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Near the Hamiltonian $H=\sum_{j=1}^{n-1}\frac{1}{2}(p_j^2+q_j^2)+(p_n^2+q_n^2)$

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Introduction

Consider a Hamiltonian system

$$(H)\qquad \dot{p}=-H_q\ ,\quad \dot{q}=H_p\ ,$$

where $p, q\in \boldsymbol{R}^n$ and $H: \boldsymbol{R}^{2n}\to\boldsymbol{R}$, or concisely

$$(H)\qquad \dot{z}=JH'(z)\ ,$$

where $z=(p, q)$ and $J=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}$ with I being the identity of $\boldsymbol{R}^n$.

Ekeland-Lasry [3] obtained

THEOREM 1. *If an energy surface S of H is a C^2 boundary of a compact, strictly convex subset C of $\boldsymbol{R}^{2n}$ and if there are positive numbers r_1, r_2 with*

$$(0.1)\qquad r_2<\sqrt{2}\,r_1$$

such that

$$(0.2)\qquad r_1B\subset C\subset r_2B$$

where B is the closed ball, then there exist at least n *distinct periodic solutions of (H) on S.*

(Hamiltonian H can be taken arbitrarily as long as it is C^2 and has S as a regular energy surface. See Lemma 1.5 of [5].)

We identify $\boldsymbol{C}^n$ with $\boldsymbol{R}^{2n}$ by $z_j=p_j+iq_j$ $(j=1, \cdots, n)$.

In this note we have

THEOREM 2. *We put*

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(0.3) $$|z|_\omega^2=\sum_{j=1}^{n-1}\frac{1}{2}|z_j|^2+|z_n|^2$$

and $Q_\omega=\{z\in C^n;\ |z|_\omega\leqq 1\}$.

Let C be a compact, strictly convex subset of R^{2n} with C^2 boundary S. Suppose there are positive numbers r_1 and r_2 with

(0.4) $$r_2<2^{1/4}r_1$$

such that

(0.5) $$r_1Q_\omega\subset C\subset r_2Q_\omega\ .$$

Then there exist at least n *distinct periodic solutions of (H) on S.*

We remark that Q_ω is a critical case which violates the condition (0.2) with (0.1). Ambrosetti-Mancini [2] gave another proof of Theorem 1 with an extention, using *Dual Action Principle* developed in [1], which will be explained in the next section. For Theorem 2, we also use the principle and count the cohomological index, proposed in [4], of invariant sets under an S^1 action a little carefully.

§ 1. Dual action principle.

This method was developed in [1] and gave another proof of Theorem 1 with an extension. We explain it briefly and collect some facts for later use.

Let S be the C^2 boundary of a compact strictly convex subset C of R^{2n} (not necessarily satisfying (0.2) or (0.5)), whose interior contains the origin.

Take $\beta>2$ and determine the Hamiltonian $H=H(z)$: $R^{2n}\to R$ by

(1.1) $$H^{-1}(1)=S$$

(1.2) $$H\text{: }\beta\text{-homogeneous}\quad (H(\lambda z)=\lambda^\beta H(z),\ \lambda>0)\ .$$

Then H is convex, so the Legendre transform $G=G(u)$ is obtained, which is α-homogeneous $(1/\alpha+1/\beta=1,\ 1<\alpha<2)$.

Put $E=\left\{u\in L^\alpha(0,\ 2\pi;\ C^n);\ \int u\equiv\int_0^{2\pi}u(t)dt=0\right\}$ and define a C^1-function f: $E\to R$ by

$$f(u)=-\frac{1}{2}\int u\cdot Lu+\int G(u)\ ,$$

where $z=Lu$ is determined by $u=-J\dot z$ and $\int z=0$. We also consider u

as a complex n-vector and $u \cdot Lu$ means the usual Euclidian inner product, considering C^n as the real $2n$-dimensional Euclidian space.

Finally put

$$M=\left\{u \in E \backslash 0;\ \int u \cdot Lu=\alpha \int G(u)\right\}.$$

Then M is a C^1 Banach submanifold of E and $f: M \rightarrow \boldsymbol{R}$ satisfies the Palais-Smale condition.

And we have a one to one correspondence between critical points of f in M and periodic orbits on S.

Futhermore we have

$$m=\min\{f(u);\ u \in M\}>0\,, \tag{1.3}$$

and for $\mu \in \boldsymbol{Z}_+=\{1, 2, \cdots\}$

$$u \in M \Rightarrow u^\mu \equiv \mu^\delta u(\mu \cdot) \in M\,, \tag{1.4}$$

$$f(u^\mu)=\mu^\vartheta f(u) \quad \text{for} \quad u \in M\,, \tag{1.5}$$

where $\delta=1/(2-\alpha)$ and $\vartheta=\alpha/(2-\alpha)=\alpha\delta$.

And for $u \in E$ with $\int u \cdot Lu>0$, there is the unique $\lambda>0$ such that $\lambda u \in M$. λ is explicitly determined by

$$\lambda^{2-\alpha}=\alpha \,⨍\, G(u) \Big/ ⨍\, u \cdot Lu\,, \qquad ((5) \text{ in } [2]) \tag{1.6}$$

where $⨍$ means $(1/2\pi)\int$.

So

$$\lambda=\left[\alpha \,⨍\, G(u)\right]^\delta \quad \text{if} \quad ⨍\, u \cdot Lu=1 \tag{1.7}$$

and because of

$$f(v)=\frac{\pi}{\vartheta}\alpha \,⨍\, G(v) \quad \text{for} \quad v \in M \qquad ((6) \text{ in } [2]) \tag{1.8}$$

we have

$$\begin{aligned} f(\lambda u) &= \frac{\pi}{\vartheta}\alpha \,⨍\, G(\lambda u) \\ &= \frac{\pi}{\vartheta}\lambda^\alpha \alpha \,⨍\, G(u) \\ &= \frac{\pi}{\vartheta}\left[\alpha \,⨍\, G(u)\right]^{2\delta} \quad \text{if} \quad ⨍\, u \cdot Lu=1\,. \end{aligned} \tag{1.9}$$

§ 2. Harmonic oscillators.

We consider the Hamiltonian

$$H_2(z)=\frac{1}{2}\sum_{j=1}^{n}\omega_j|z_j|^2\,, \tag{2.1}$$

where $0<\omega_1\leqq\omega_2\leqq\cdots\leqq\omega_n=1$ are angular frequencies.

Since the complex version of (H) is $\dot{z}=2i(\partial/\partial\bar{z})H(z)$ and $2(\partial/\partial\bar{z}_j)H_2(z)=\omega_j z_j$, (H) becomes componentwisely

$$\dot{z}_j=i\omega_j z_j\,,\quad j=1,2,\cdots,n\,. \tag{2.2}$$

Hence the j-th periodic solution with multiplicity $\mu\in Z_+$ is

$$c_j e^{i\omega_j t}a_j\,;\quad c_j\in C\backslash 0\,,\quad 0\leqq t\leqq 2\mu\pi/\omega_j \tag{2.3}$$

where a_j is the j-th vector of the usual orthogonal basis of C^n, that is, $a_j=(0,\cdots,\overset{j}{1},\cdots,0)$.

We put

$$H_\beta(z)=\frac{1}{\beta}|z|_\infty^\beta \tag{2.4}$$

for $\beta>2$.

H_β is β-homogeneous and satisfies (1.1) if S is

$$\{z\in C^n;\ |z|_\infty=\beta^{1/\beta}\}\,. \tag{2.5}$$

The Legendre transform $G_0(u)$ of $H_\beta(z)$ is

$$G_0(u)=\frac{1}{\alpha}|u|_\tau^\alpha \tag{2.6}$$

where $|u|_\tau^2=\sum_{j=1}^{n}\tau_j|u_j|^2$, $\tau_j=1/\omega_j$.

We attach the suffix 0 for the notations as G, f, M, m, etc. derived from H_β. An elementary calculation gives

LEMMA 1. *The corresponding critical point of f_0 in M_0 to* (2.3) *is*

$$v_j^\mu(t)\equiv\mu^s\tau_j^{\vartheta/2}e^{\mu i t}a_j \tag{2.7}$$

and, writing v_j^1 as v_j, we have

$$f_0(v_j^\mu)=\mu^\vartheta f_0(v_j)=(\mu\tau_j)^\vartheta\frac{\pi}{\vartheta} \tag{2.8}$$

We also have

(2.9) $$Lv_j^\mu = \frac{1}{\mu} v_j^\mu ,$$

(2.10) $$\int v_j^\mu L v_k^\nu = \delta^{\mu\nu}\delta_{jk}\mu^\vartheta \tau_j^\vartheta .$$

Thus, for S defined by (2.5)

$$m_0 = \min\{f_0(u);\ u \in M_0\} = f_0(v_n) = \frac{\pi}{\vartheta} ,$$

hence

(2.11) $$f_0(v_j) = \tau_j^\vartheta m_0 \quad \textit{and} \quad f_0(v_j^\mu) = (\mu\tau_j)^\vartheta m_0 .$$

We write G_i, f_i, M_i, m_i for $r_i Q_\omega$ in § 1 ($i=1, 2$) as G, f, M, m for C. Then (0.5) implies

(2.12) $$G_1(u) \leqq G(u) \leqq G_2(u) .$$

Also we have ($i=1, 2$)

(2.13) $$G_i(u) = R_i^\alpha G_0(u) ,$$

where

(2.14) $$R_i = r_i \beta^{-1/\beta} ,$$

hence (0.4) becomes

(2.15) $$R_2/R_1 < 2^{1/4} .$$

Further we have

LEMMA 2. $m_i = m_0 R_i^{2\vartheta}$ *and* $R_i^\vartheta e^{it} a_n \in M_i$ *attains* m_i ($i=1, 2$).

PROOF. By Lemma 1, $w = ce^{it}a_n$, with some $c>0$, attains m_i. Since $w \in M_i$, we have

$$\begin{aligned} \int w \cdot Lw &= \alpha \int G_i(w) \\ &= \alpha \int R_i^\alpha G_0(w) && \text{(by (2.13))} \\ &= R_i^\alpha \int |w_n|^\alpha && \text{(by (2.6))} \\ &= R_i^\alpha c^\alpha . \end{aligned}$$

On the other hand $Lw = w$, hence $\int w \cdot Lw = \int |c|^2 = c^2$.

Thus we have $c^2=R_i^\alpha c^\alpha$, hence $c=R_i^\vartheta$. So

$$\begin{aligned} m_i &= f_i(w) \\ &= m_0\alpha \int G_i(w) && \text{(by (1.8))} \\ &= m_0\alpha \int R_i^\alpha G_0(w) && \text{(by (2.13))} \\ &= m_0 R_i^\alpha c^\alpha \\ &= m_0 R_i^{2\vartheta} . && \text{Q.E.D.} \end{aligned}$$

Also we have

LEMMA 3. $m_1 \leqq m \leqq m_2$.

PROOF. Let $w=R_2^\vartheta e^{it}a_n \in M_2$ be the point which attains m_2 by Lemma 2. Then for some $\lambda>0$, $\lambda w \in M$.

$$\begin{aligned} \lambda^{2-\alpha} &= \alpha \int G(w) \Big/ \int w\cdot Lw && \text{(by (1.6))} \\ &\leqq \alpha \int R_2^\alpha G_0(w)/R_2^{2\vartheta} && \text{(by (2.12) and (2.13))} \\ &= R_2^\alpha R_2^{\vartheta\alpha}/R_2^{2\vartheta} \\ &= 1 . \end{aligned}$$

Hence

$$\begin{aligned} m &\leqq f(\lambda w) \\ &= m_0\alpha \int G(\lambda w) && \text{(by (1.8))} \\ &\leqq m_0\alpha \int R_2^\alpha G_0(\lambda w) && \text{(by (2.12))} \\ &= m_0 R_2^\alpha \lambda^\alpha \alpha \int G_0(w) \\ &\leqq m_0 R_2^\alpha R_2^{\vartheta\alpha} \\ &= m_0 R_2^{2\vartheta} \\ &= m_2 . && \text{(by Lemma 2)} \end{aligned}$$

Now, since $R_1^\alpha G_0(u) \leqq G(u)$, we have

$$\min\{G(u);\ |u|=1\} \geqq \frac{1}{\alpha} R_1^\alpha ,$$

thus this R_1 plays the role of r in Lemma 3 of [2].

Because b in the lemma equals π, we have $m \geqq (\pi/\vartheta)R_1^{2\vartheta} = m_0 R_1^{2\vartheta} = m_1$ by (1.5) in the proof of the lemma. Q.E.D.

§ 3. Proof of Theorem 2.

By (2.15), we can take a number $\nu > 1$ with

$$\nu^{(\vartheta+1)/2\vartheta} \frac{R_2}{R_1} < 2^{1/4} . \tag{3.1}$$

First we obtain

LEMMA 4. *If $u \in M$ with $f(u) \leqq (\sqrt{2}\,\nu)^{\vartheta} m$, then $f_1(\lambda u) \leqq 2^{\vartheta} m_1 \nu^{-1}$, where $\lambda > 0$ is determined so as to $\lambda u \in M_1$.*

PROOF.

$$\begin{aligned}
\lambda^{2-\alpha} &= \alpha \int G_1(u) \Big/ \int u \cdot Lu && \text{(by (1.6))} \\
&= \alpha \int G_1(u) \Big/ \alpha \int G(u) && (u \in M) \\
&\leqq 1 , && \text{(by (2.12))}
\end{aligned}$$

so $\lambda \leqq 1$. Hence

$$\begin{aligned}
f_1(\lambda u) &= m_0 \alpha \int G_1(\lambda u) && \text{(by (1.8))} \\
&= \lambda^{\alpha} m_0 \alpha \int G_1(u) \\
&\leqq m_0 \alpha \int G(u) && \text{(by (2.12))} \\
&= f(u) && \text{(by (1.8))} \\
&\leqq (\sqrt{2}\,\nu)^{\vartheta} m \\
&\leqq (\sqrt{2}\,\nu)^{\vartheta} m_2 && \text{(by Lemma 3)} \\
&= (\sqrt{2}\,\nu)^{\vartheta} m_0 R_2^{2\vartheta} && \text{(by Lemma 2)} \\
&\leqq (\sqrt{2}\,\nu)^{\vartheta} m_0 (\nu^{-(\vartheta+1)/2\vartheta} R_1 2^{1/4})^{2\vartheta} && \text{(by (3.1))} \\
&= \sqrt{2}^{\,\vartheta} \nu^{\vartheta} m_0 \nu^{-\vartheta-1} R_1^{2\vartheta} 2^{\vartheta/2} \\
&= 2^{\vartheta} \nu^{-1} m_0 R_1^{2\vartheta} \\
&= 2^{\vartheta} m_1 \nu^{-1} . && \text{(by Lemma 2)}
\end{aligned}$$

Q.E.D.

We denote $M^c = \{u \in M;\ f(u) \leqq c\}$ and $M_i^c = \{u \in M_i;\ f_i(u) \leqq c\}$, $i = 1, 2$.

Now we use the theory of cohomological index [4]. For the definition and the properties, we refer [5], [3]. The index of an invariant set K under an S^1 action is denoted by $i(K)$.

On E, hence on M or M_1, we consider the usual S^1 action

$$A_s u = u(s+\cdot) \quad \text{for} \quad s \in S^1 = R/2\pi Z .$$

We put $\eta = (\sqrt{2}\,\nu)^{\vartheta} m$ and $\eta_1 = 2^{\vartheta} m_1 \nu^{-1}$. Then we have

LEMMA 5. $i(M^{\eta}) = 1$.

PROOF. There is a periodic solution in M^{η} attaining m, hence $1 \leqq i(M^{\eta})$ by 2° and 5° of Lemma 1.13 of [5].

Lemma 4 gives an equivariant map from M^{η} into $M_1^{\eta_1}$, so 2° of the lemma gives $i(M^{\eta}) \leqq i(M_1^{\eta_1})$.

From Lemma 1, m_1 is a critical value of multiplicity 1 and next critical value is $2^{\vartheta} m_1$. Since $\eta_1 < 2^{\vartheta} m_1$, we have $i(M_1^{\eta_1}) \leqq 1$. Q.E.D.

For $i = 1, 2, \cdots$, we define $\Gamma_i = \{K \subset M;\ K\text{: compact invariant subset of } M,\ i(K) \geqq i\}$ and

$$\kappa_i = \inf_{K \in \Gamma_i} \operatorname{Max} f(K) .$$

Then $\kappa_1, \kappa_2, \cdots$ are critical values of f. If $\kappa_i = \kappa_j$, $i < j$, then i (the set of critical points of the level $\kappa_i = \kappa_j) \geqq j - i + 1$ (III of [3]).

LEMMA 6. $\kappa_2 \geqq \eta$.

PROOF. For any $K \in \Gamma_2$, by 2° of Lemma 1.13 of [5], K cannot be involved in M^{η}. Hence $\operatorname{Max} f(K) > \eta$. Therefore $\kappa_2 \geqq \eta$. ($i(K) \geqq 2$, $i(M^{\eta}) = 1$ by Lemma 5.) Q.E.D.

LEMMA 7. $\kappa_{n+1} \leqq (2\sqrt{2})^{\vartheta} m$.

The proof is given in § 4.

PROOF OF THEOREM 2. If, in the critical values $\kappa_2, \cdots, \kappa_{n+1}$, at least two of them coincide, there exist infinitely many geometrically distinct critical points on the level, so the theorem is given.

Therefore we only consider the case

$$\sqrt{2}^{\,\vartheta} m < \eta \leqq \kappa_2 < \kappa_3 < \cdots < \kappa_{n+1} \leqq (2\sqrt{2})^{\vartheta} m .$$

In this case we have geometrically distinct n critical points, we shall prove the theorem, as follows.

If critical points c_i and c_j corresponding to κ_i and κ_j respectively are geometrically same, that is, $c_i=c^{\mu_i}$ and $c_j=c^{\mu_j}$ for some $c\in M$ and $\mu_i<\mu_j$.

Since $f(c_j)=\kappa_j\leqq(2\sqrt{2})^{\vartheta}m<3^{\vartheta}m$, μ_j must be 2 and $\mu_1=1$. Therefore $c_j=c_i^2$. This is a contradiction because

$$\kappa_j=2^{\vartheta}\kappa_i\geqq 2^{\vartheta}\eta>(2\sqrt{2})^{\vartheta}m\,. \qquad \text{Q.E.D.}$$

§ 4. Proof of Lemma 7.

For $\xi=(\xi_1, \cdots, \xi_n, \xi_{n+1})\in \boldsymbol{C}^{n+1}$, we put

$$u_\xi=\xi_1 v_1+\cdots+\xi_n v_n+\xi_{n+1}v_n^2\in E\,. \tag{4.1}$$

Then, from (2.10), we have

$$\begin{aligned}\int u_\xi\cdot Lu_\xi &=2^{\vartheta}|\xi_1|^2+\cdots+2^{\vartheta}|\xi_{n-1}|^2+|\xi_n|^2+2^{\vartheta}|\xi_{n+1}|^2\\ &\equiv\|\xi\|^2\,.\end{aligned} \tag{4.2}$$

We put $\Sigma=\{\xi\in\boldsymbol{C}^{n+1};\ \|\xi\|=1\}$ and for $\xi\in\Sigma$, we define

$$\lambda(\xi)=\left[\alpha\int G_0(u_\xi)\right]^{\delta}, \tag{4.3}$$

then (1.7) implies $\varphi_0(\xi)\equiv\lambda(\xi)u_\xi\in M_0$ and we have

$$f_0\circ\varphi_0(\xi)=m_0\left[\alpha\int G_0(u_\xi)\right]^{2\delta}\,. \qquad \text{(by (1.9))} \tag{4.4}$$

Now we have

Lemma 8. $\operatorname{Max} f_0\circ\varphi_0(\Sigma)=2^{\vartheta}m_0$.

Proof. This will be done by only a little careful change of notations as follows:

We put

$$h(\xi)\equiv\alpha G_0(u_\xi)=(2^{\vartheta+1}|\xi_1|^2+\cdots+2^{\vartheta+1}|\xi_{n-1}|^2+|\xi_n+\xi_{n+1}2^{\delta}e^{it}|^2)^{\alpha/2}\,,$$

then the following estimate gives the lemma by (4.4):

$$\operatorname{Max}\left\{\int h(\xi);\ \xi\in\Sigma\right\}=2^{\alpha/2}\,. \tag{4.5}$$

From the shape of h, we may consider only real ξ_j (the phase of ξ_{n+1} is cancelled under the integration from 0 to 2π), so we change ξ_j to

x_j. For $x=(x_1, x_2, \cdots, x_{n+1})\in \boldsymbol{R}^{n+1}$, we write

$$h(x, t)=(2^{\vartheta+1}x_1^2+\cdots+2^{\vartheta+1}x_{n-1}^2+x_n^2+2^{2\delta}x_{n+1}^2+2^{\delta+1}x_nx_{n+1}\cos t)^{\alpha/2}$$

and

$$H(x)=\int h(x, t)dt .$$

We seek the maximum of $H(x)$ under the constraint

$$\|x\|^2=2^{\vartheta}x_1^2+\cdots+2^{\vartheta}x_{n-1}^2+x_n^2+2^{\vartheta}x_{n+1}^2=1 .$$

Let $b=(b_1, \cdots, b_{n+1})$ be the point with $\|b\|=1$ which attains the maximum of $H(x)$.

Then there is a Lagrange multiplier λ with

$$H_{x_1}=\int\frac{\alpha}{2}(\ \ )^{\alpha/2-1}\cdot 2^{\vartheta+1}\cdot 2b_1=\lambda\cdot 2^{\vartheta+1}b_1$$

$$\vdots$$

$$H_{x_{n-1}}=\int\frac{\alpha}{2}(\ \ )^{\alpha/2-1}\cdot 2^{\vartheta+1}\cdot 2b_{n-1}=\lambda\cdot 2^{\vartheta+1}b_{n-1}$$

$$H_{x_n}=\int\frac{\alpha}{2}(\ \ )^{\alpha/2-1}(2b_n+2^{\delta+1}b_{n+1}\cos t)=\lambda\cdot 2b_n \tag{4.6}$$

$$H_{x_{n+1}}=\int\frac{\alpha}{2}(\ \ )^{\alpha/2-1}(2^{2\delta}\cdot 2b_{n+1}+2^{\delta+1}b_n\cos t)=\lambda\cdot 2^{\vartheta+1}b_{n+1} . \tag{4.7}$$

First we consider the case

(i) $b_j\neq 0$ for *some* $j=1, 2, \cdots, n-1$.

Then we have

$$\lambda=2\int\frac{\alpha}{2}(\ \ )^{\alpha/2-1} . \tag{4.8}$$

Remarking that $2\delta=\vartheta+1$, from (4.7), we have

$$b_n\int\frac{\alpha}{2}(\ \ )^{\alpha/2-1}\cdot 2^{\delta+1}\cos t=0 .$$

If the integral part $\int\cdots=0$, then (4.6) implies

$$2b_n\int\frac{\alpha}{2}(\ \ )^{\alpha/2-1}=b_n\cdot 2\lambda .$$

Thus, since $\lambda\neq 0$, we have $b_n=0$ by (4.8).
If the integral part $\int\cdots\neq 0$, we have also $b_n=0$.
Then we have

$$\begin{aligned}H(b)&=\int h(b,\,t)\\&=\int(2^{\vartheta+1}(b_1^2+\cdots+b_{n-1}^2+b_{n+1}^2))^{\alpha/2}\\&=2^{\alpha/2}\cdot\|b\|^{\alpha}\\&=2^{\alpha/2}\ .\end{aligned}$$

Next we consider the case

(ii) $b_j=0$ for *any* $j=1, 2, \cdots, n-1$.

In this case, the problem becomes to maximize

$$\int(x_n^2+2^{2\delta}x_{n+1}^2+2^{\delta+1}x_nx_{n+1}\cos t)^{\alpha/2}$$

under the constraint $x_n^2+2^{\vartheta}x_{n+1}^2=1$.
We put $y=x_n$ and $z=2^{\vartheta/2}x_{n+1}$. Then

$$\begin{aligned}H(y,\,z)&=\int(y^2+2z^2+2\sqrt{2}\,yz\cos t)^{\alpha/2}\\&=\int|y+\sqrt{2}\,ze^{it}|^{\alpha}\end{aligned}$$

and the constraint is

$$y^2+z^2=1\ .$$

In this case

$$\begin{aligned}H(y,\,z)&=\int|y+\sqrt{2}\,ze^{it}|^{\alpha}\\&=\|y+\sqrt{2}\,ze^{it}\|_{L^\alpha}^{\alpha}\\&\leqq(\|y+ze^{it}\|_{L^\alpha}+(\sqrt{2}-1)\|ze^{it}\|_{L^\alpha})^{\alpha}\\&=(\|y+ze^{it}\|_{L^\alpha}+(\sqrt{2}-1)|z|)^{\alpha}\ .\end{aligned}\tag{4.9}$$

We put $y=\cos\sigma$ and $z=\sin\sigma$. Then

$$\|y+ze^{it}\|_{L^\alpha}^{\alpha}=\int|y+ze^{it}|^{\alpha}$$

$$=\int (y^2+z^2+2yz\cos t)^{\alpha/2}$$
$$=\int (1+\sin 2\sigma\cdot\cos t)^{\alpha/2}\ .$$

We define for $-1\leqq a\leqq 1$

$$f(a)=\int (1+a\cos t)^{\alpha/2}dt\ .$$

Then

$$f'(a)=\int \frac{\alpha}{2}(1+a\cos t)^{\alpha/2-1}\cos t dt$$

and

$$f''(a)=\int \frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)(1+a\cos t)^{\alpha/2-2}\cos^2 t dt$$
$$<0\ .$$

Also we have $f'(0)=0$.
This means $f(a)$, $-1\leqq a\leqq 1$, attains its maximum 1 at $a=0$.
Thus we have, under $y^2+z^2=1$,

$$\|y+ze^{it}\|_{L^\alpha}^\alpha\leqq 1\ .$$

Therefore, by (4.9), we have

$$H(y, z)\leqq(1+(\sqrt{2}-1)|z|)^\alpha$$
$$\leqq 2^{\alpha/2}\ ,$$

proving (4.5). Q.E.D.

LEMMA 9. *We define* $\varphi\colon \Sigma\to M$ *as* $\varphi_0\colon \Sigma\to M_0$. *Then*

$$\operatorname{Max} f\circ\varphi(\Sigma)\leqq(2\sqrt{2})^{\vartheta} m\ .$$

PROOF. For any $\xi\in\Sigma$, we have

$$f\circ\varphi(\xi)=m_0\left[\alpha\int G(u_\xi)\right]^{2\delta} \qquad \text{(by (1.9))}$$
$$\leqq m_0\left[\alpha\int G_2(u_\xi)\right]^{2\delta} \qquad \text{(by (2.12))}$$
$$=m_0\left[\alpha\int G_0(u_\xi)\right]^{2\delta}R_2^{\alpha 2\delta} \qquad \text{(by (2.13))}$$
$$=f_0\circ\varphi_0(\xi)\cdot R_2^{2\vartheta}$$

$$\begin{aligned}
&\leqq 2^{\vartheta} m_0 R_2^{2\vartheta} && \text{(by Lemma 8)}\\
&< 2^{\vartheta} m_0 (R_1 \cdot 2^{1/4})^{2\vartheta} && \text{(by (2.15))}\\
&= (2\sqrt{2})^{\vartheta} m_0 R_1^{2\vartheta} &&\\
&= (2\sqrt{2})^{\vartheta} m_1 &&\\
&\leqq (2\sqrt{2})^{\vartheta} m\,. && \text{(by Lemma 3)}
\end{aligned}$$

Q.E.D.

PROOF OF LEMMA 7. To compute the index of Σ, we use 6° of Lemma 1.13 of [5]. We can find $2(n+1)$-dimensional invariant subspace F and Σ is equivariantly isomorphic to $F \cap \mathcal{S}$ (for $\mathcal{S}$, see 6°).

Hence $i(\Sigma) = (1/2)\dim F = n+1$.

We consider the equivariant map $\varphi : \Sigma \to M$ and put $K = \varphi(\Sigma)$. Then 2° of Lemma 1.13 in [5] gives $K \in \Gamma_{n+1}$ and $\mathrm{Max}(K) \leqq (2\sqrt{2})^{\vartheta} m$ by Lemma 9.

Hence $\kappa_{n+1} \leqq \mathrm{Max}\, f(K) \leqq (2\sqrt{2})^{\vartheta} m$. Q.E.D.

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