Lemma 4, we have

$$\sum_{j=t}^{i}\begin{bmatrix} d-k \\ i-j \end{bmatrix}\begin{bmatrix} k-t \\ j-t \end{bmatrix}q^{(j-t)(d-k-i+j)}=\sum_{j=0}^{i-t}\begin{bmatrix} d-k \\ i-t-j \end{bmatrix}\begin{bmatrix} k-t \\ j \end{bmatrix}q^{j(d-k-i+t+j)}=\begin{bmatrix} d-t \\ i-t \end{bmatrix}.$$

Therefore (7) is equal to

$$\sum_{i=0}^{e}\sum_{t=0}^{i}q^{t(d-i)}\begin{bmatrix} d-t \\ i-t \end{bmatrix}(q^n-1)\cdots(q^n-q^{i-1})\frac{(q^{-x};q)_t q^t}{(q^{-n};q)_t(q;q)_t}$$

$$=\sum_{i=0}^{e}\sum_{t=0}^{i}(q^n;q^{-1})_i(-1)^i q^{i(i-1)/2+t(d-i)}\frac{(1-q^d)\cdots(1-q^{d-i+1})}{(1-q^d)\cdots(1-q^{d-t+1})}$$

$$\times\frac{(1-q^{i-t+1})\cdots(1-q^i)}{(1-q)\cdots(1-q^i)}\frac{(q^{-x};q)_t q^t}{(q^{-n};q)_t(q;q)_t}$$

$$=\sum_{i=0}^{e}\frac{(q^n;q^{-1})_i(q^d;q^{-1})_i}{(q;q)_i}q^{i(i-1)/2}(-1)^i\times\sum_{t=0}^{i}\frac{(q^{-i};q)_t(q^{-x};q)_t q^t}{(q^{-d};q)_t(q^{-n};q)_t(q;q)_t}$$

$$=\sum_{i=0}^{e}K_i^{Aff}(x,q^{-d},n;q).$$

Hence we have the desired identity.

Now we shall consider the eigenvalues of A. Since A_j is the j-th adjacency matrix of the generalized Hamming scheme with parameters (k, n, q), the eigenvalues of A_j are $K_j^{Aff}(x, q^{-k}, n; q)$, where $x=0, 1, \cdots, k$ (see [2], 3.5, 3.6). Hence the eigenvalues of the matrix A are

$$\sum_{j=0}^{e}\sum_{i=0}^{e}\begin{bmatrix} d-k \\ i-j \end{bmatrix}q^{j(d-k)}(q^{n-j}-1)\cdots(q^{n-j}-q^{i-j-1})K_j^{Aff}(x,q^{-k},n;q),$$

where $x=0, 1, \cdots, k$, which are equal to the values of the function discribed in (2) at x by virtue of Lemma 5.

Since u is the minimum zero of the function (2) and $k<u$, the matrix A is nonsingular. Since the row sum of A_j is the j-th valency

$K_j^{Aff}(0, q^{-k}, n; q)$, Lemma 5 shows that every row sum of A is equal to

$$\sum_{i=0}^{e} K_i^{Aff}(0, q^{-d}, n; q) = |S_e(c)| .$$

Therefore the equation (5) has the unique solution

$$|C \cap (b+Y)| = \frac{|Y|}{|S_e(c)|} \quad \text{for any } k \times n \text{ matrix } b .$$

This implies that C is distributed evenly to the cosets of X by Y. Putting $k=u-1$, we have,

$$|C \cap (b+Y)| = \frac{q^{(d-u+1)n}}{|S_e(c)|} .$$

Since $|C \cap (b+Y)|$ is an integer, we have Theorem 2.

ADDED IN PROOF. Recently Laura Chihara proved in her thesis (The University of Minnesota, 1985) that there exist no perfect e-dode in generalized Hamming scheme.

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