TOKYO J. MATH.
VOL. 8, NO. 2, 1985

On the Range of Wave Operators

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(Communicated by K. Kojima)

Introduction

In this article we shall study the nonlinear wave equation:

$$(1)\qquad \begin{aligned} &u_{tt}-u_{xx}=g(u)+h(x,t)\ , && (x,t)\in(0,\pi)\times\boldsymbol{R}\ ,\\ &u(0,t)=u(\pi,t)=0\ , && t\in\boldsymbol{R}\ ,\\ &u(x,t+2\pi/\alpha)=u(x,t)\ , && (x,t)\in(0,\pi)\times\boldsymbol{R}\ ,\end{aligned}$$

where $g(s)$ is a continuous function on $\boldsymbol{R}$ such that $g(0)=0$ and that $h(x,t)$ is a given $(2\pi/\alpha)$-periodic function of t. Here $\alpha>0$ is a number such that the set:

$$(2)\qquad \{j^2-\alpha^2k^2;\ j\in\boldsymbol{N},\ k\in\boldsymbol{Z}\}$$

is a discrete set of $\boldsymbol{R}$.

We shall consider the existence of weak solutions of (1) without assumptions of monotonicity and Lipschitz continuity of $g(s)$. Let A be the differential operator $\partial_t^2-\partial_x^2$ acting on functions which satisfy the Dirichlet boundary conditions in x and which are $(2\pi/\alpha)$-periodic in t. Assume that $g(s)$ has at most linear growth and let N be the Nemytskii operator defined by

$$(Nu)(x,t)=g(u(x,t))\qquad \text{for all}\quad u\in L^2\ .$$

Now our main theorem is as follows.

THEOREM 1. *Let $\beta,\gamma\in\boldsymbol{R}$, $\beta<\gamma$ be two and consecutive eigenvalues of A, and suppose that there exist numbers $\delta>0$ and $\rho\geqq 0$ such that*

$$(3)\qquad \beta+\delta\leqq\frac{g(s)}{s}\leqq\gamma-\delta\qquad \text{for}\quad |s|\geqq\rho\ .$$

Received April 9, 1984

Then for all h in a dense subset Ξ of L^2, (1) has a weak solution (or equivalently, L^2-closure of $\{Au-Nu;\ u\in D(A)\}=L^2$).

When α is a rational number, H. Hofer [3] proved the denseness of the range $\{Au-Nu; u\in D(A)\}$ under the assumption of *global Lipschitz continuity of* $g(s)$. He reduced this problem to an abstract variational one in a Hilbert space and applied his general theory. Our result is an extension of his result, because we don't assume any Lipschitz condition on $g(s)$. Instead of the variational methods, we approximate the wave equation (1) by telegraph equations:

$$(4)\qquad \begin{aligned} &u_{tt}-u_{xx}+\varepsilon u_t=g(u)+h(x,t)\ , && (x,t)\in(0,\pi)\times\boldsymbol{R}\ ,\\ &u(0,t)=u(\pi,t)=0\ , && t\in\boldsymbol{R}\ ,\\ &u(x,t+2\pi/\alpha)=u(x,t)\ , && (x,t)\in(0,\pi)\times\boldsymbol{R}\ ,\end{aligned}$$

for $\varepsilon>0$. The advantages of this method are as follows: under weak condition on $g(s)$ we can construct approximate solutions and we don't need the decomposition of the space L^2 as the other methods.

When $\alpha\in\boldsymbol{Q}$ and $g(s)$ is monotone, we can prove that the operator $Au-Nu$ is *surjective*. That is, we have the following theorem.

THEOREM 2. *In addition to the hypotheses of Theorem 1, assume $\alpha\in\boldsymbol{Q}$ and $g(s)$ or $-g(s)$ is nondecreasing. Then the operator $Au-Nu$ is surjective, that is,*

$$L^2=\{Au-Nu;\ u\in D(A)\}\ .$$

More precisely, for $h\in L^2$, let u^ε be the solutions of the telegraph equations (4). Then there exists a subsequence u^{ε_n} ($\varepsilon_n\to 0$) which converges weakly in L^2 to a weak solution $u\in L^2$ of the wave equation (1).

§1. Telegraph equations.

We first introduce some function spaces. Let C_∞ be the real vector space of arbitrarily often continuously differentiable functions in $(0,\pi)\times\boldsymbol{R}$, which are $(2\pi/\alpha)$-periodic in $t\in\boldsymbol{R}$ and satisfy $u(0,t)=u(\pi,t)=0$ for all $t\in\boldsymbol{R}$. We denote by L^2 the completion of C_∞ with respect to the norm

$$\|u\|=(u,u)^{1/2}\ ,$$

where $(u,v)=\int_\Omega uv\,dx\,dt$, $\Omega=(0,\pi)\times(0,2\pi)$, and by H^1_ω the completion of C_∞ with respect to the norm

$$\|u\|_{H^1}=(\|u\|^2+\|u_t\|^2+\|u_x\|^2)^{1/2}\ .$$

We can observe that the set of functions:

$$\{\sin jx \exp i\alpha kt;\ j \in \boldsymbol{N},\ k \in \boldsymbol{Z}\}$$

is dense in L^2 and in H^1_ω. Then we can define a selfadjoint linear operator A and a closed linear operator A_ε ($\varepsilon>0$) in L^2 by

$$D(A)=\{u \in L^2;\ u(x, t)=\sum_{(j,k)\in \boldsymbol{N}\times\boldsymbol{Z}} u_{jk} \sin jx\, e^{i\alpha kt} \quad (u_{j,-k}=\bar{u}_{jk}),$$
$$\text{where } \sum_{(j,k)\in \boldsymbol{N}\times\boldsymbol{Z}} (j^2-\alpha^2k^2)^2|u_{jk}|^2<\infty\}\ ,$$

$$Au=\sum_{(j,k)\in \boldsymbol{N}\times\boldsymbol{Z}} (j^2-\alpha^2k^2)u_{jk} \sin jx\, e^{i\alpha kt}$$
$$\text{for all } \quad u(x, t)=\sum u_{jk} \sin jx e^{i\alpha kt} \in D(A)\ ,$$

$$D(A_\varepsilon)=D(A)\cap H^1_\omega\ ,$$
$$A_\varepsilon u=Au+\varepsilon u_t \qquad \text{for all} \quad u \in D(A_\varepsilon)\ .$$

By the assumption (2), the set of the eigenvales of A:

$$\sigma(A)=\{j^2-\alpha^2k^2;\ j \in \boldsymbol{N},\ k \in \boldsymbol{Z}\}$$

is a discrete set of $\boldsymbol{R}$.

DEFINITION.
(i) For a given $h \in L^2$, a function $u \in L^2$ is said to be *a weak solution of the wave equation* (1), if and only if

$$u \in D(A)\ , \qquad Au=Nu+h\ .$$

(ii) For a given $h \in L^2$, a function $u \in L^2$ is said to be *a weak solution of the telegraph equation* (4), if and only if

$$u \in D(A)\cap H^1_\omega\ , \qquad A_\varepsilon u=Nu+h\ .$$

These are equivalent to the usual definition of weak solutions in the distribution sense.

Let L_ε be a closed linear operator in L^2 defined by

$$D(L_\varepsilon)=D(A)\cap H^1_\omega\ ,$$
$$L_\varepsilon u=A_\varepsilon u-\frac{1}{2}(\beta+\gamma)u \qquad \text{for all} \quad u \in D(L_\varepsilon)\ ,$$

where $\beta, \gamma \in \boldsymbol{R}$, $\beta<\gamma$ are two consecutive elements of $\sigma(A)$. Then we have

LEMMA 3. *L_ε has a bounded inverse*

$$L_\varepsilon^{-1}\colon L^2 \longrightarrow D(A)\cap H^1_\omega$$

and satisfies

$$(5)\qquad \|L_\varepsilon^{-1}u\| \leqq \frac{2}{\gamma-\beta}\|u\| \qquad \text{for all} \quad u \in L^2\,,$$

$$(6)\qquad \|L_\varepsilon^{-1}u\|_{H^1} \leqq C_\varepsilon \|u\| \qquad \text{for all} \quad u \in L^2\,.$$

SKETCH OF PROOF. We find

$$L_\varepsilon^{-1}u = \sum_{(j,k)\in N\times Z} \frac{1}{(j^2-\alpha^2k^2-(1/2)(\beta+\gamma))+i\varepsilon\alpha k} u_{jk} \sin jx e^{i\alpha kt}\,,$$

for all $u=\sum u_{jk}\sin jx\, e^{i\alpha kt} \in L^2$.

Since β, γ are two consecutive numbers in $\sigma(A)=\{j^2-\alpha^2k^2;\, j\in N,\, k\in Z\}$, we have

$$\left| j^2-\alpha^2k^2-\frac{1}{2}(\beta+\gamma)\right| \geqq \frac{1}{2}(\gamma-\beta) \qquad \text{for all} \quad (j,k)\in N\times Z\,,$$

and there exists a number $C_\varepsilon>0$ such that

$$\frac{j^2+k^2}{(j^2-\alpha^2k^2-(1/2)(\beta+\gamma))^2+\varepsilon^2\alpha^2k^2} \leqq C_\varepsilon \qquad \text{for all} \quad (j,k)\in N\times Z\,.$$

Hence we can get the desired inequalities.

Now we can solve the telegraph equations.

THEOREM 4. *Assume that $g(s)$, β, γ and δ satisfy the hypotheses of Theorem 1.*
Then for all $h\in L^2$, the telegraph equation (4) has a weak solution $u^\varepsilon \in D(A)\cap H^1_\omega$. Moreover, there exists a number $R>0$ that is independent of $\varepsilon>0$ and satisfies

$$(7)\qquad \|u^\varepsilon\| \leqq R \qquad \textit{for all} \quad \varepsilon>0\,.$$

PROOF. We define a nonlinear operator $T_\varepsilon\colon L^2\to L^2$ by

$$T_\varepsilon u = L_\varepsilon^{-1}\Big(Nu-\frac{1}{2}(\beta+\gamma)u+h\Big) \qquad \text{for all} \quad u\in L^2\,.$$

Then (4) is equivalent to $u=T_\varepsilon u$.
Since (6) holds, it follows from Sobolev's lemma that T_ε is compact and continuous in L^2. And by the assumption (3), we get

$$\left\|Nu-\frac{1}{2}(\beta+\gamma)u\right\| \leqq \Big(\frac{\gamma-\beta}{2}-\delta\Big)\|u\|+C\,.$$

Hence by (5), we find

$$\|T_\varepsilon u\| = \left\|L_\varepsilon^{-1}\Big(Nu - \frac{1}{2}(\beta+\gamma)u + h\Big)\right\|$$
$$\leqq \frac{2}{\gamma-\beta}\Big(\frac{\gamma-\beta}{2} - \delta\Big)\|u\| + C \,.$$

If we choose $R>0$ large enough, for all $\varepsilon>0$ the ball of radius R centered at the origin is mapped into itself by the compact operator T_ε. Then the results follows from Schauder's fixed point theorem.

§2. Proof of Theorem 1 and a remark.

To prove our theorems we shall use the following lemma.

LEMMA 5. *For all* $u \in D(A) \cap H_\omega^1$ *and* $h \in C_\infty$, *we have*

(8) $$(Au, u_t) = (u, u_t) = 0 \,,$$

(9) $$(Nu, u_t) = 0 \,,$$

(10) $$(h, u_t) = -(h_t, u) \,.$$

Now we shall prove Theorem 1.

PROOF OF THEOREM 1. By Theorem 4, for a given $h \in C_\infty$ there exists $u^\varepsilon \in D(A) \cap H_\omega^1$ such that

$$Au^\varepsilon + \varepsilon u_t^\varepsilon = Nu^\varepsilon + h \,.$$

Taking L^2 scalar product with u_t^ε, we find

$$(Au^\varepsilon, u_t^\varepsilon) + \varepsilon\|u_t^\varepsilon\|^2 = (Nu^\varepsilon, u_t^\varepsilon) + (h, u_t^\varepsilon) \,.$$

Using Lemma 5 and (7), we get

$$\varepsilon\|u_t\|^2 = -(h_t, u^\varepsilon)$$
$$\leqq \|h_t\|\,\|u^\varepsilon\|$$
$$\leqq R\|h_t\| \,.$$

Therefore it is easily seen that

$$\varepsilon u_t^\varepsilon \longrightarrow 0 \qquad \text{strongly in } L^2 \text{ as } \varepsilon \longrightarrow 0 \,,$$

$$Au^\varepsilon - Nu^\varepsilon = h - \varepsilon u_t^\varepsilon \longrightarrow h \qquad \text{strongly in } L^2 \text{ as } \varepsilon \longrightarrow 0 \,.$$

Hence any $h \in C_\infty$ belongs to L^2-*closure of* $\{Au - Nu;\, u \in D(A)\}$. Since C_∞

is dense in L^2, the proof is completed.

REMARK 1. Even if $g(s)$ doesn't satisfy the growth condition (3), if the solutions u^ϵ of the telegraph equation (4) are a priori bounded in L^2, we can prove the denseness of the range $\{Au-Nu; u\in D(A)\}$. For example, using P. J. McKenna's a priori estimates, we have the following.

THEOREM (cf. P. J. McKenna [4]). *Let $\alpha=1$ and let $\beta,\gamma\in\boldsymbol{R}$, $\beta<\gamma<0$ be two consecutive eigenvalues of A. Then there exists a constant $\lambda>0$ with following property. If $g(s)$ is continuous and satisfies*

$$\beta-\lambda+\delta\leqq\frac{g(s)}{s}\leqq\frac{\beta+\gamma}{2}-\delta \qquad \textit{for all} \quad s\leqq-\rho\ ,$$

$$\frac{\beta+\gamma}{2}+\delta\leqq\frac{g(s)}{s}\leqq\gamma+\lambda-\delta \qquad \textit{for all} \quad s\geqq\rho\ ,$$

for some constants $\delta>0$ and $\rho\geqq0$, then

$$L^2=L^2\textit{-closure of } \{Au-Nu;\ u\in D(A)\}\ .$$

P. J. McKenna [4] assumed the *monotonicity* of $g(s)$ in addition to the hypotheses of the above theorem and proved the existence of a solution of (1) for *all* $h\in L^2$.

§3. Proof of Theorem 2 and remarks.

To prove Theorem 2 we shall use the following lemma.

LEMMA 6. *Suppose that α is a rational number. Let $\{u_n\}_{n=1}^\infty$ be a sequence in $D(A)$ such that*

$$Au_n\longrightarrow Au\ ,\quad u_n\longrightarrow u \quad \textit{weakly in } L^2 \textit{ as } n\longrightarrow\infty\ ,$$

then we have

$$(Au_n, u_n)\longrightarrow(Au, u) \quad \textit{as} \quad n\longrightarrow\infty\ .$$

PROOF. We can see that A is a self-adjoint operator with the closed range:

$$R(A)=\overline{\mathrm{span}}\{\sin jx\, e^{i\alpha kt};\ j^2-\alpha^2k^2\neq0,\ (j,k)\in\boldsymbol{N}\times\boldsymbol{Z}\}\ .$$

Thus L^2 admits an orthogonal decomposition $L^2=R(A)\oplus N(A)$ where

$$N(A)=\overline{\mathrm{span}}\{\sin jx\, e^{i\alpha kt};\ j^2-\alpha^2k^2=0\ ,\quad (j,k)\in\boldsymbol{N}\times\boldsymbol{Z}\}$$

is the kernnel of A. Hence A^{-1} is well-defined from $R(A)$ into $R(A)$ and

$$A^{-1}v=\sum_{j^2-\alpha^2k^2\neq 0}\frac{1}{j^2-\alpha^2k^2}v_{jk}\sin jx\, e^{i\alpha kt}$$
$$\text{for all}\quad v=\sum_{j^2-\alpha^2k^2\neq 0}v_{jk}\sin jx\, e^{i\alpha kt}\in R(A)\ .$$

Since α is a rational number, we have

$$\frac{j+|k|}{|j^2-\alpha^2k^2|}\leqq C_\alpha \qquad \text{for all}\quad (j,k)\in \boldsymbol{N}\times\boldsymbol{Z} \text{ with } j^2-\alpha^2k^2\neq 0\ .$$

So we find

$$\|A^{-1}v\|_{H^1}\leqq C\|v\| \qquad \text{for all}\quad v\in R(A)\ ,$$

i.e.,

$$\|Pu\|_{H^1}\leqq C\|Au\| \qquad \text{for all}\quad u\in D(A)\ ,$$

where $P: L^2\to R(A)$ is the orthogonal projection. Suppose that $u_n\to u$, $Au_n\to Au$ weakly in L^2 as $n\to\infty$. Then there exists a constant $C>0$ which is independent of n such that

$$\|Pu_n\|_{H^1}\leqq C \qquad \text{for all}\quad n\in\boldsymbol{N}\ .$$

It follows from Sobolev's lemma that

$$Pu_n\longrightarrow Pu \qquad \text{strongly in } L^2 \text{ as}\quad n\longrightarrow\infty\ .$$

Since A is a self-adjoint operator, we obtain

$$(Au_n, u_n)=(Au_n, Pu_n)$$
$$\longrightarrow (Au, Pu)=(Au, u) \quad \text{as}\quad n\longrightarrow\infty\ .$$

Thus the proof is completed.

PROOF OF THEOREM 2. By Theorem 4 there exists a solution $u^\varepsilon\in D(A)\cap H^1_\omega$ of

$$Au^\varepsilon+\varepsilon u^\varepsilon_t=Nu^\varepsilon+h\ . \tag{11}$$

Taking L^2 scalar product with Au^ε, and by (8), we have

$$\|Au^\varepsilon\|^2=(Nu^\varepsilon+h, Au^\varepsilon)\ .$$

Hence by (7) we get

(12)
$$\|Au^\varepsilon\| \leqq \|Nu^\varepsilon + h\| \\ \leqq C .$$

By (11) and (12) we obtain

(13)
$$\|\varepsilon u_t^\varepsilon\| \leqq C .$$

Using the estimates (7), (12) and (13), we can extract a subsequence $\varepsilon_n \to 0$ such that

$$u^{\varepsilon_n} \longrightarrow u , \qquad Au^{\varepsilon_n} \longrightarrow Au , \qquad \varepsilon_n u_t^{\varepsilon_n} \longrightarrow v \qquad \text{weakly in } L^2$$

for some $u \in D(A)$ and $v \in L^2$. Then we have for any $\phi \in C_\infty$,

$$(\varepsilon_n u_t^{\varepsilon_n}, \phi) = -\varepsilon_n (u^{\varepsilon_n}, \phi_t) .$$

Letting $\varepsilon_n \to 0$, we have

$$(v, \phi) = 0 \qquad \text{for all} \quad \phi \in C_\infty .$$

Since C_∞ is dense in L^2, we have

$$v = 0 ,$$

i.e.,

$$\varepsilon_n u_t^{\varepsilon_n} \longrightarrow 0 \qquad \text{weakly in } L^2 .$$

From Lemma 6 we get

(14)
$$(Au^{\varepsilon_n}, u^{\varepsilon_n}) \longrightarrow (Au, u) .$$

We set $\lambda = \text{sign}(\beta + \gamma)$, then $\lambda g(s)$ is a nondecreasing function. Hence we have

$$\lambda(Nu^\varepsilon - N\xi, u^\varepsilon - \xi) \geqq 0 \qquad \text{for all} \quad \xi \in L^2 ,$$

i.e.,

(15)
$$\lambda(Au^\varepsilon + \varepsilon u_t^\varepsilon - h - N\xi, u^\varepsilon - \xi) \geqq 0 \qquad \text{for all} \quad \xi \in L^2 .$$

Passing to the limit in (15), we obtain by (8) and (14)

$$\lambda(Au - h - N\xi, u - \xi) \geqq 0 \qquad \text{for all} \quad \xi \in L^2 .$$

Now we shall use Minty's device. For $\psi \in L^2$ and $\tau > 0$, set $\xi = u - \tau\psi$. After dividing by τ we get

$$\lambda(Au - h - N(u - \tau\psi), \psi) \geqq 0 .$$

Letting $\tau\to 0$, we have

$$\lambda(Au-h-Nu,\psi)\geqq 0 \quad \text{for all} \quad \psi\in L^2 .$$

Hence we conclude that

$$Au=Nu+h .$$

REMARK 2. Using the telegraph equations, we can prove many existence theorems, which is obtained by the variational methods, under the condition where $g(s)$ *interacts with one eigenvalue of* A. For example, we can prove the following theorem due to H. Brezis [1].

THEOREM (H. Brezis [1]). *Assume* $g(s)$ *is nondecreasing and satisfies*

$$|g(s)|\leqq\gamma|s|+C , \quad s\in\boldsymbol{R} , \quad \textit{for some constants } \gamma<3 \textit{ and } C .$$

Assume $f(x,t)\in L^\infty$ *admits a decomposition of the form*

$$f(x,t)=f^*(x,t)+f^{**}(x,t)$$

with

$$\int_0^{2\pi}\int_0^{\pi} f^*(x,t)(p(x+t)-p(t-x))dxdt=0$$

for all 2π*-periodic* $p(t)\in L^2_{\mathrm{loc}}(\boldsymbol{R})$, *and*

$$g(-\infty)+\delta\leqq f^{**}(x,t)\leqq g(\infty)-\delta , \quad \textit{for some } \delta>0, \textit{ for all } x,\ t .$$

Let u^ε $(\varepsilon>0)$ *be a weak solution of the telegraph equation:*

$$\begin{aligned} &u^\varepsilon_{tt}-u^\varepsilon_{xx}+\varepsilon u^\varepsilon_t+g(u^\varepsilon)=f(x,t) , && (x,t)\in(0,\pi)\times\boldsymbol{R} ,\\ &u^\varepsilon(0,t)=u^\varepsilon(\pi,t)=0 , && t\in\boldsymbol{R} ,\\ &u^\varepsilon(x,t+2\pi)=u^\varepsilon(x,t) , && (x,t)\in(0,\pi)\times\boldsymbol{R} . \end{aligned}$$

Then there exists a subsequence u^{ε_n} $(\varepsilon_n\to 0)$ *which converges weakly in* L^2 *to a weak solution* $u\in L^\infty$ *of the wave equation:*

$$\begin{aligned} &u_{tt}-u_{xx}+g(u)=f(x,t) , && (x,t)\in(0,\pi)\times\boldsymbol{R} ,\\ &u(0,t)=u(\pi,t)=0 , && t\in\boldsymbol{R} ,\\ &u(x,t+2\pi)=u(x,t) , && (x,t)\in(0,\pi)\times\boldsymbol{R} . \end{aligned}$$

REMARK 3. Theorem 2 and Remark 2 can be extended to the equations of the form

$$u_{tt}-u_{xx}+F(x, t, u)=0 .$$

REMARK 4. If we assume *the Lipschitz condition*:

$$\beta+\delta\leqq\frac{g(u)-g(v)}{u-v}\leqq\gamma-\delta \qquad \text{for all} \quad u\neq v ,$$

then the sequence of solutions u^ε of the telegraph equations (4) converges *strongly* in L^2 to the unique solution $u\in L^2$ of the wave equation (1).

REMARK 5. There exists a solution of the wave equation (1) that is not a limit of solutions u^ε of the telegraph equations (4). For example, let $\alpha=1$ and we shall consider

$$Au+Nu=0 , \tag{16}$$

$$Au^\varepsilon+\varepsilon u_t^\varepsilon+Nu^\varepsilon=0 , \tag{17}$$

where $g(s)$ is nondecreasing and satisfies

$$g(0)=0 ,$$

$$\underline{\lim}_{s\to 0}\frac{g(s)}{s}>3 ,$$

$$\overline{\lim}_{|s|\to\infty}\frac{g(s)}{s}<3 .$$

Taking L^2 scalar product of (17) and u_t^ε, we get by (8), (9) and the monotonicity of $g(s)$

$$u^\varepsilon=0 \qquad \text{for all} \quad \varepsilon>0 .$$

On the other hand, J. M. Coron [2] proved the existence of a nontrivial solution of the wave equation (16).

Acknowledgement. The author would like to thank Prof. Haruo Sunouchi for his advice and encouragement.

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