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Schur Orthogonality Relations for Certain Non Square Integrable Representations of Real Semisimple Lie Groups.

Hisaichi MIDORIKAWA

Tsuda College

Introduction

Let G be a noncompact connected real semisimple Lie group with finite center. An irreducible unitary representation is called square integrable if its matrix coefficients are square integrable with respect to the Haar measure on G. Let (π, H) be a square integrable irreducible unitary representation of G (see, for instance, Theorem 4.5.9.3 in [8]). There exists a positive constant d_π such that

$$\int_G (\pi(x)\phi, \psi)\overline{(\pi(x)\phi', \psi')}dx = d_\pi^{-1}(\phi, \phi')\overline{(\psi, \psi')} \tag{0.1}$$

for all ϕ, ψ, ϕ', ψ' in H, where dx is the Haar measure on G. The identity (0.1) is called the Schur orthogonality relation for the representation π.

Our main purpose in this paper is to give an analogous result to this relation for certain non square integrable unitary representations of G. We shall state our results more precisely. Let K be a maximal compact subgroup of G. The coset space G/K is a Riemannian symmetric space. Let $d(p, q)$ $(p, q \in G/K)$ be the distance from p to q. We define $d(x)$, $x \in G$ by $d(x) = d(xo, o)$ where o is the origin of G/K.

THEOREM I. *Let (π, H) be an irreducible unitary representation of G. We assume that there exists a K-finite vector ϕ_0 in H such that*

$$0 < \lim_{\varepsilon \to +0} \varepsilon \int_G |(\pi(x)\phi_0, \phi_0)|^2 e^{-\varepsilon d(x)} dx < \infty \ .$$

Then for each K-finite vectors ϕ, ψ, ϕ', ψ' in H, we have

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$$\lim_{\varepsilon\to+0}\varepsilon\int_G(\pi(x)\phi,\psi)\overline{(\pi(x)\phi',\psi')}e^{-\varepsilon d(x)}dx=d_\pi^{-1}(\phi,\phi')\overline{(\psi,\psi')}$$

where d_π *is a positive constant which is independent of* $\phi, \psi, \cdots$.

REMARK 1. When G is of real rank one, the irreducible components of a reducible principal series representation of G satisfy the assumption in Theorem I (cf. Lemma 1.4 in [4]). Applying the same arguments as in §3 [4], we can prove that all unitary principal series representations with regular infinitesimal character satisfy our assumption.

We shall state our proof briefly after the following preparations. Let $\mathfrak{g}_C$ be the complexification of the Lie algebra $\mathfrak{g}$ of G. Canonically the universal enveloping algebra $\mathfrak{u}(\mathfrak{g})$ of $\mathfrak{g}_C$ acts on $C^\infty(G)$ to the left and right. We denote the left (resp. right) action on $C^\infty(G)$ by bf (resp. fb) for f in $C^\infty(G)$. Let us define a subspace $H(G, \chi)$ of $C^\infty(G)$, for a given character χ, by the following;

(0.2) a function f in $C^\infty(G)$ belongs to $H(G,\chi)$ if f satisfies (1) $zf=\chi(z)f$ for all z in $\mathfrak{z}$, (2) $\lim_{\varepsilon\to+0}\varepsilon\int_G|(b_1fb_2)(x)|^2e^{-\varepsilon d(x)}dx$ is finite for each b_1, b_2 in $\mathfrak{u}(\mathfrak{g})$.

The space $H(G, \chi)$ is a topological G-module with the seminorm $|\ \ |$ defined by $|f|^2=\lim_{\varepsilon\to+0}\varepsilon\int_G|f(x)|^2e^{-\varepsilon d(x)}dx$, $f\in C^\infty(G)$ (see Lemma 2.4). Consider an irreducible unitary representation (π, H) of G satisfying the assumption in Theorem I. We denote by $H_\pi(G)$ the subspace of $C^\infty(G)$ which is generated by K-finite elements of π, and define a Hermitian form $(\ ,\ )$ on $H_\pi(G)$ by

$$(f,g)=\lim_{\varepsilon\to+0}\varepsilon\int_G f(x)\overline{g(x)}e^{-\varepsilon d(x)}dx \quad \textit{for } f,\ g \textit{ in } H_\pi(G)\ . \tag{0.3}$$

In our proof of Theorem I, one difficulty is to show that $(\ ,\ )$ is positive definite. To overcome this difficulty, we characterize $H(G, \chi)$ in the following Theorem II and Theorem III. We define $N=\{f\in H(G,\chi);\ |f|=0\}$, $L^2(G)=$the set of all square integrable functions on G.

THEOREM II. *Let* $H(G,\chi)$ *be the subspace of* $C^\infty(G)$ *defined by* (0.2). *Then* $N=H(G,\chi)\cap L^2(G)$.

THEOREM III. *Assume that there exists an element* f_0 *in* $H(G,\chi)$ *such that* $|f_0|>0$. *Then the character* χ *is not real regular* (*for the definition of real character, see* §7).

Theorem II and III are proved by an estimation for the solutions of ordinary first order differential equations combining with Tauberian theorem of Hardy-Littlewood.

Let us now state our proof of the fact: $(\ ,\ )$ is positive definite on $H_\pi(G)$. Let χ be the infinitesimal character of π. Since $H(G, \chi)$ is a $\mathfrak{g}$-module (cf. Lemma 5 in [4]), we see that $H_\pi(G) \subseteq H(G, \chi)$. Let N be the null space as in Theorem II, and assume that $N \neq 0$. Then the G-module N is decomposed into the square integrable irreducible representations of G. Consequently, by a result of Harish-Chandra (cf. [7], Proposition 15.13), χ is real. Therefore, by Theorem II, we get $N=0$. Hence the Hermitian form $(\ ,\ )$ in (0.3) is positive definite.

Finally we shall state an application of Theorem I. Let $\mathfrak{C}(G)$ be the Schwartz space in the sense of Harish-Chandra. A distribution T on G is called tempered if it extends to a continuous linear form on $\mathfrak{C}(G)$.

THEOREM IV. *Let (π, H) be the same as in Theorem* I. *We define a distribution θ_π on G by $\theta_\pi(f) = \operatorname{trace} \int_G f(x)\pi(x)dx$, for all C^∞-functions f on G with compact support. Then θ_π is tempered.*

REMARK 2. For the case where (π, H) is square integrable, Harish-Chandra proves that the character of π is tempered. By Theorem I, we can prove Theorem IV applying the same arguments as in Theorem 10.2.1.1 in [9].

The contents of this paper are as follows. After the preparations in §1, we introduce the function spaces $H(G, \chi)$ and $H_\tau(G, \chi)$ in §2 which are closely related to the representations considered in Theorem I. We shall prove in this section that $H(G, \chi)$ is a topological G-module with a seminorm. We also study in §3, the differential equations concerning the functions in $H(G, \chi)$ and $H_\tau(G, \chi)$. Throughout in §3, §4 and §5, we estimate the asymptotic behaviour at infinity of K-finite functions in $H_\tau(G, \chi)$. Furthermore in §4, we shall give the proofs of Theorems II and III. In §6, we study the Schur orthogonality relations of a non square integrable irreducible unitary representation of G. Finally in §7, we shall show that the parabolic subgroup of G along which some K-finite function in $H(G, \chi)$ has nonzero constant term is uniquely determined by χ.

§1. Notations and preliminaries.

Let G_C be a connected complex semisimple Lie group and G a connected noncompact real form of G_C. By $\mathfrak{g}_C$ and $\mathfrak{g}$, we denote the Lie algebras of G_C and G respectively. For each subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we denote the complexification of $\mathfrak{h}$ by $\mathfrak{h}_C$ and the universal enveloping algebra of $\mathfrak{h}_C$ by $\mathfrak{u}(\mathfrak{h})$. Then the algebra $\mathfrak{u}(\mathfrak{h})$ acts on the set of all C^∞-functions $C^\infty(H)$ on H (H is the analytic Lie subgroup of G with Lie algebra $\mathfrak{h}$) to the left (resp. to the right) as follows; for each X in $\mathfrak{h}$,

$$(Xf)(x)=\frac{d}{dt}f(\exp tXx)|_{t=0} \quad \left(\text{resp. } (fX)(x)=\frac{d}{dt}f(x\exp tX)|_{t=0}\right)$$

for x in H and f in $C^\infty(H)$. We shall denote these actions of $\mathfrak{u}(\mathfrak{h})$ by bf and fb (or resp. $f(b;x)$ and $f(x;b)$) for b in $\mathfrak{u}(\mathfrak{h})$. Let $\tilde{\mathfrak{a}}$ be a Cartan subalgebra of $\mathfrak{g}$. We denote the root system of $(\mathfrak{g}_C, \tilde{\mathfrak{a}}_C)$ by $\Phi(\tilde{\mathfrak{a}})$, the Weyl group of $(\mathfrak{g}_C, \tilde{\mathfrak{a}}_C)$ by $W(\tilde{\mathfrak{a}}_C)$ and the ring of all $W(\tilde{\mathfrak{a}}_C)$-invariant polynomial functions on the dual space of $\tilde{\mathfrak{a}}_C$ by $I(\tilde{\mathfrak{a}})$. For each root α in $\Phi(\tilde{\mathfrak{a}})$, we put $\mathfrak{g}_\alpha=\{X\in\mathfrak{g}_C;\ \mathrm{ad}(H)X=\alpha(H)X,\ H\in\tilde{\mathfrak{a}}_C\}$. Then we have $\mathfrak{g}_C=\tilde{\mathfrak{a}}_C\oplus\sum_{\alpha\in\Phi(\tilde{\mathfrak{a}})}\mathfrak{g}_\alpha$ and $\dim g_\alpha=1$.

Consider a positive root system $\Phi^+(\tilde{\mathfrak{a}})$ of $\Phi(\tilde{\mathfrak{a}})$, and put $\mathfrak{n}^+=\sum_{\alpha\in\Phi^+(\tilde{\mathfrak{a}})}\mathfrak{g}_\alpha$, $\mathfrak{n}^-=\sum_{-\alpha\in\Phi^+(\tilde{\mathfrak{a}})}\mathfrak{g}_\alpha$. Then we have $\mathfrak{g}_C=\mathfrak{n}^+\oplus\tilde{\mathfrak{a}}_C\oplus\mathfrak{n}^-$. Let $\mathfrak{z}$ be the center of $\mathfrak{u}(\mathfrak{g})$. Then there is a unique isomorphism $\gamma=\gamma_{\mathfrak{g}/\tilde{\mathfrak{a}}}$ of $\mathfrak{z}$ to $\mathfrak{u}(\tilde{\mathfrak{a}})$ which is called the Chevalley isomorphism, such that

$$(1.1)\qquad z-\gamma(z)\in\mathfrak{u}(\mathfrak{g})\mathfrak{n}^+ \quad \text{for all } z \text{ in } \mathfrak{z}\,.$$

Let ρ be one half the sum of all positive roots in $\Phi(\tilde{\mathfrak{a}})$. Regard $\mathfrak{u}(\tilde{\mathfrak{a}})$ as the algebra of polynomial functions on the dual space $\tilde{\mathfrak{a}}_C$, and define

$$(1.2)\qquad \mu_{\mathfrak{g}/\tilde{\mathfrak{a}}}(z)(\lambda)=\gamma_{\mathfrak{g}/\tilde{\mathfrak{a}}}(z)(\lambda-\rho) \quad \text{for each linear form } \lambda \text{ on } \tilde{\mathfrak{a}}_C \text{ and } z \text{ in } \mathfrak{z}.$$

Then $\mu_{\mathfrak{g}/\tilde{\mathfrak{a}}}$ is an algebra isomorphism of $\mathfrak{z}$ onto $I(\tilde{\mathfrak{a}})$ which is called the Harish-Chandra isomorphism.

Let P be a parabolic subgroup of G (for simplicity P is said to be a p.s.g.r. of G). Then we have the Langlands decomposition $P=MAN$, where A is a split component of P, N is the unipotent radical of P, $Z(A)$ is the centralizer of A in P and M is a closed subgroup of G which is isomorphic to $Z(A)/A$. We shall denote the root system of (P, A), the Lie algebras of M, A, N respectively by $\Phi(A)$, $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{n}$. Let p be an element in P and $\mathrm{Ad}(p)|_{\mathfrak{n}}$ the restriction of $\mathrm{Ad}(p)$ to $\mathfrak{n}$. Define a function d_P on P by

$$(1.3)\qquad d_P(p)=(\det(\mathrm{Ad}(p)|_{\mathfrak{n}}))^{1/2}\,.$$

We see that there is a unique linear form ρ_P on $\mathfrak{a}$ such that

$$(1.4)\qquad d_P(a)=\exp(\rho_P(\log a)) \quad \text{for all } a \text{ in } A\,.$$

Let K be a maximal compact subgroup of G with Lie algebra $\mathfrak{k}$ and θ the Cartan involution corresponding to K. We fix a minimal p.s.g.r. P_0 of G with θ-stable split component A_0 and denote the Langlands decomposition of P_0 by $P_0=M_0A_0N_0$. A p.s.g.r. P of G is called standard if A is contained in A_0. We see that any p.s.g.r. of G is conjugate to a standard p.s.g.r. under an inner automorphism of K (cf, for instance G. Warner [8]).

Let $P=MAN$ be a standard p.s.g.r. of G. We put $\mathfrak{m}_1=\mathfrak{m}\oplus\mathfrak{a}$. Then $\mathfrak{m}_1$ is reductive having the same rank as $\mathfrak{g}$. Consider a Cartan subalgebra $\tilde{\mathfrak{a}}$ of $\mathfrak{g}$ containing $\mathfrak{a}$. Choosing a positive root system $\Phi^+(\tilde{\mathfrak{a}})$ suitably we have $\Phi(A)=\{\alpha|_{\mathfrak{a}};\, \alpha\in\Phi^+(\tilde{\mathfrak{a}})$ and $\alpha\not\equiv 0$ on $\mathfrak{a}\}$ where $\alpha|_{\mathfrak{a}}$ is the restriction of α to $\mathfrak{a}$.

Let $\gamma_{\mathfrak{m}_1/\tilde{\mathfrak{a}}}$ be the Harish-Chandra isomorphism of $\mathfrak{z}(\mathfrak{m}_1)$ into $I_{\mathfrak{m}_1}(\tilde{\mathfrak{a}})$, $I_{\mathfrak{m}_1}(\tilde{\mathfrak{a}})$ is the ring of all polynomial functions on the dual space of $\tilde{\mathfrak{a}}_C$ invariant under the Weyl group of $((\mathfrak{m}_1)_C, \tilde{\mathfrak{a}}_C)$. Then we have the following two lemmas (cf. p. 120 Proposition 29, part II and p. 60 Corollary 10, part I in [7]).

LEMMA 1. *There exists a unique isomorphism* $\gamma_{\mathfrak{g}/\mathfrak{m}_1}$ *of* $\mathfrak{z}$ *into* $\mathfrak{z}(\mathfrak{m}_1)$ *such that* $\gamma_{\mathfrak{g}/\tilde{\mathfrak{a}}}=\gamma_{\mathfrak{m}_1/\tilde{\mathfrak{a}}}\circ\gamma_{\mathfrak{g}/\mathfrak{m}_1}$. *Furthermore for each* z *in* $\mathfrak{z}$, $z-\gamma_{\mathfrak{g}/\mathfrak{m}_1}(z)\in\theta(\mathfrak{n}_C)\mathfrak{u}(\mathfrak{g})$.

LEMMA 2. *The algebra* $\mathfrak{z}(\mathfrak{m}_1)$ *is a free* $\gamma_{\mathfrak{g}/\mathfrak{m}_1}(\mathfrak{z})$ *module with finite index.*

Following Harish-Chandra we define an isomorphism $\mu_P=\mu_{\mathfrak{g}/\mathfrak{m}_1}$ of $\mathfrak{z}$ into $\mathfrak{z}(\mathfrak{m}_1)$ by

$$\mu_{\mathfrak{g}/\mathfrak{m}_1}(z)=d_P\circ\mu_{\mathfrak{g}/\mathfrak{m}_1}(z)\circ d_P^{-1}\quad\text{for } z \text{ in } \mathfrak{z}\,. \tag{1.5}$$

Then we have

$$\mu_{\mathfrak{g}/\tilde{\mathfrak{a}}}=\mu_{\mathfrak{m}_1/\tilde{\mathfrak{a}}}\circ\mu_{\mathfrak{g}/\mathfrak{m}_1}\,. \tag{1.6}$$

Finally we shall state the following two integral formulae on G which will be applied frequently to our arguments in this paper. For the minimal p.s.g.r. $P_0=M_0A_0N_0$ of G, we choose a Cartan subalgebra $\tilde{\mathfrak{a}}_0$ containing $\mathfrak{a}_0$ and a positive root system $\Phi^+(\tilde{\mathfrak{a}}_0)$ of $(\mathfrak{g}_C, (\tilde{\mathfrak{a}}_0)_C)$ satisfying $\Phi(A_0)=\{\alpha|_{\mathfrak{a}_0};\, \alpha\in\Phi^+(\tilde{\mathfrak{a}}_0)$ and $\alpha\not\equiv 0$ on $\mathfrak{a}_0\}$. Define a function $D=D_G$ on A_0 by

$$D(a)=\prod_{\substack{\alpha\in\Phi^+(\tilde{\mathfrak{a}}_0)\\ \alpha\not\equiv 0 \text{ on } \mathfrak{a}_0}}|\exp\alpha(\log a)-\exp(-\alpha(\log a))|\,,\quad a\in A_0\,. \tag{1.7}$$

Let dx, da, dn and dk be the Haar measures on G, A_0, N_0 and K respectively. We normalize dk as $\int dk=1$. Then we have the following two lemmas (cf. Lemma 10.1.16 and Proposition 10.1.7 in [3]).

LEMMA 3. *For each C^∞-function f on G with compact support, we have* $\int_G f(x)dx=\int_K\int_{A_0}\int_{N_0} f(kan)d_{P_0}(a)^2 dndadk$.

LEMMA 4. *There exists a positive constant c_G such that*

$$\int_G f(x)dx=c_G\int_{A_0^+}\int_{K\times K} f(kak')D(a)dkdk'da$$

where A_0^+ is the positive Weyl chamber of (P_0, A_0).

REMARK 1. The set $G_0=KA_0^+K$ is open dense in G.

§2. Topological G-module $H(G, \chi)$.

Let K be a fixed maximal compact subgroup of G. Then the homogeneous space G/K is a Riemannian symmetric space of noncompact type. Let $d(p, q)$ $(p, q\in G/K)$ be the distance which induced from the Killing form on $\mathfrak{g}$ and define a function d on G by

$$d(x)=d(xo, o)\ , \quad x\in G \quad \text{where } o \text{ is the origin in } G/K\ . \tag{2.1}$$

Let $C^\infty(G)$ be the ring of C^∞-functions on G. For a given character χ of $\mathfrak{z}$, we define a subspace $H(G, \chi)$ of $C^\infty(G)$ as follows.

(2.2) A function f in $C^\infty(G)$ belongs to $H(G, \chi)$ if f satisfies
(1) $\lim_{\varepsilon\to+0}\varepsilon\int_G |f(b_1; x; b_2)|^2\exp(-\varepsilon d(x))dx<\infty$ for b_1, b_2 in $\mathfrak{u}(\mathfrak{g})$,
(2) $zf=\chi(z)f$ for all z in $\mathfrak{z}$.

We put

$$|f|^2=\lim_{\varepsilon\to+0}\varepsilon\int_G |f(x)|^2\exp(-\varepsilon d(x))dx \quad \text{for } f \text{ in } H(G, \chi)\ . \tag{2.3}$$

Then $H(G, \chi)$ is a topological vector space with the seminorm $|\ \ |$. Define two actions R and L on $C^\infty(G)$ by

$$(R_x f)(y)=f(yx)\ , \qquad (L_x f)(y)=f(x^{-1}y) \quad \text{for } x,\ y \text{ in } G\ .$$

In this section we shall prove that $H(G, \chi)$ is a topological G-module by these actions R and L.

LEMMA 1. *Let $H(G, \chi)$ be the subspace of $C^\infty(G)$ defined by (2.2).*

Then $H(G, \chi)$ is a L (resp. R) module (i.e., for each f in $H(G, \chi)$ and x in G, $|R_x f|=|L_x f|=|f|$ and $R_x f$ (resp. $L_x f$) belongs to $H(G, \chi)$).

PROOF. It will be seen that the arguments for L-module $H(G, \chi)$ also can be applied to the R-module. Thus it is enough to show that $|L_x f|=|f|$ and $L_x f$ belongs to $H(G, \chi)$. Immediately we have $|L_x f|^2=\lim_{\varepsilon\to+0} \varepsilon\int_G |f(y)|^2 \exp(-\varepsilon d(xy))dy$. We put $V=\{y\in G; d(y)\leqq d(x)\}$. Since $d(xy)\geqq|d(x)-d(y)|$, we obtain

$$|L_x f|^2\leqq\lim_{\varepsilon\to+0} \varepsilon\left\{\int_{G-V} |f(y)|^2 \exp(-\varepsilon d(y))dy+\int_V |f(y)|^2 \exp(-\varepsilon d(xy))dy\right\} .$$

Bearing in mind V is compact, the first term$\leqq|f|^2$ and the second term$=0$ in the above inequality. Therefore $|L_x f|\leqq|f|$ for any x in G. Replacing $L_{x^{-1}}f$ by f we have $|f|\leqq|L_{x^{-1}}f|$.

Consequently we get $|L_x f|=|f|$ for each x in G. It remains to prove that $L_x f$ belongs to $H(G, \chi)$. Since $L_x\circ z=z\circ L_x$, we have $z(L_x f)=\chi(z)L_x f$, $z\in\mathfrak{z}$. Let b_1 and b_2 be two elements in $\mathfrak{u}(\mathfrak{g})$. A direct calculation verifies that $(L_x f)(b_1; y; b_2)=L_x(f(\mathrm{Ad}(x^{-1})b_1; y; b_2))$ for all y in G. The element $\mathrm{Ad}(x^{-1})b_1$ is expressed as a finite linear combination of the elements in $\mathfrak{u}(\mathfrak{g})$. This implies that $|b_1(L_x f)b_2|=|(\mathrm{Ad}(x^{-1})b_1)fb_2|<\infty$ as desired.

A function f in $C^\infty(G)$ is called left (resp. right) K-finite if the subspace in $C^\infty(G)$ generated by the set $\{L_k f; k\in K\}$ (resp. $\{R_k f; k\in K\}$) is finite dimensional. Especially if f is left and right K-finite f is called K-finite. Let $\mathscr{E}(K)$ be the set of all equivalence classes of irreducible unitary representations of K. We put for each τ in $\mathscr{E}(K)$, $\chi_\tau(k)=\mathrm{trace}\,\tau(k)$, $k\in K$. Let us now define the convolution operators $\chi_\tau *$ and $*\chi_\tau$ on $C^\infty(G)$ by

$$(2.4)\qquad (\chi_\tau * f)(x)=\deg\tau\int_K \chi_\tau(k)f(k^{-1}x)dk ,$$

$$(f*\chi_\tau)(x)=\deg\tau\int_K \chi_\tau(k^{-1})f(xk)dk .$$

We remark that $\chi_\tau*\chi_\tau=\chi_\tau$ by this definition.

LEMMA 2. *Let $H_K(G, \chi)$ be the set of all K-finite functions in $H(G, \chi)$. Then the space $H_K(G, \chi)$ is topologically dense in $H(G, \chi)$.*

PROOF. We fix f in $H(G, \chi)$ and x in G. By Peter-Weyl theorem on the compact group K, we have

$$(2.5)\qquad f(x)=\sum_{\sigma,\tau\in\mathscr{E}(K)} \deg\tau\deg\sigma(\chi_\tau*f*\chi_\sigma)(x) .$$

On the other hand by using the Schur orthogonality relations on K, we have $|\chi_\tau * f|^2 \leqq \deg \tau |f|^2$ and $|f * \chi_\sigma|^2 \leqq \deg \sigma |f|^2$. Therefore

$$(2.6) \qquad |\chi_\tau * f * \chi_\sigma| \leqq (\deg \sigma \deg \tau)^{1/2} |f| .$$

Similarly we can prove that $|b_1(\chi_\tau * f * \chi_\sigma) b_2|$ is finite for τ, σ in $\mathscr{E}(K)$ and b_1, b_2 in $\mathfrak{u}(\mathfrak{g})$. Hence we have that $\chi_\tau * f * \chi_\sigma$ belongs to $H_K(G, \chi)$. It remains to prove that the series in (2.5) converges to f under the topology in $H(G, \chi)$. Let Ω_K be the Casimir operator on K. Then for a given τ in $\mathscr{E}(K)$ there exists a positive constant $\tau(\Omega_K)$ such that $\Omega_K \chi_\tau = \tau(\Omega_K)\chi_\tau$. Moreover choosing a positive integer m suitably, we have

$$(2.7) \qquad \sum_{\tau \in \mathscr{E}(K)} \tau(\Omega_K)^{-m} \quad \text{is finite} .$$

Let m be the same as in (2.7). In view of (2.6) we have that

$$|\chi_\tau * f * \chi_\sigma| \leqq (\deg \tau \deg \sigma)^{1/2} (\tau(\Omega_K)\sigma(\Omega_k))^{-m} |\Omega_K^m f \Omega_K^m| .$$

Combining this inequality with (2.5) and (2.6) (See Lemma 2 in [4].), we conclude that the series (2.5) converges to f in the topological space $H(G, \chi)$. This completes our proof.

Let $C_c^\infty(G)$ be the set of all C^∞-functions on G with compact support and consider an element ϕ in $C_c^\infty(G)$. We define the operators $\phi *$ and $* \phi$ on $C_c(G)$ by

$$(2.8) \qquad (\phi * f)(x) = \int_G \phi(xy) f(y^{-1}) dy , \qquad (f * \phi)(x) = \int_G f(xy)\phi(y^{-1}) dy .$$

The following lemma (cf. Theorem 1 in [1]) will be play an essential role to prove that $H(G, \chi)$ is a topological G-module.

LEMMA 3. *Let f be a fixed element in $C^\infty(G)$. Assume that f is K-finite and the dimension of $\mathfrak{z}f$ is finite. Then there exists ϕ(resp. ψ) in $C_c^\infty(G)$ such that $f * \phi = f$ (resp. $\psi * f = f$).*

Let us now prove that the actions $x \to R_x f$ and $x \to L_x f$ ($f \in H(G, \chi)$) are continuous.

LEMMA 4. *Let χ be a character of $\mathfrak{z}$. We define the space $H(G, \chi)$ by (2.2). Then G continuously acts on $H(G, \chi)$ to the left (resp. right).*

PROOF. Let f be an element in $H(G, \chi)$. We shall prove that the mapping $x \to R_x f$ of G to $H(G, \chi)$ is continuous at the identity 1 in G. In view of Lemma 2.2 we have

(2.9) for each positive real number δ, there exists g in $H_K(G, \chi)$ such that $|f-g|<\delta$.

Applying Lemma 2.3 to g we get $g*\phi=g$ for an element ϕ in $C_c^\infty(G)$. Let V be a compact neighbourhood of 1 and U the support of ϕ. Then for any x in G we have

$$|(R_x g-g)(y)|^2=\left|\int_G(\phi(z^{-1}x)-\phi(z^{-1}))g(yz)dz\right|^2$$
$$\leqq\int_W|\phi(z^{-1}x)-\phi(z^{-1})|^2dz\int_W|g(yz)|^2dz$$

where $W=VU^{-1}\cup U^{-1}$.

Since W is compact, it follows from Fubini theorem that

$$|R_x g-g|^2\leqq \mathrm{vol}(W)|g|^2\int_W|\phi(z^{-1}x)-\phi(z^{-1})|^2dz\ .$$

Hence we have

(2.10) there exists a compact neighbourhood V_0 of 1 such that $|R_x g-g|<\delta$ for all x in V_0.

Bearing in mind Lemma 2.1, (2.9) and (2.10) imply that for each positive real number δ there exists a neighbourhood V_0 of 1 such that $|R_x g-g|\leqq |R_x(f-g)|+|R_x g-g|+|g-f|<3\delta$ for all x in V_0. Hence the mapping $x\rightarrow R_x f$ is continuous at the identity in G. Since $H(G, \chi)$ is a R-module (See Lemma 1.) the mapping $x\rightarrow R_x f$ $(f\in H(G, \chi))$ of G to $H(G, \chi)$ is continuous. Replacing the above arguments for R to L we have the conclusion of the lemma.

REMARK 1. The space $H(G, \chi)$ and $H_K(G, \chi)$ are the algebraic $\mathfrak{g}$-modules (for a proof of this fact see p. 440, Lemma 5 in [4]).

§3. Differential equation associated with a function in $H_K(G, \chi)$.

Let f be an element in $H(G, \chi)$ for a given infinitesimal character χ of $\mathfrak{z}$ (see (2.2) for the definition of $H(G, \chi)$). It will be seen that the norm of f is determined by the asymptotic behaviour of f at the infinity. This fact will be proved by reducing to the evaluation of a solution of first order differential equation associated with f. Our arguments are a modefication for the theory, which is due to Harish-Chandra, of asymptotic behaviour for K-finite and $\mathfrak{z}$-finite functions on G.

In this section we shall state the first step of these procedure following Harish-Chandra.

DEFINITION 1. Let $\tau=(\tau_1, \tau_2)$ be a double representation of K on V_τ. A V_τ-valued function f on G is τ-spherical if f satisfies

$$(3.1) \qquad f(kxk')=\tau_1(k)f(x)\tau_2(k') \quad \text{for } x \text{ in } G,\ k,\ k' \text{ in } K.$$

We know that for each g in $H_K(G, \chi)$ there exists a double representation (τ, V_τ) of K, a vector v in V_τ and a τ-spherical function f such that $g(x)=(f(x), v)$, $x\in G$. Let us consider a double representation $\tau=(\tau_1, \tau_2)$ of K and a character χ of $\mathfrak{z}$. We shall use the following notations;

$C^\infty(G; V_\tau; \tau)$: the linear space of all τ-spherical C^∞-mappings of G to V_τ,

$H_\tau(G, \chi)$: the subspace of $C^\infty(G; V_\tau; \tau)$ consisting of those f such that
(1) $\lim_{\varepsilon\to+0} \varepsilon\int_G |f(b_1; x; b_2)|^2 e^{-\varepsilon d(x)}dx<\infty$ for all b_1, b_2 in $\mathfrak{u}(\mathfrak{g})$ and
(2) $zf=\chi(z)f$ for all z in $\mathfrak{z}$.

Let $P_0A_0N_0$ be a minimal p.s.g.r. of G with θ-stable split component A_0. For a standard p.s.g.r. $P=MAN$ of G, we denote the Lie algebras of M, A, N and $M_1=MA$ respectively by $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{n}$ and $\mathfrak{m}_1$. We fix an element f in $H_\tau(G, \chi)$ and put

$$(3.2) \qquad z(f)=\{z\in\mathfrak{z};\ zf=0\}, \qquad \mathfrak{z}(\mathfrak{m}_1; f)=\mathfrak{z}(\mathfrak{m}_1)\mu_P(\mathfrak{z}(f))$$
where μ_P is the same as in (1.5).

We know that these two sets in (3.2) form the ideals in $\mathfrak{z}$ and $\mathfrak{z}(\mathfrak{m}_1)$ respectively. Since $\mathfrak{z}(\mathfrak{m}_1)$ is a free $\mu_P(\mathfrak{z})$-module of finite rank, the dimension of the residue ring $\mathfrak{z}(\mathfrak{m}_1)/\mathfrak{z}(\mathfrak{m}_1; f)$ is finite. We denote the canonical projection of u in $\mathfrak{z}(\mathfrak{m}_1)$ to $\mathfrak{z}(\mathfrak{m}_1)/\mathfrak{z}(\mathfrak{m}_1; f)$ by $\bar{u}$. Choose a base $\{\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_p\}$ $(u_1=1)$ in $\mathfrak{z}(\mathfrak{m}_1)/\mathfrak{z}(\mathfrak{m}_1, f)$. Then for each fixed u in $\mathfrak{z}(\mathfrak{m}_1)$ there exist the complex numbers $\Gamma(u)_{i,j}$ and $\eta_{u,j}\in\mathfrak{z}(\mathfrak{m}_1; f)$ such that $uu_j=\sum_{i=1}^p \Gamma(u)_{j,i}u_i+\eta_{u,j}$ $(j=1, 2, \cdots, p)$.

Let C^p be the p-dimensional cartesian product over C and $\{e_1, e_2, \cdots, e_p\}$ be the canonical base of C^p. We define a linear endomorphism $\Gamma(u)$ of $V_\tau\otimes C^p$ for u in $\mathfrak{z}(\mathfrak{m}_1)$ and a $V_\tau\otimes C^p$-valued C^∞-function Φ on M_1 by

$$(3.3) \qquad \Gamma(u)(v\otimes e_i)=\sum_{j=1}^p v\otimes\Gamma(u)_{i,j}e_j \quad \text{for } v \text{ in } V_\tau,$$

$$(3.4) \qquad \Phi(m)=\sum_{j=1}^p (d_Pf)(m; u_j)\otimes e_j \quad \text{for } m \text{ in } M_1$$

where f is the same as in (3.2) and $d_P(p)=|\det(\mathrm{Ad}(p))_{\mathfrak{n}}|^{1/2}$. We now state the following two lemmas due to Harish-Chandra.

LEMMA 1. *Let notations being as above. Then we have* $\Phi(m; u) = (\Gamma(u)\Phi)(m) + \Psi_u(m)$ *for* u *in* $\mathfrak{z}(\mathfrak{m}_1)$ *and* m *in* M_1 *where* $\Psi_u(m) = \sum_{j=1}^{p} (d_P f)(m; \eta_{u,j}) \otimes e_j$.

This lemma is a direct consequence of the definitions of (3.3) and (3.4).

Let f be a fixed function in $H_\tau(G, \chi)$. Then the function $d_P f$ on M_1 can be estimated as follows.

LEMMA 2. *Let* η *be an element in* $z(\mathfrak{m}_1; f)$ *and* $\{X_1, X_2, \cdots, X_q\}$ *a base of* $\theta(\mathfrak{n})$. *Then there exists* b *in* $\mathfrak{u}(\mathfrak{g})$ *such that*

$$|(d_P f)(m; \eta)| \leq \text{a const.} |\mathrm{Ad}(m)|_{\theta(\mathfrak{n})}| \sup_{1 \leq i \leq q} |d_P(m) f(X_i; m; b)|$$

for all m *in* M_1 *where* $|\mathrm{Ad}(m)|_{\theta(\mathfrak{n})}|$ *is the norm of* $\mathrm{Ad}(m)|_{\theta(\mathfrak{n})}$.

PROOF. Let $\eta = \zeta \mu_P(z)$ ($\zeta \in \mathfrak{z}(\mathfrak{m}_1)$, $z \in \mathfrak{z}(f)$) be an element in $\mathfrak{z}(\mathfrak{m}_1; f)$. Bearing in mind $d_P^{-1} \circ \mu_P(z) \circ d_P = \gamma_{\mathfrak{g}/\mathfrak{m}_1}(z)$, we get

(3.5) $\quad (d_P f)(m; \eta) = d_P(m) f(m; \zeta' \gamma_{\mathfrak{g}/\mathfrak{m}_1}(z))$
where $\zeta' = d_P^{-1} \circ \zeta \circ d_P$ and m is an element in M_1.

On the other hand by Lemma 1.1, we have $z - \gamma_{\mathfrak{g}/\mathfrak{m}_1}(z) \in \theta(\mathfrak{n}_G)\mathfrak{u}(\mathfrak{g})$. Since $\mathfrak{m}_1$ normalizes $\theta(\mathfrak{n}_G)$, we see that there exists Y in $\theta(\mathfrak{n}_G)$ and b in $\mathfrak{u}(\mathfrak{g})$ such that $\zeta'(\gamma_{\mathfrak{g}/\mathfrak{m}_1}(z) - z) = Yb$. Therefore (3.5) is rewriten as $(d_P f)(m; \eta) = d_P(m) f(m; \zeta' z) + d_P(m; Yb)$. Hence, by $zf = 0$, we have $(d_P f)(m; \eta) = d_P(m) f(\mathrm{Ad}(m) Y; m; b)$. This implies the conclusion of Lemma 2.

We keep the same notations as above. Let $\Phi(A)$ be the root system of (P, A). In the following we shall assume that P is of split rank one p.s.g.r. of G (i.e. $\dim A = 1$). Let β be a unique simple root in $\Phi(A)$. Choose an element H_0 in $\mathfrak{a}$ satisfying $\beta(H_0) = 1$. Then each element a in A is parametrized by

$$a = a_t, \quad a_t = \exp(tH_0), \quad t \in \boldsymbol{R}. \tag{3.6}$$

Let f be a function in $H_\tau(G, \chi)$, and define $\Gamma(\ )$ and Φ corresponding to f by (3.3) and (3.4) respectively. We now put $\Gamma = \Gamma(H_0)$ and $\Psi = \Psi_{H_0}$. Therefore we have

$$\frac{\partial}{\partial t} \Phi(a_t m) = (\Gamma\Phi)(a_t m) + \Psi(a_t m), \quad m \in M_1, \quad t \in \boldsymbol{R}. \tag{3.7}$$

The endomorphism Γ on $V_\tau \otimes C^p$ has the Jordan decomposition; $\Gamma = S + T$, S and T are the linear endomorphisms of $V \otimes C^p$, T is nilpotent, S is semisimple, T and S are commute to each other. We define two $V_\tau \otimes C^p$-valued C^∞-functions on M_1 by

$$(3.8) \qquad \tilde{\Phi}(a_t m) = \exp(-tT)\Phi(a_t m) , \qquad \tilde{\Psi}(a_t m) = \exp(-tT)\Psi(a_t m) .$$

From (3.7) it follows that

$$(3.9) \qquad \frac{\partial}{\partial t}\tilde{\Phi}(a_t m) = S\tilde{\Phi}(a_t m) + \tilde{\Psi}(a_t m) \quad \text{for } t \text{ in } \boldsymbol{R} \text{ and } m \text{ in } M_1 .$$

Let $\Phi(\Gamma)$ be the set of all eigenvalues of Γ. We denote E_λ the projection to the eigenspace of S for $\lambda \in \Phi(\Gamma)$ and write $\tilde{\Phi}_\lambda = E_\lambda \tilde{\Phi}$, $\tilde{\Psi}_\lambda = E_\lambda \tilde{\Psi}$. Then we have

$$(3.10) \qquad \frac{\partial}{\partial t}\tilde{\Phi}_\lambda = \lambda\tilde{\Phi}_\lambda + \tilde{\Psi}_\lambda , \qquad \tilde{\Phi} = \sum_{\lambda \in \Phi(\Gamma)} \tilde{\Phi}_\lambda \quad \text{and} \quad \tilde{\Psi} = \sum_{\lambda \in \Phi(\Gamma)} \tilde{\Psi}_\lambda .$$

§4. An estimation for $\tilde{\Phi}$.

Let f be a $\mathfrak{z}$-finite and K-finite function on G. Assume that f satisfies "the weak inequality". We know that the asymptotic behaviour of f at the infinity can be calculated by using the evaluation in Lemma 3.2 concerning with first order of linear differential equations in Lemma 3.1. In this paper we shall calculate, for a given function f in $H_\tau(G, \chi)$, the asymptotic behaviour of f at the infinity without the assumption; f satisfies the weak inequality. Speaking more precisely, the asymptotic behaviour of f in $H_\tau(G, \chi)$ can be calculated by using the Tauberian theorem of Hardy and Littlewood combining with the evaluation for the solution of first order differential equation in (3.9).

Let $P_0 = M_0 A_0 N_0$, $\mathfrak{m}_0$, $\mathfrak{a}_0$, $\mathfrak{n}_0$, $\Phi(A_0)$ be the same as in the previous section. Let $\tilde{\mathfrak{a}}_0$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_0$. Choosing a positive root system $\Phi^+(\tilde{\mathfrak{a}}_0)$ of $(\mathfrak{g}_C, (\tilde{\mathfrak{a}}_0)_C)$ suitably, we may assume that $\Phi(A_0) = \{\alpha|_{\mathfrak{a}_0}; \alpha \in \Phi^+(\tilde{\mathfrak{a}}_0) \text{ and } \alpha \not\equiv 0 \text{ on } \mathfrak{a}_0\}$. We denote the positive Weyl chamber of A_0 by A_0^+ under this ordering. Let $\mathrm{cl}(A_0^+)$ be the closure of A_0^+. For each simple root β in $\Phi(A_0)$ and a real positive number r, we put

$$(4.1) \qquad A(\beta, r) = \{a \in \mathrm{cl}(A_0^+); \beta(\log a) \geqq r\rho(\log a)\} \quad \text{where } \rho \text{ is one half the sum of all roots in } \Phi^+(\tilde{\mathfrak{a}}_0).$$

LEMMA 1. *There exists a positive real number r_0 such that* $\mathrm{cl}(A_0^+) \subseteqq \bigcup_{\alpha \in \Psi_0} A(\beta, r_0)$ *where Ψ_0 is the simple root system of $\Phi(A_0)$.*

(For the proof of this lemma, see Theorem 14.8, Part II in [7]).

Let β be an element in Ψ_0. We define a p.s.g.r. $P=P_\beta$ of G as follows. Let $\mathfrak{a}_\beta$ be the subspace of $\mathfrak{a}_0$ defined by $\mathfrak{a}_\beta=\{H\in\mathfrak{a}_0;\ \alpha(H)=0$ for all α in $\Psi-\{\beta\}\}$. We put the set of all roots in $\Phi(A_0)$ which does not vanish identically on $\mathfrak{a}_\beta$ by $\Phi_\beta(A_0)$, $\mathfrak{u}_\alpha=\{X\in\mathfrak{n}_0;\ \mathrm{ad}\,(H)X=\alpha(H)X$ for all H in $\mathfrak{a}_\beta\}$ for α in $\Phi_\beta(A_0)$, $\mathfrak{n}_\beta=\sum_{\alpha\in\Phi_\beta(A_0)}\mathfrak{u}_\alpha$, $N_\beta=$the analytic subgroup of G with Lie algebra $\mathfrak{n}_\beta$, $A_\beta=$the analytic subgroup of G corresponding to $\mathfrak{a}_\beta$, $Z(A_\beta)=$the centralizer of A_β in G. Then there exists a reductive subgroup M_β of G such that $Z(A_\beta)=M_\beta A_\beta$ and $M_\beta\cap A_\beta=\{1\}$. Therefore $P_\beta=M_\beta A_\beta N_\beta$ is a Langlands decomposition of the p.s.g.r. P_β of G. The group P_β is called the p.s.g.r. of G corresponding to β.

Let us now fix a simple root β in $\Phi(A_0)$, and denote $P=P_\beta$, $M=M_\beta$, $A=A_\beta$, $N=N_\beta$. We put $\mathfrak{a}_0^*=\mathfrak{a}_0\cap\mathfrak{m}$, $A_0^*=\exp(\mathfrak{a}_0^*)$. Then we have $A_0=AA_0^*$ and $A\cap A_0^*=\{1\}$.

Let us consider a minimal p.s.g.r. P_0^* of M with split component A_0^*. Then the root system $\Phi(A_0^*)$ of (P_0^*, A_0^*) is given by $\Phi(A_0^*)=\{\alpha\in\Phi(A_0);\ \alpha\equiv 0$ on $\mathfrak{a}\}$. Define a function D_M on A_0^* by (1.7) and extend it to A_0 by

$$D_M(aa^*)=D_M(a^*)\quad\text{for } a \text{ in } A \text{ and } a^* \text{ in } A_0^*\ . \tag{4.2}$$

Let r and r' be two nonnegative real numbers. For each measurable function g on A_0 we define $|g|_{P,r,r'}$ and $|g|_{P,r}$ by

$$|g|_{P,r,r'}^2=\int_{A(\beta,r)}|g(a)|^2D_M(a)(1+d(a))^{r'}da\ , \tag{4.3}$$

$$|g|_{P,r}^2=\lim_{\varepsilon\to+0}\varepsilon\int_{A(\beta,r)}|g(a)|^2D_M(a)\exp(-\varepsilon d(a))da \tag{4.4}$$

where β is a fixed simple root in $\Phi(A_0)$ and $P=P_\beta$ is the same as above.

LEMMA 2. *Let τ be a double representation of K with finite dimension and χ is a character of $\mathfrak{z}$. Then, for each g in $H_\tau(G,\chi)$ and positive real number r, we have that $|d_Pg|_{P,r}$ is finite.*

PROOF. By using the integral formula in (1.4), we get

$$|g|^2=\lim_{\varepsilon\to+0}\varepsilon c_G\int_{\mathrm{cl}(A_0^+)}|g(a)|^2D(a)e^{-\varepsilon d(a)}da\ . \tag{4.5}$$

On the other hand we see that

(4.6) there exists a positive constant C such that $D(a)\geqq Cd_P(a)^2D_M(a)$ for all a in $A(\beta,r)$ satisfying $\beta(\log a)\geqq 1$.

Hence, from (4.5) it follows that the integral

$$\lim_{\varepsilon\to+0} \varepsilon \int_{\substack{A(\beta,r)\\ \beta(\log a)\geq 1}} |(d_P g)(a)|^2 D_M(a) e^{-\varepsilon d(a)} da \quad \text{is finite} .$$

Furthermore, since the set $\{a \in A(\beta, r);\ \beta(\log a) \leqq 1\}$ is compact we conclude that $|d_P g|_{P,r}$ is finite.

Let $P=MAN$ be the p.s.g.r. of G corresponding to a fixed simple root β in Ψ_0. Then by the definition of A, we have $\dim A=1$. We enumerate the simple root system Ψ_0 of $\Phi(A_0)$ by $\Psi_0=\{\beta=\beta_1, \beta_2, \cdots, \beta_l\}$, and define the dual base $\{H_i; 1\leqq i\leqq l\}$ of Ψ_0 by

$$\beta_i(H_j)=\delta_{i,j}\ , \quad 1\leqq i,\ j\leqq l\ . \tag{4.7}$$

Then A is parametrized by $A=\{a_t; a_t=\exp(tH_1), t\in \boldsymbol{R}\}$. Let P_0^* be a minimal p.s.g.r. of M. We can assume that the split A_0^* of P_0^* satisfies $A_0=AA_0^*$. We define a linear form $\bar\beta$ on $\mathfrak{a}_0$ by $\bar\beta=\beta$ on $\mathfrak{a}$ and $\bar\beta=0$ on $\mathfrak{a}_0^*$.

LEMMA 3. *Let r be a positive real number and β a fixed simple root in Ψ_0. Assume that $A(\beta, r)\neq\emptyset$. Then r satisfies the following two properties;* (1) $1-r\rho(H_1)>0$, (2) *the set $A(\beta, r)$ is given by* $A(\beta, r)=\{a_t a^*; a^*\in \mathrm{cl}((A_0^*)^+),\ t\geqq 0$ *and* $(1-r\rho(H_1))t\geqq(r\rho-\beta)(\log a^*)\}$ *where $(A_0^*)^+$ is the positive Weyl chamber of A_0^* and* $\mathrm{cl}(A_0^{*+})$*is the closure of* $(A_0^*)^+$.

PROOF. First we shall prove that

$$\beta(\log a)\leqq\bar\beta(\log a) \quad \text{for all } a \text{ in } \mathrm{cl}(A_0^+)\ . \tag{$*$}$$

Choose a base $\{A_1, A_2, \cdots, A_l\}$ of $\mathfrak{a}_0$ satisfying $(A_i, H)=\beta_i(H)$ for all i and H in $\mathfrak{a}_0$ where $(\ ,\ )$ is the inner product on $\mathfrak{a}_0$ induced from the Killing form on $\mathfrak{g}$. Let a be an element in $\mathrm{cl}(A_0^+)$. Then a is written as $\log a=\sum_{j=1}^l t_j A_j$ for some real numbers t_j. Since $\beta_i(\log a)=\sum_{j=1}^l t_j(\beta_i, \beta_j)$ and $(\beta_i, \beta_j)\leqq 0$ for $i\neq j$, we get $t_j\geqq 0$ for $j\neq 1$ and $\beta_1(\log a)\leqq t_1|\beta|^2$. Since $\beta=\beta_1$, we have the conclusion of (*).

Finally we shall prove the lemma by using (*). Let $a_t a^*$ be an element in $A(\beta, r)$. Then, by (*), we have $t\beta(H_1)=\beta(\log a_t)\geqq \beta(\log a_t a^*)\geqq 0$. Hence we have $t\geqq 0$. For $j\neq 1$, we have $\beta_j(\log a^*)= \beta_j(\log a_t a^*)\geqq 0$. Consequently a^* belongs to $\mathrm{cl}((A_0^*)^+)$. Since $\beta(\log a_t a^*)\geqq r\rho(\log a_t a^*)$, we conclude (**); $a_t a^*$ satisfies $t\geqq 0$, $a^*\in \mathrm{cl}((A_0^*)^+)$ and $(1-r\rho(H_1))t\geqq(r\rho-\beta)(\log a^*)$. Conversely assume that an element $a_t a^*$ in A_0 satisfies the properties (**). Immediately we have $t\geqq r\rho(\log a_t a^*)$, for $j\geqq 2$, $\beta_j(\log a^*)\geqq 0$. We see that ρ is expressed as $\rho=\sum_{j=1}^l m_j\beta_j$ on $\mathfrak{a}_0$ for suitable nonnegative real numbers m_j. Consequently the last property of $a_t a^*$ in (**) is equivalent to the fact; $(1-r\rho(H_1))\beta(\log a_t a^*)\geqq \sum_{j=2}^l r m_j\beta_j(\log a^*)$. Therefore $\beta_j(\log a_t a^*)\geqq 0$ for all $j=1, 2, \cdots, l$. (since

$A(\beta, r)\neq\emptyset$, we get $1-r\rho(H_1)>0$). This completes our proof.

Let χ and τ be the same as in Lemma 4.2. We define the space $H_\tau(G, \chi)$ by (3.2) and a maximal p.s.g.r. $P=MAN$ of G corresponding to a fixed simple root β in $\Phi(A_0)$. Define Φ, Ψ, $\Phi(\Gamma)$, T, $\tilde{\Phi}_\lambda$ and $\tilde{\Psi}_\lambda$ for the pair (P, f) $(f\in H_\tau(G, \chi))$ by (3.4), (3.7), $\cdots$,. Since T is a unipotent endomorphism of the finite dimensional vector space $V_\tau\otimes C^p$, there exist a positive constant C and a number k such that $|\exp(-tT)|\leqq C(1+|t|)^k$ for all t in $\boldsymbol{R}$. Hence, by the definitions of $\tilde{\Phi}_\lambda$ and $\tilde{\Psi}_\lambda$, we have

(4.8) there exist two positive constants C_1 and C_2 such that $|\tilde{\Phi}_\lambda(a_t a^*)|\leqq C_1(1+|t|)^k|\Phi(a_t a^*)|$ and $|\tilde{\Psi}_\lambda(a_t a^*)|\leqq C_2(1+|t|)^k|\Psi(a_t a^*)|$ for all $a_t a^*$ in A_0 .

Let r be a positive real number satisfying $A(\beta, r)\neq\emptyset$. We put a subset $B_r(t)$ $(t\geqq 0)$ in A_0^* by $B_r(t)=\{a^*\in \mathrm{cl}((A_0^*)^+);\ (1-r\rho(H_1))t\geqq(r\rho-\beta)(\log a^*)\}$. We now define a function g_Φ on the interval $(0, \infty)$ by $g_\Phi(t)=\int_{B_r(t)}|\Phi(a_t a^*)|^2 D_M(a^*)da^*$.

Lemma 4. *Let the notations and assumptions being as above. Then we have that* $\lim_{N\to\infty}(1/N)\int_0^N g_\Phi(t)dt<\infty$.

Proof. By Tauberian theorem of Hardy-Littlewood (cf. Chapter 7 in [6]), it is enough to prove that $\lim_{\varepsilon\to+0}\varepsilon\int_0^\infty g_\Phi(t)e^{-\varepsilon t}dt<\infty$. In view of the definition of Φ, we see that there exist $u_1, u_2, \cdots, u_p$ in $\mathfrak{z}(\mathfrak{m}_1)$ such that $|\Phi(m)|^2=\sum_{j=1}^p|(d_P f)(m; u_j)|^2$ for all m in M_1. Therefore $\int_0^\infty g_\Phi(t)e^{-\varepsilon t}dt=\sum_{j=1}^p\int_{A(\beta,r)}|(d_P f)(a; u_j)|^2 e^{-\bar{\beta}(\log a)}D_M(a)da$. Since $H_K(G, \chi)$ is a $\mathfrak{g}$-module (See Remark 2.1.) and $d_P(a; u_j)=a$ const. $d_P(a)$, our proof is reduced to the following; $\lim_{\varepsilon\to+0}\varepsilon\int_{A(\beta,r)}|(d_P f)(a)|^2 D_M(a)e^{-\bar{\beta}(\log a)}da<\infty$ for any function f in $H_\tau(G, \chi)$. Let B be the Killing form on $\mathfrak{g}$. By the Riemannian structure on G/K (See [2], Theorem 3.3 and Proposition 3.4, IV.) we have $d(a)^2=B(\log a, \log a)$ for all a in A_0. Since $B(H, H)=\sum_{\alpha\in\Phi(\tilde{\mathfrak{a}}_0)}\alpha(H)^2$ for H in $\tilde{\mathfrak{a}}_0$, we have

(4.9) there exist two positive constants c_1 and c_2 such that $c_1\rho(\log a)\leqq d(a)\leqq c_2\rho(\log a)$ for all a in A_0^+.

Consequently, by $\beta(\log a)\geqq r\rho(\log a)$ for a in $A(\beta, r)$, we have $e^{-\bar{\beta}(\log a)}\leqq e^{-\beta(\log a)}\leqq e^{-r\rho(\log a)}\leqq e^{-cd(a)}$ for a in $A(\beta, r)$ where $c=c_2^{-1}r$ is positive. This implies that $\lim_{\varepsilon\to+0}\varepsilon\int_{A(\beta,r)}|(d_P f)(a)|^2 D_M(a)e^{-\varepsilon\bar{\beta}(\log a)}da\leqq c^{-1}|d_P f|_{P,r}^2$ (see (4.4)). Hence by Lemma 4.2, we can prove the lemma.

LEMMA 5. *We keep the same notations as above. Let* λ *be an element in* $\Phi(\Gamma)$. *Assume that* $\mathrm{Re}(\lambda)=\sigma>0$. *Then we have* $|\tilde{\Phi}_\lambda|_{P,r,r'}<\infty$ *for all nonnegative real numbers* r *and* r'.

PROOF. Since $\tilde{\Phi}_\lambda$ satisfies the differential equation in (3.10), we see that $(\partial/\partial u)e^{-\lambda u}\tilde{\Phi}_\lambda(a_{t+u}a^*)=e^{-\lambda u}\tilde{\Psi}_\lambda(a_{t+u}a^*)$ for t in $\boldsymbol{R}$ and a^* in A_0^*. Integrating the both side of this identity over the interval $(0, N)$ we obtain

$$\tilde{\Phi}_\lambda(a_ta^*)=e^{-\lambda N}\tilde{\Phi}_\lambda(a_{t+N}a^*)-\int_0^N e^{-\lambda u}\tilde{\Psi}_\lambda(a_{t+u}a^*)du\ . \tag{4.10}$$

In view of Lemma 4.4 and (4.10), it is enough to verify that

$$\lim_{N\to\infty}\int_0^N dt\int_{B_r(t)} e^{-2\sigma N}|\tilde{\Phi}_\lambda(a_{t+N}a^*)|^2 D_M(a^*)(1+t)^{r'}da^*<\infty\ , \tag{4.11}$$

$$\lim_{N\to\infty}\int_0^N dt\int_{B_r(t)}\left|\int_0^N e^{-\lambda u}\tilde{\Psi}_\lambda(a_{t+u}a^*)du\right|^2 D_M(a^*)(1+t)^{r'}da^*<\infty\ . \tag{4.12}$$

Let $I(N)$ be the integral of (4.12). Using Lemma 4.3 and (4.8), we have $I(N)\leqq C_1e^{-2\sigma N}(1+2N)^{r'+k}\int_0^N\int_{B_r(t)}|\Phi(a_{t+N}a^*)|^2D_M(a^*)da^*dt$. By the definition of $B_r(t)$, we have $B_r(t+N)\ni a_{t+N}a^*$ for all a_ta^* in $B_r(t)$. Hence $I(N)$ is evaluated by

$$\begin{aligned}I(N)&\leqq C_1e^{-2\sigma N}(1+2N)^{r'+k}\int_N^{2N}dt\int_{B_r(t)}|\Phi(a_ta^*)|^2D_M(a^*)da\\&\leqq C_1e^{-2\sigma N}(1+2N)^{r'+k}\int_0^{2N}g_\Phi(t)dt\end{aligned}$$

where k and r' are positive.

Since σ is positive, Lemma 4.4 implies that the limit of (4.11) is finite. It remains to prove the fact (4.12).

Combining Lemma 3.2 with (4.8), we have

$$|\tilde{\Psi}_\lambda(a_ta^*)|\leqq \text{a const.}\,|\mathrm{Ad}(a_ta^*)|_{\theta(\mathfrak{n})}\,|d_P(a_ta^*)\sup_{1\leqq j\leqq q}|f(X_j;\,a_ta^*;\,b)|(1+|t|)^k$$

for some X_i in $\theta(\mathfrak{n}_o)$ and b in $\mathfrak{n}(\mathfrak{g})$.

Since $|\mathrm{Ad}(a_ta^*)|_{\theta(\mathfrak{n})}\leqq e^{-\beta(\log a_ta^*)}\leqq e^{-r\rho(\log a_ta^*)}\leqq e^{-\kappa t}$ $(\kappa=r\rho(H_1))$ for all a_ta^* in $A(\beta, r)$, the proof of (4.12) is reduced the following;

$$\begin{aligned}\lim_{N\to\infty}\int_0^N dt\int_{B_r(t)}&\left\{\int_0^N d_P(a_{t+u}a^*)\,|f(X_j;\,a_{t+u}a^*;\,b)e^{-\kappa(t+u)}|du\right\}^2\\&\times D_M(a^*)(1+d(a_ta^*))^{r'}da^*<\infty\ .\end{aligned}\tag{4.13}$$

By Schwarz inequality, the integrant { } of (4.13) is estimated as follows;

$$|\{\quad\}|^2 \leqq \left(\int_0^N e^{-\kappa(t+u)}du\right)^2 \left(\int_0^N e^{-\kappa(t+u)}\,|d_P(a_{t+u}a^*)f(X_j;\, a_{t+u}a^*;\, b)|\right)^2 \times D_M(a^*)(1+d(a_ta^*))^{r'}du\ .$$

Hence we have

$$\int_{B_r(t)} da^*\left(\int_0^N |d_P(a_{t+u}a^*)f(X_j;\, a_{t+u}a^*;\, b)e^{-\kappa(t+u)}|\,du\right)^2 D_M(a^*)(1+d(a_ta^*))^{r'}$$
$$\leqq \kappa^{-1}e^{-\kappa t}\int_0^N du\int_{B_r(u)} e^{-\kappa u}\,|f(X_j;\, a_ua^*;\, b)|^2 d_P(a_ua^*)^2 D_M(a^*)(1+d(a_ta^*))^{r'}da^*\ .$$

By (4.9) and (*) in the proof of Lemma 4.3, we have $d(a_ta^*) \leqq c_2\rho(\log a_ta^*) \leqq rc_2\beta(\log a_ta^*) \leqq rc_2\bar{\beta}(\log a_t) = rc_2t$ for all a_ta^* in $A(\beta, r)$. Hence the above integral $\leqq \kappa^{-1}(1+c_2t)^{r'}e^{-\kappa t}\int_{A(\beta,r)} e^{-c\kappa d(a)}|(d_P(X_jfb))(a)|^2 D_M(a)da$ where $c=c_2^{-1}r$. Therefore the term in (4.13)

$$\leqq \text{a const.}\int_0^\infty (1+c_2t)^{r'}e^{-\kappa t}dt\int_{A(\beta,r)} e^{-c\kappa d(a)}\,|(d_P(X_jfb))(a)|^2 D_M(a)da\ .$$

By Lemma 4.2, the last integral in the above integrals is finite. Consequently the term in (4.13) is also finite.

REMARK 1. In view of the proof of (4.11) we see that for each λ in Γ, $\int_{A(\beta,r)} |\Psi_\lambda(a_ta^*)|^2 D_M(a^*)e^{wt}da^*dt < \infty$ for suitable real number $w>0$.

LEMMA 6. *Assume that an eigenvalue λ of Γ satisfies* $\mathrm{Re}\,(\lambda)=\sigma<0$. *Then we have $|\Phi_\lambda|_{P,r,r'}<\infty$ for positive real numbers r and r'.*

PROOF. Since $(\partial/\partial u)e^{-\lambda ut}\tilde{\Phi}_\lambda(a_{ut}a^*) = te^{-ut}\tilde{\Psi}_\lambda(a_{ut}a^*)$, we have $\tilde{\Phi}_\lambda(a_ta^*) = e^{\sigma(1-\delta)t}\tilde{\Phi}_\lambda(a_{ut}a^*) + te^{\lambda t}\int_\delta^1 e^{-\lambda ut}\tilde{\Psi}_\lambda(a_{ut}a^*)du$ for each positive real number δ satisfying $\delta<1$. We put

$$J_1 = \int_{A(\beta,r)} e^{\sigma(1-\delta)t}|\tilde{\Phi}_\lambda(a_ta^*)|^2 D_M(a^*)(1+t)^{r'}da^*dt\ ,$$

$$J_2 = \int_{A(\beta,r)} \left\{e^{\sigma t}\int_\delta^1 te^{-\sigma ut}|\tilde{\Psi}_\lambda(a_{ut}a^*)|du\right\}^2 D_M(a^*)(1+t)^{r'}tda^*dt\ .$$

Therefore it is enough to verify J_1 and J_2 are finite for a positive number δ.

Put $s=\rho(H_1)^{-1}(1-\delta(1-r\rho(H_1)))$. Since $1>r\rho(H_1)$ (See (1) in Lemma 4.3.) we have $s>0$. Furthermore we get (See (2) in Lemma 4.3.)

(4.14) $$a_{\delta t}a^* \in A(\beta, s) \quad \text{for all } a_t a^* \text{ in } A(\beta, r)\,.$$

Hence we have

$$J_1 \leqq \delta^{-(r'+1)} \int_0^\infty dt \int_{B_s(t)} e^{-\kappa' t} |\widetilde{\Phi}_\lambda(a_t a^*)|^2 D_M(a^*)(\delta+t)^{r'} da^*$$

where $\kappa' = (1-\delta)\delta^{-1}(-\sigma)$ is positive. Combining Lemma 4.2 with Lemma 4.3 and (4.8), we see that J_1 is finite

Let us now consider the integral J_2. From $\sigma(1-u)t \leqq 0$ for $\delta \leqq u < 1$ and $t \geqq 0$, it follows that

$$\left\{e^{\sigma t} \int_\delta^1 e^{-\sigma u t} t |\widetilde{\Psi}_\lambda(a_{ut}a^*)| du\right\}^2 \leqq \int_\delta^1 |t\widetilde{\Psi}_\lambda(a_{ut}a^*)|^2 du\,.$$

Therefore

$$J_2 \leqq \text{a const.} \int_\delta^1 \int_0^\infty \int_{B_r(t)} |\widetilde{\Psi}_\lambda(a_{ut}a^*)|^2 D_M(a^*)(1+t)^{r'+1} du\,dt\,da^*\,.$$

Let $a_t a^*$ be one of each element in $A(\beta, r)$. By (4.14), we have $a_{\delta t}a^* \in A(\beta, s)$. Since $(1-s\rho(H_1))ut \geqq (1-s\rho(H_1))\delta t \geqq (r\rho-\beta)(\log a^*)$ (See Lemma 4.3.), we get $a_{ut}a^* \in A(\beta, s)$ for all $a_t a^*$ in $A(\beta, r)$. Hence we have

$$J_2 \leqq \text{a const.} \int_\delta^1 du \int_{A(\beta,s)} |\widetilde{\Psi}_\lambda(a_t a^*)|^2 \delta^{-1}(1+\delta^{-1}t)^{r'+1} D_M(a^*) da^* dt\,.$$

By Remark 4.1 the integral on $A(\beta, s)$ in the above inequality is finite. Consequently J_2 is finite. This implies our conclusion.

LEMMA 7. *Let λ be an element in $\Phi(\Gamma)$, and assume that λ is a purely imaginary number. Then there exists a $V_\tau \otimes C^p$-valued measurable function Z_λ on* $\mathrm{cl}((A_0^*)^+)$ *such that $|\widetilde{\Phi}_\lambda - e^\lambda Z_\lambda|_{P,r,r'} < \infty$ for all positive real numbers r and r', where $e^\lambda Z_\lambda$ is defined by $(e^\lambda Z_\lambda)(a_t a^*) = e^{\lambda t} Z_\lambda(a^*)$ for a_t in A and a^* in* $\mathrm{cl}((A_0^*)^+)$.

PROOF. Let n be a positive integer. Then, for each m in M, we have $(\partial/\partial t)\{e^{-\lambda t}\widetilde{\Phi}(a_t m) + \int_t^n e^{-\lambda u}\widetilde{\Psi}_\lambda(a_u m) du\} = 0$. We put $Z_n(m) = e^{-\lambda t}\widetilde{\Phi}_\lambda(a_t m) + \int_t^n e^{-\lambda u}\widetilde{\Psi}_\lambda(a_u m) du$. We claim that the series $\{Z_n(a^*)\}_n$ $(a^* \in \mathrm{cl}((A_0^*)^+))$ is a Cauchy sequence. In view of Remark 4.1, we have $\int_{A(\beta,r)} |\widetilde{\Psi}_\lambda(a)|^2 D_M(a) da < \infty$. Consequently

(4.15) $$\int_0^\infty |\widetilde{\Psi}_\lambda(a_t a^*)|^2 dt < \infty \quad \text{for almost everywhere } a^* \text{ in } \mathrm{cl}((A_0^*)^+)\,.$$

Since the set $\{a^* \in A_0^*; D_M(a^*) \neq 0\}$ is open dense in A_0^*, we have $\int_0^\infty |\tilde{\Psi}_\lambda(a_t a^*)|^2 dt$ is finite. From this fact the sequence Z_n converges to a measurable function Z_λ (i.e., $Z_\lambda(a^*) = e^{-\lambda t}\tilde{\Phi}_\lambda(a_t a^*) + \int_t^\infty e^{-\lambda u}\tilde{\Psi}_\lambda(a_u a^*)du$. Let r and r' be two positive real numbers. Then we have (See (4.3).)

$$|\tilde{\Phi}_\lambda - e^\lambda Z|_{P,r,r'} \leqq \int_{A(\beta,r)} |(\tilde{\Phi}_\lambda - e^\lambda Z_\lambda)(a)|^2 D_M(a)(1+d(a))^{r'} da \ ,$$

hence by Lemma 4.3,

$$\leqq \int_0^\infty dt \int_{B_r(t)} \left(\int_t^\infty |\tilde{\Psi}_\lambda(a_u a^*)| du\right)^2 D_M(a)(1+d(a_t a^*))^{r'} da^* dt \ .$$

Let δ be a positive real number. By Schwarz inequality, we have

$$\begin{aligned}\left(\int_t^\infty |\tilde{\Psi}_\lambda(a_u a^*)| du\right)^2 &\leqq \left(\int_t^\infty e^{-\delta u} du\right)\left(\int_t^\infty e^{\delta u} |\tilde{\Psi}_\lambda(a_u a^*)|^2 du\right) \\ &\leqq e^{-\delta t}\left(\int_t^\infty e^{-\delta u} du\right)\left(\int_t^\infty e^{2\delta u} |\tilde{\Psi}_\lambda(a_u a^*)|^2 du\right) .\end{aligned}$$

Consequently

$$\begin{aligned}|\tilde{\Phi}_\lambda - e^\lambda Z|_{P,r,r'} \leqq \int_0^\infty e^{-\delta t}\left(\int_t^\infty e^{-\delta u} du\right)\bigg(\int_t^\infty \int_{B_r(t)} & e^{2\delta u} |\tilde{\Psi}_\lambda(a_u a^*)|^2 D_M(a^*) \\ & \times (1+d(a_t a^*))^{r'} da^*\bigg) dt \ .\end{aligned}$$

By Lemma 4.3, we have $\{a_u a^*; u \geqq t, a^* \in B_r(t)\} \subset A(\beta, r)$. Hence

$$\begin{aligned}|\tilde{\Phi}_\lambda - e^\lambda Z_\lambda|_{P,r,r'} \leqq & \left(\int_0^\infty e^{-\delta t} dt\right)\left(\int_{A(\beta,r)} e^{2\delta t} |\tilde{\Psi}_\lambda(a_t a^*)|^2 D_M(a^*) da^* dt\right) \\ & \times \left(\int_0^\infty e^{-\delta t}(1+c_2 r^{-1} t)^{r'} dt\right)\end{aligned}$$

where c_2 is the same constant as in (4.9). We now choose a sufficiently small positive real number δ. Then the second integral is finite (see Remark (4.1)). This completes our proof.

Let us summarize Lemmas 4.5, 4.6 and 4.7. Let β be a simple root in $\Phi(A_0)$ and $P = MAN$ be the p.s.g.r. of G corresonding to β. For a function f in $H_\tau(G, \chi)$, we define Φ, Ψ as in §3. Let $P_0^* = M_0^* A_0^* N_0^*$ be a minimal p.s.g.r. of M satisfying $A_0^* \subset A_0$. We put $K_M = K \cap M$. Then $M = K_M P_0^* = K_M \operatorname{cl}((A_0^*)^+) K_M$. By the definition of Φ and Ψ, we see that

(4.16) $\quad \Phi(k_1 m k_2) = \tau_1(k_1)\Phi(m)\tau_2(k_2)$ and $\Psi(k_1 m k_2) = \tau_1(k_1)\Psi(m)\tau_2(k_2)$

for all k_1, k_2 in K_M and m in M.

Define a double unitary representation τ_{K_M} of K_M by

(4.17) $\tau_{K_M}=(\tau_1|_{K_M}, \tau_2|_{K_M})$ where $\tau_i|_{K_M}$ is the restriction of τ_i of K to K_M.

Then $\tilde{\Phi}_\lambda$ and $\tilde{\Psi}_\lambda$ ($\lambda \in \Phi(\Gamma)$) are τ_{K_M}-spherical. So Z_λ in Lemma 4.7 can be extended canonically to a function on M which is τ_{K_M}-spherical. We define a $V_\tau \otimes C^p$-valued τ_{K_M}-spherical function Φ_P on M by

(4.18) $$\Phi_P(a_t m)=\sum_{\substack{\lambda\in\Phi(\Gamma)\\ \mathrm{Re}(\lambda)=0}} e^{\lambda t}\exp(tT)Z_\lambda(m) \quad \text{for } t \text{ in } \boldsymbol{R} \text{ and } m \text{ in } M.$$

Then we have

(4.19) $$\Phi(a_t m)=\sum_{\substack{\lambda\in\Phi(\Gamma)\\ \mathrm{Re}(\lambda)\neq 0}} \exp(tT)\tilde{\Psi}_\lambda(a_t m)+\Phi_P(a_t m) \quad \text{(see (3.8) and (3.10))}.$$

Combining Lemma 4.5, Lemma 4.6 with Lemma 4.7 we conclude the following.

LEMMA 8. *Let notations being as above, and Ω be a compact subset in M. Then, for each two positive real numbers r and r', we have*

(1) $$\int_0^\infty \int_\Omega |\Phi(a_t m)-\Phi_P(a_t m)|^2 D_M(a)(1+d(a_t m))^{r'} dm dt<\infty,$$

(2) $$\int_{A(\beta,r)} |\Phi(a)-\Phi_P(a)|^2 D_M(a)(1+d(a))^{r'} da<\infty.$$

§5. The constant term of a function in $H_\tau(G, \chi)$.

For a fixed simple root β in $\Phi(A_0)$, we consider the maximal p.s.g.r. $P=MAN$ corresponding to β (see §3). Let χ be a character of $\mathfrak{z}$ and (τ, V_τ) be a finite dimensional double unitary representation of K. For a function f in $H_\tau(G, \chi)$ and P, we define $V_\tau\otimes C^p$-valued functions Φ and Φ_P by (3.8) and (4.18) respectively. Then Φ_P is of the form; $\Phi_P(a_t m)= f_P(a_t m)\otimes e_1+\sum_{j=2}^p f_{P,j}(a_t m)\otimes e_j$ for the suitable τ_{K_M}-spherical functions $f_{P,j}$ on AM. Consequently, by the definition of Φ_P, there exist a finite number of purely imaginary numbers $\lambda_1, \lambda_2, \cdots, \lambda_q$, the polynomials $p_1, p_2, \cdots, p_q$ in t and τ_{K_M}-spherical measurable functions $f_1, f_2, \cdots, f_q$ on M such that

(5.1) $$f_P(a_t m)=\sum_{j=1}^q e^{(\lambda_i t)} p_i(t) f_i(m) \quad \text{for all } t \text{ in } \boldsymbol{R} \text{ and } m \text{ in } M.$$

Our main purpose in this section is to prove that f_P is a $\mathfrak{z}(\mathfrak{m}_1)$-finite, τ_{K_M}-spherical C^∞-function on M.

The following lemma is a direct consequence of Lemma 4.8.

LEMMA 1. *Let Ω be a compact subset in M and r (resp. r') be a positive real number. Then we have*

(1) $\int_0^\infty \int_\Omega |(d_P f - f_P)(a_t m)|^2 D_M(a)(1+d(a_t m))^{r'} dm dt < \infty$,

(2) $\int_{A(\beta,r)} |(d_P f - f_P)(a)|^2 D_M(a)(1+d(a))^{r'} da < \infty$.

LEMMA 2. *Let Ω be a compact subset in M and f_i be the same as in* (5.1). *Then f_i is square integrable on each compact subset in M.*

PROOF. In view of the definitions for f_i, f_P and Φ_P, it is sufficient to prove that $\int_\Omega |Z_\lambda(m)|^2 dm$ are finite for all purely imaginary numbers λ in $\Phi(\Gamma)$. Since $Z_\lambda(m) = e^{-\lambda t}\widetilde{\Phi}_\lambda(a_t m) + \int_t^\infty e^{-\lambda u}\widetilde{\Psi}_\lambda(a_u m)du$, so what must be shown is that there exists a positive real number δ such that

$$\int_\Omega |\widetilde{\Phi}_\lambda(a_\delta m)|^2 dm < \infty \quad \text{and} \quad \int_\Omega \left| \int_\delta^\infty e^{-\lambda u}\widetilde{\Psi}_\lambda(a_u m)du \right|^2 dm < \infty . \tag{5.2}$$

Since $\widetilde{\Phi}_\lambda$ is a C^∞-function on AM and Ω is compact the first integral is finite for all positive real number δ. Let Ω^* be a compact subset in $\mathrm{cl}((A_0^*)^+)$ satisfying $\Omega \subset K_M \Omega^* K_M$. Choosing sufficiently large positive real number δ, we get $a_\delta \Omega^* \subseteqq A(\beta, r)$. Therefore, since $\widetilde{\Psi}_\lambda$ is τ_{K_M}-spherical we have

$$\begin{aligned}
\int_\Omega \left| \int_\delta^\infty e^{-\lambda u}\widetilde{\Psi}_\lambda(a_u m)du \right|^2 dm &\leqq \int_{\Omega^*} \left| \int_\delta^\infty |\widetilde{\Psi}_\lambda(a_u a^*)| du \right|^2 D_M(a^*)da^* \\
&\leqq \int_{\Omega^*} \Big(\int_\delta^\infty e^{-wu} du \Big)\Big(\int_\delta^\infty e^{wt} |\widetilde{\Psi}_\lambda(a_u a^*)|^2 du \Big) da^* \\
&\leqq \text{a const.} \int_{A(\beta,r)} e^{wt} |\widetilde{\Psi}_\lambda(a_t a^*)|^2 D_M(a^*) da^* dt .
\end{aligned}$$

where w is a sufficiently small positive real number. Hence by Remark 4.1, the second integral in (5.2) is finite as desired.

LEMMA 3. *For the functions f_i and p_i in* (5.1), *we have* (1) *all p_i's are constant,* (2) *all f_i's are square integrable on M.*

PROOF. We put, for each positive integer n, $M_n = \{m \in M; d(m) \leqq n\}$. Combining Lemma 4.1 with Lemma 5.1 (See also the integral formula in Lemma 1.4), we get

$$|f|^2 \geqq \text{a const.} \lim_{\varepsilon \to +0} \varepsilon \int_0^\infty \int_{M_n} \sum_{i,j=1}^q e^{(\lambda_i - \lambda_j - \varepsilon\delta)t} p_i(t)\overline{p_j(t)} f_i(m)\overline{f_j(m)} dm dt$$

where δ is some positive real number. Since

$$\lim_{\varepsilon\to+0}\varepsilon\int_0^\infty e^{(\lambda_i-\lambda_j-\varepsilon\delta)t}p_i(t)\overline{p_j(t)}dt=0 \quad \text{for } i\neq j \text{ and } \int_{M_n}|f_i(m)|^2dm<\infty$$

for all i (cf.Lemma 5.2), the above inequality leads to the following; $|f|^2\geqq$ a const. $\sum_{i=1}^q\left(\lim_{\varepsilon\to+0}\varepsilon\int_0^\infty e^{-\varepsilon\delta t}|p_i(t)|^2dt\int_{M_n}|f_i(m)|^2dm\right.$. This implies that p_i's are constant and $|f|^2\geqq$ a const. $\int_{M_n}|f_i(m)|^2dm$ for all positive integer n, where the constant is independent on n. Consequently, since $\lim_{n\to\infty}M_n=M$, we have our conclusion.

REMARK 1. For the above functions f and f_i, $|f|\geqq a$ const. $\int_M|f_i(m)|^2dm$. Let $\langle\ ,\ \rangle$ be a positive definite Hermitian form on V_τ, and λ a fixed imaginary number. For a given f in $H_\tau(G,\chi)$, we define a linear form f_λ by

$$f_\lambda(\phi)=\lim_{\varepsilon\to+0}\varepsilon\int_0^\infty dt\int_M\langle(d_Pf)(a_tm),(e^\lambda\phi)(a_tm)\rangle e^{-(\varepsilon d(a_tm))}dm \tag{5.3}$$

where $\phi\in C_c^\infty(M\colon V_\tau)$, $e^\lambda\phi$ is defined by $(e^\lambda\phi)(a_tm)=e^{\lambda t}\phi(m)$. We have immediately $|f_\lambda(\phi)|^2\leqq$ a const. $|f|^2\int_M|\phi(m)|^2dm$ for all ϕ in $C_c^\infty(G\colon V_\tau)$. Therefore f_λ is a V_τ-valued distribution on M. Bearing in mind (1) in Lemma 5.1 and $\lim_{t\to\infty}\rho(\log a_t)d(a_t)^{-1}=1$, the same arguments as in the proof of Lemma 5.3 implies

$$f_{\lambda_i}(\phi)=p_i\cdot\rho(H_0)^{-1}\int_M\langle f_i(m),\phi(m)\rangle dm \quad \text{for } \phi \text{ in } C_c^\infty(G\colon V_\tau)\ . \tag{5.4}$$

We shall prove that f_{λ_i} is a $\mathfrak{z}(\mathfrak{m})$-finite and K_M-finite distribution on G after the following lemma.

LEMMA 4. *We define a function g_ε on G by $g_\varepsilon(x)=e^{-\varepsilon d(x)}$ for a given positive real number ε. Let X be an element in the Lie algebra $\mathfrak{g}$ of G. Then we have $g_\varepsilon(X;x)\leqq$ a const. $g_\varepsilon(x)$ for all x in KA_0^+K where the constant does not depend on ε.*

PROOF. Let $\tilde{\mathfrak{a}}_0$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_0$, and $\Phi(\tilde{\mathfrak{a}}_0)$, $\mathfrak{g}_\alpha,\cdots$ be the same as in §1. We choose X_α in $\mathfrak{g}_\alpha$ satisfying $B(X_\alpha,X_{-\alpha})=1$ (B is the Killing form on $\mathfrak{g}_c$). Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. Then X_α can be expressed as $X_\alpha=Y_\alpha+Z_\alpha$, $Y_\alpha\in\mathfrak{k}$ and $Z_\alpha\in\mathfrak{p}$. Therefore, for each root α in $\Phi(\tilde{\mathfrak{a}}_0)$, $\mathrm{Ad}(a)Y_\alpha=(1/2)\mathrm{Ad}(a)(X_\alpha+\theta(X_\alpha))=\cosh(\alpha(\log a))Y_\alpha+\sinh(\alpha(\log a))Z_\alpha$, $a\in A_0$. We now assume that a is a regular element in A_0. Then

$$Z_\alpha=(\sinh(\alpha(\log a)))^{-1}(\mathrm{Ad}(a)Y_\alpha-\cosh(\alpha(\log a))Y_\alpha)\ . \tag{5.5}$$

Put $G_0=KA_0^+K$. We see that the mapping $(k_1, a, M_0k_2)\to k_1ak_2$ of $K\times A_0^+\times(M_0\backslash K)$ into G is an analytic diffeomorphism. Consequently if $k_1ak_2=k_1'a'k_2'$, then $a=a'$ and $k_1^{-1}k_1'=k_2'k_2^{-1}\in M_0$ where $k_1, k_2, k_1', k_2'\in K$, $a, a'\in A_0^+$. Let us prove Lemma 5.4. Let $x=k_1ak_2$ be an element in KA_0^+K. Then the value $(d/dt)g_\varepsilon(\exp(t\operatorname{Ad}(k_1)^{-1}Xx)|_{t=0}$ is uniquely determined by x. Select a base $\{H_1, H_2, \cdots, H_m\}$ of $\tilde{\mathfrak{a}}_0$ as follows; $H_1, H_2, \cdots, H_l\in\mathfrak{a}_0$ and $H_{l+1}, H_{l+2}, \cdots, H_m\in\tilde{\mathfrak{a}}_0\cap\mathfrak{k}$. Then $\operatorname{Ad}(k_1)^{-1}X$ is of the form $\operatorname{Ad}(k_1)^{-1}X=\sum_{i=1}^m u_i(k_1)H_i+\sum_{\alpha\in\Phi(\tilde{\mathfrak{a}}_0)}u_\alpha(k_1)X_\alpha$ where u_i and u_α are analytic on K. Bearing in mind the function g_ε is K-invariant, we have immediately $g_\varepsilon(Y_\alpha; a)=g_\varepsilon(a; Y_\alpha)=0$ for all roots α, $g_\varepsilon(H_i; a)=g_\varepsilon(a; H_i)=0$ for $i>l$ and $g_\varepsilon(X_\alpha; a)=g_\varepsilon(a; X_\alpha)=0$ for all roots α in $\Phi(\tilde{\mathfrak{a}}_0)$ which vanish identically on $\mathfrak{a}_0$. Therefore

$$g_\varepsilon(X; x)=g_\varepsilon((\operatorname{Ad}(k_1)^{-1}X); a)=\sum_{i=1}^{l}u_i(k_1)g_\varepsilon(H_i; a)+\sum_{\substack{\alpha\in\Phi(\tilde{\mathfrak{a}}_0)\\ \alpha\neq 0 \text{ on } \mathfrak{a}_0}}u_\alpha(k_1)g_\varepsilon(X_\alpha; a)\ .$$

For $i\leqq l$, $g_\varepsilon(H_i\colon a)=d(a)^{-1}B(H_i, \log a)$.

Hence, we have $|g_\varepsilon(H_i; x)|\leqq\varepsilon|H_i|$ for all $i\leqq l$. Let α be an element in $\Phi(\tilde{\mathfrak{a}}_0)$ which does not vanish on $\mathfrak{a}_0$. By (5.5), we get $g_\varepsilon(X_\alpha; a)=g_\varepsilon(Z_\alpha; a)=g_\varepsilon(a; \operatorname{Ad}(a^{-1})Z_\alpha)=(\sinh\alpha(\log a^{-1}))g_\varepsilon(a; Y_\alpha)=0$. Consequently we have $|g_\varepsilon(X; a)|\leqq$ a const. $\times\,\varepsilon g_\varepsilon(x)$ for all x in KA_0^+K.

Let σ be the conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$. We define an antilinear automorphism * of $\mathfrak{g}_c$ by

$$X^*=-\sigma(X)\quad\text{for } X \text{ in } \mathfrak{g}_c\ . \tag{5.6}$$

We extend * canonically to the universal enveloping algebra $\mathfrak{u}(\mathfrak{g})$ of $\mathfrak{g}_c$.

LEMMA 5. *Let f be an element in $H_\tau(G, \chi)$, and f_P, f_i be the same as in* (5.2). *Then we have that f_i is $\mathfrak{z}(\mathfrak{m})$-finite, and f_P satisfies $\mu_P(z)f_P=\chi(z)f_P$ for all z in $\mathfrak{z}$.*

PROOF. We fix a number i $(1\leqq i\leqq q)$, and let f_{λ_i} be the distribution on M defined by (5.4). It is easy to see that f_{λ_i} is τ_{K_M}-spherical. Let z be an element in $\mathfrak{z}$. By using Lemma 1.1, we have $b=z-\gamma_{\mathfrak{g}/\mathfrak{m}_1}(z)\in\theta(\mathfrak{n}_c)\mathfrak{u}(\mathfrak{g})$. From $\gamma_{\mathfrak{g}/\mathfrak{m}_1}(z)=d_P^{-1}\circ\mu_P(z)\circ d_P$, it follows that $d_P(a_tm)f(a_tm; z)=(d_Pf)(a_tm; \mu_P(z))+d_P(a_tm)f(a_tm; b)$. Since b is of the form $b=Xb'$ for some X in $\theta(\mathfrak{n}_c)$, b' in $\mathfrak{u}(\mathfrak{g}_c)$, $f(a_tm; b)$ is estimated as follows;

$$|f(a_tm; b)|\leqq\text{a const.}|\operatorname{Ad}(a_tm)|_{\theta(\mathfrak{n})}|\sup_{1\leqq i\leqq k}|f(X_i; a_tm; b')|$$

where $X_1, X_2, \cdots, X_k$ is a base of $\theta(\mathfrak{n})$. Since X_ifb' belongs to $H_{\tau'}(G, \chi)$

for a suitable double representation τ' (See Remark 2.1.), the above estimation for fb' and Lemma 4.2 imply that

$$\lim_{\varepsilon\to+0} \varepsilon \int_0^\infty dt \int_M \langle d_P(a_t m)f(a_t m; b'), e^{(\lambda_i t)}\phi(m)\rangle e^{-\varepsilon d(a_t m)} dm = 0 .$$

Hence we have

$$(5.7) \qquad (zf)_{\lambda_i}(\phi) = \lim_{\varepsilon\to+0} \varepsilon \int_0^\infty dt \int_M \langle (d_P f)(a_t m), (e^{\lambda_i t}\phi)(a_t m; \mu_P(z)^*)\rangle e^{-\varepsilon d(a_t m)} dm .$$

Define a ring homomorphism η_i of $\mathfrak{z}(\mathfrak{m}_1)$ onto $\mathfrak{z}(\mathfrak{m})$ as follows; $\eta_i(z)=z$ for z in $\mathfrak{z}(\mathfrak{m})$ and $\eta_i(H)=(\lambda_i-\rho_P)(H)$ for H in $\mathfrak{a}$. By definition, we have $\chi(z)f_{\lambda_i}(\phi)=f_{\lambda_i}((\eta_i\circ\mu_P(z))\phi)$ for all z in $\mathfrak{z}$. On the other hand, $\mathfrak{z}(\mathfrak{m}_1)$ is a finitely generated free $\mu_P(\mathfrak{z})$-module, that is, there exist $z_1, z_2, \cdots, z_n$ in $\mathfrak{z}(\mathfrak{m}_1)$ such that $\mathfrak{z}(\mathfrak{m}_1)=\mu_P(\mathfrak{z})z_1+\cdots+\mu_P(\mathfrak{z})z_n$. This implies $\mathfrak{z}(\mathfrak{m})=\eta_i(\mathfrak{z}(\mathfrak{m}_1))=(\eta_i\circ\mu_P)(\mathfrak{z})\eta_i(z_1)+\cdots+(\eta_i\circ\mu_P)(\mathfrak{z})\eta_i(z_n)$. Hence the dimension of $\mathfrak{z}(\mathfrak{m})f_{\lambda_i}$ is finite. It remains to prove f_i is analytic on M and f_P satisfies $\mu_P(z)f_P=\chi(z)f_P$. In view of the formula in (5.5), the finiteness for $\dim\mathfrak{z}(\mathfrak{m})f_i$ is obvious. Consequently f_i is a finite linear combination of the eigenfunctions for a certain elliptic differential operator on M. Therefore f_i is real analytic on M, and by (5.7) $\mu_P(z)f_P=\chi(\mathfrak{z})f_p$ for all z in $\mathfrak{z}$. This completes our proof.

Sumarizing the previous five lemmas we have the following.

THEOREM 1. *Let f be an element in $H_\tau(G, \chi)$ and $P=MAN$ the p.s.g.r. of G corresponding to a fixed simple root β in $\Phi(A_0)$. Then there is an analytic function f_P on M with the following properties;*
(5.8) (1) *f_P is of the form $f_P(a_t m)=\sum_{i=1}^q e^{\lambda_i t}f_i(m)$ for a_t in A, m in M where all λ_i's are purely imaginary and f_i's are square integrable on M,*

(2) *for each positive real numbers r and r',*

$$\int_{A(\beta,r)} |(d_P f-f_P)(a)|^2 D_M(a)(1+d(a))^{r'} da < \infty ,$$

(3) *$\mu_P(z)f_P=\chi(z)f_P$ for all z in $\mathfrak{z}$.*

REMARK 2. The function f_P on M has the similar properties to the constant term along P in the sense of Harish-Chandra (see Lemmas 57, 58 in [1], or Theorems 14.1, 14.2, Part II in [7]). In [1], the constant term of f was calculated under the assumption "f satisfies the weak inequality". In the above theorem, we assume that f is a τ-spherical eigenfunction of $\mathfrak{z}$ and $\lim_{\varepsilon\to+0}\varepsilon\int_G |f(x)|^2 e^{-\varepsilon d(x)}dx$ is finite.

Let f be an element in $H_\tau(G, \chi)$. We shall prove f satisfies the weak inequality which is an application of Theorem 5.1. From this fact we see that f_P is actually the constant term of f along P (cf., [7]).

Let ρ be one half the sum of all positive roots in $\Phi(\mathfrak{a}_0)$. We define a function Ξ on G, which is called elementary spherical function on G, by

$$\Xi(x)=\int_K e^{-\rho(H(xk))}dk\ , \quad x\in G,\ k\in K\ , \tag{5.9}$$

where $H(xk)$ is determined by $xk\in K\exp(H(xk))N_0$, $H(xk)\in\mathfrak{a}_0$. It is known that there exists a positive constant c and a number $m>0$ such that

$$e^{-\rho(\log a)}\leqq\Xi(a)\leqq ce^{-\rho(\log a)}(1+d(a))^m \quad \text{for all } a \text{ in } \mathrm{cl}(A_0^+) \tag{5.10}$$

(cf. Proposition 8.3.7.3 and Proposition 8.7.7.4 in [8]).

The Schwarz space $\mathfrak{C}(G)$ on G is defined by

$$\mathfrak{C}(G)=\{f\in C^\infty(G);\ \nu_{r,b_1,b_2}(f)<\infty \ \text{ for all } r\geqq 0,\ b_1,\ b_2\in\mathfrak{u}(\mathfrak{g})\} \tag{5.11}$$

where

$$\nu_{r,b_1,b_2}(f)=\sup_{x\in G}|(b_1fb_2)(x)|\,\Xi(x)^{-1}(1+d(x))^r\ .$$

We see that $\mathfrak{C}(G)$ is a topological vector space.

DEFINITION. A distribution T on G is called tempered if T is extended to a continuous linear form on $\mathfrak{C}(G)$.

THEOREM 2. *Let χ be a character of $\mathfrak{z}$ and f a C^∞-function on G with the properties; $zf=\chi(z)f$ for all z in $\mathfrak{z}$, f is K-finite and $\lim_{\varepsilon\to+0}\varepsilon\int_G|f(x)|^2e^{-\varepsilon d(x)}dx$ is finite. Define a distribution $T=T_f$ by $T(\phi)=\int_G f(x)\phi(x)dx$, $\phi\in C_c^\infty(G)$. Then T is tempered.*

REMARK 3. Since T is tempered, the function f in the above theorem satisfy the weak inequality (see for example, Lemma 8.3.8.7 in [9]), and consequently f_P is actually the constant term of f along P.

PROOF OF THEOREM 2. By the assumption for f, there exists a finite dimensional double representation (τ, V_τ) of K such that $f(x)=(h(x), v)$ for some h in $H_\tau(G,\chi)$ and v in V_τ. On the other hand, since $D(a)\leqq e^{2\rho(\log a)}$ for all a in $\mathrm{cl}(A_0^+)$, (5.10) implies that for a positive number p the integral $L=\int_{A_0^+}|\Xi(a)|^2D(a)(1+d(a))^{-p}da$ is finite. We put $\nu_p(\phi)=$

$\sup_{x\in G}|\phi(x)|\Xi(x)^{-1}(1+d(x))^{p}$ for ϕ in $C_c(G)$. Using the integral formula in Lemma 1.4, we get

$$\begin{aligned}|T(\phi)|&\leqq\nu_p(\phi)C_G|v|\int_{A_0^+}|h(a)|\Xi(a)(1+d(a))^{-p}D(a)da\\&\leqq\nu_p(\phi)C_G|v|L\int_{A_0^+}|h(a)|^2(1+d(a))^{-p}D(a)da\,.\end{aligned}$$

Therefore our proof is reduced the following: The integrals $I=\int_{A(\beta,r)}|h(a)|^2D(a)(1+d(a))^{-p}da$ are finite for all β in Ψ_0. (See Lemma 4.1.) Let P be the maximal p.s.g.r. of G corresponding to a fixed element β in Ψ_0. Then we have

$$\begin{aligned}I&\leqq\int_{A(\beta,r)}|(d_Pf)(a)|^2D(a)(1+d(a))^{-2}da\\&\leqq|d_Pf-f_P|^2_{P,r,1}+2|d_Pf-f_P|_{P,r,1}\left(\int_{A(\beta,r)}|f_P(a)|^2(1+d(a))^{-4}da\right)^{1/2}\\&\quad+\int_{A(\beta,r)}|f_P(a)|^2D_M(a)(1+d(a))^{-4}da\,.\end{aligned}$$

By Theorem 1, we have

$$|d_Pf-f_P|_{P,r,1}<\infty\quad\text{and}\quad\int_{A(\beta,r)}|f_P(a)|^2D_M(a)(1+d(a))^{-2}da<\infty\,.$$

Hence the theorem follows.

THEOREM 3. *Let f be an element in $H_\tau(G,\chi)$. Then the following three conditions are equivalent*

(1) $|f|=0$

(2) *f is square integrable on G*

(3) *$f_{P_\beta}=0$ for all β in Ψ_0.*

REMARK 3. The equivalence between (2) and (3) is a result of Harish-Chandra.

PROOF. By Theorem 5.1, f is square integrable if and only if $f_{P_\beta}=0$ for all β in Ψ_0. Therefore (2) is equivalent to (3). We shall prove that (1) is equivalent to (2). If f is square integrable on G, then by the definition of $|f|$ we have immediately $|f|=0$. Conversely assume that $|f|=0$. By Remark 5.1 and (1) in Theorem 5.1, we get $f_{P_\beta}=0$ for all β in Ψ_0. Hence f is square integrable on G. This completes our proof.

§6. Schur orthogonality relations.

Let us consider an irreducible unitary representation (π, H) of G with the following property;

(6.1) there exists a K-finite vector ϕ_0 in H such that

$$0<\lim_{\varepsilon\to+0}\varepsilon\int_G |(\pi(x)\phi_0, \phi_0)|^2 e^{-\varepsilon d(x)}dx<\infty .$$

In this section we shall state the Schur orthogonality relations for a nonsquare integrable representation π of G satisfying (6.1).

THEOREM 1. *Let χ be a character of $\mathfrak{z}$. We define $H(G, \chi)$ by (2.2) and a bilinear form $(\ ,\ )$ on $H(G, \chi)$ by $(f, g)=\lim_{\varepsilon\to+0}\varepsilon\int_G f(x)\overline{g(x)}e^{-\varepsilon d(x)}dx$. Assume that there exists a K-finite function f_0 in $H(G, \chi)$ such that $|f_0|=(f_0, f_0)^{1/2}>0$. Then $(\ ,\ )$ is a positive definite Hermitian form on $H(G, \chi)$.*

PROOF. We first show that $L^2(G)\cap H(G, \chi)=\{0\}$. Suppose that $L^2(G)\cap H(G, \chi)\neq\{0\}$. Then we have $L^2(G)\cap H_K(G, \chi)\neq\{0\}$. Let f be a nontrivial element in $L^2(G)\cap H_K(G, \chi)$. Since f is square integrable $\mathfrak{z}$-finite and K-finite function on G, χ is real regular (see §7, for the fact and definition of real regular character of $\mathfrak{z}$). On the other hand $H(G, \chi)$ contains K-finite function f_0 on G satisfying $|f_0|>0$. Consequently by Theorem 5.3, f_0 has a nontrivial constant term $(f_0)_P$ for a maximal p.s.g.r. of G. Therefore by (3) in Theorem 5.1, χ is not real regular. Hence we have a contradiction. This implies that $L^2(G)\cap H(G, \chi)=\{0\}$ as claimed. Finally we shall prove the lemma. Let f be an element in $H(G, \chi)$, and assume $|f|=0$. In view of (2.6), we have $|\chi_\tau * f * \chi_\sigma|=0$ for all σ, τ in $\mathscr{E}(K)$. Using Theorem 5.3, $\chi_\tau * f * \chi_\sigma$ belongs to $L^2(G)\cap H(G, \chi)$. Hence by the above fact $f=\sum_{\sigma,\tau\in\mathscr{E}(K)}\chi_\tau * f * \chi_\sigma=0$. This completes our proof.

Let χ and $H(G, \chi)$ be the same as above theorem. We denote the completion of $H(G, \chi)$ by H^χ. For each x in G and ϕ in H^χ, we define $\pi^\chi(x)$ by $(\pi^\chi(x)\phi)(y)=\phi(xy)$ for y in G. Since $H(G, \chi)$ is a unitary G-module (See Lemmas 2.1 and 2.4), (π^χ, H^χ) is a unitary representation of G.

Let (π, H) be an irreducible unitary representation of G and ϕ_0 an element in H satisfying (6.1). We denote the set of all K-finite functions in H by H_K. Since π is irreducible, there exists an infinitesimal character χ of $\mathfrak{z}$ such that $z\phi=\chi(z)\phi$ for all z in $\mathfrak{z}$. For this character χ, we define a representation (π^χ, H^χ) as above. Then the function $f_0(x)=(\pi(x)\phi_0, \phi_0)$ belongs to $H(G, \chi)$ (see the proof of Lemma 13 in [4]). We shall prove that π is equivalent to a subrepresentation of (π^χ, H^χ) after the following preparations. Let τ be an element in $\mathscr{E}(K)$. Define a projection operator of H to H_K by

$$E(\tau)\phi=\int_K \deg(\tau)\pi(k)\phi dk . \tag{6.2}$$

Let τ be an element in $\mathscr{E}(K)$ appearing in the restriction of π to K. We choose an element ϕ_0 ($\neq 0$) in H_K with the property; $E(\tau)\phi_0=\phi_0$. We put $\pi(f)=\int_G f(x)\pi(x)dx$ for f in $C_c^\infty(G)$ and $H(\phi_0)=\{\pi(f)\phi_0; f\in C_c^\infty(G)\}$. Since π is irreducible, the space $H(\phi_0)$ is dense in H. For a given ψ in H_K, we define a linear operator S_ψ of $H(\phi_0)$ to $C_c^\infty(G)$ by

$$S_\psi(\pi(f)\phi_0)(y)=(\pi(y)\pi(f)\phi_0, \psi) , \quad y\in G . \tag{6.3}$$

Then the image of S_ψ is contained in $H(G, \chi)$, and we have $S_\psi\circ\pi(x)=\pi^\chi(x)\circ S_\psi$ for all x in G (see Lemma 2.1).

LEMMA 1. *Let notations and assumptions being as above. We denote the minimal invariant subspace of H^χ containing $(\pi(y)\phi_0, \psi)$ by H_ψ and the restriction of π^χ to H_ψ by π_ψ. Then (π_ψ, H_ψ) is unitary equivalent to (π, H).*

PROOF. By definition, it is enough to show that (π_ψ, H_ψ) is irreducible and (π, H) is infinitesimal equivalent to (π_ψ, H_ψ) (cf. corollary 4.5.5.3 in [7]). We first prove π_ψ is irreducible. From $H(\phi_0)$ is dense in H and π is irreducible, it follows that S_ψ is injective. We put $H(\tau)=E(\tau)H$ and

$$V(\tau)=\left\{f\in C_c^\infty(G); \chi_\tau * f=f \text{ and } \int_K f(kxk^{-1})dk=f(x) \text{ for all } x \text{ in } G\right\} .$$

Then $V(\tau)$ is an algebra with the convolution product, and each element f in $V(\tau)$ operates on $H(\tau)$ canonically. It is known (cf. [2], Theorem 6, §2) that the representation $f\to\pi(f)$ of $V(\tau)$ is irreducible. Consequently, since $H(\tau)$ is finite dimensional, the representation $f\to\pi_\psi(f)$ of $V(\tau)$ on $S_\psi(H(\tau))$ is irreducible. Let W be a nontrivial closed invariant subspace of H_ψ and $W^\perp$ the orthogonal complement of W in H_ψ. Then we have $S_\psi(H(\tau))\subseteq E(\tau)W+E(\tau)W^\perp$, and hence by the above fact, $E(\tau)W\supseteq S_\psi(H(\tau))$ or $E(\tau)W^\perp\supseteq S_\psi(H(\tau))$. But H_ψ is the minimal closed invariant subspace of H^χ containing $S_\psi(\phi_0)$. Therefore $E(\tau)W\supseteq S_\psi(H(\tau))$. Thus $W=H_\psi$ and (π_ψ, H_ψ) is irreducible. Since $S_\psi\circ\pi(x)=\pi_\psi(x)\circ S_\psi$ on $H(\phi_0)$, the infinitesimal equivalence is obvious. Hence the lemma follows.

THEOREM 2. *Let (π, H) be an irreducible unitary representation of*

G, and assume that there exists a K-finite vector ϕ_0 such that $0 < \lim_{\varepsilon\to+0} \varepsilon \int_G |(\pi(x)\phi_0, \phi_0)|^2 e^{-\varepsilon d(x)} dx < \infty$. Then for a suitable constant d_π, we have

$$\lim_{\varepsilon\to+0} \varepsilon \int_G |(\pi(x)\phi, \psi)|^2 e^{-\varepsilon d(x)} dx = d_\pi^{-1} |\phi|^2 |\psi|^2$$

for all K-finite vectors ϕ, ψ in H_K.

PROOF. Let ϕ_0 be a fixed element in H_K. We define S_ψ and (π_ψ, H_ψ) as in (7.2) for ϕ_0 in H_K. Since π and π_ψ are unitary equivalent to each other, there is a unitary mapping Q_ψ of H_ψ to H such that $\pi(x)\circ Q_\psi = Q_\psi \circ \pi_\psi(x)$ for all x in G. We put $D_\psi = Q_\psi \circ S_\psi$. Therefore D_ψ and $\pi(x)$ are commutative. Since π is irreducible, $D_\psi = \lambda_\psi I$ for a suitable complex number λ_ψ where I is the identity mapping of $H(\phi_0)$. Consequently we have

$$|\lambda_\psi|^2 |\pi(f)\phi_0|^2 = |S_\psi(\pi(f)\phi_0)|^2 = \lim_{\varepsilon\to+0} \varepsilon \int_G |(\pi(x)\pi(f)\phi_0, \psi)|^2 e^{-\varepsilon d(x)} dx ,$$

for all f in $C_c^\infty(G)$. By using the same argument as above, we can prove that for a fixed element ψ_0 in H_k,

$$|\lambda_\phi|^2 |\pi(f)\psi_0|^2 = \lim_{\varepsilon\to+0} \varepsilon \int_G |(\pi(x)\pi(f)\phi, \psi_0)|^2 e^{-\varepsilon d(x)} dx,\ f \in C_c^\infty(G),\ \phi \in H_K ,$$

where λ_ϕ is a complex number. Combining these two equations, we obtain $\lim_{\varepsilon\to+0} \varepsilon \int_G |(\pi(x)\phi, \psi)|^2 e^{-\varepsilon d(x)} dx = |\lambda_\phi|^2 |\psi|^2 = |\lambda_\psi|^2 |\phi|^2$ for all ϕ, ψ in H_K. We now put $d_\pi = |\lambda_{\phi_0}|^{-2} |\phi_0|^2$. Since $d_\pi^{-1} |\phi_0|^2 |\psi|^2 = |\lambda_\psi|^2 |\phi_0|^2 = (|\lambda_\psi|^2 |\psi|^{-2})(|\phi_0|^2 |\psi|^2)$, we get $d_\pi = |\lambda_\psi|^{-2} |\psi|^2$ for all ψ in H_K. Hence we have our conclusion.

THEOREM 3. *Let (π_i, H_i) $(i=1, 2)$ be two irreducible unitary representation of G satisfying (6.1). Suppose π_1 and π_2 are inequivalent. Then we have, for each ϕ, ϕ' in $H_{1,K}$ and ψ, ψ' in $H_{2,K}$,*

$$\lim_{\varepsilon\to+0} \varepsilon \int_G (\pi_1(x)\phi, \phi')\overline{(\pi_2(x)\psi, \psi')} e^{-\varepsilon d(x)} dx = 0$$

where $H_{i,K}$ is the set of all K-finite vectors in H_i.

PROOF. Let ϕ_0 and ψ_0 be two fixed elements in $H_{1,K}$ and $H_{2,K}$ respectively. We define a bilinear form $\langle\ ,\ \rangle$ on $H_{1,K} \times H_{2,K}$ by $\langle \phi, \psi \rangle = \lim_{\varepsilon\to+0} \varepsilon \int_G (\pi_1(x)\phi, \phi_0)\overline{(\pi_2(x)\psi, \psi_0)} e^{-\varepsilon d(x)} dx$ for ϕ in $H_{1,K}$ and ψ in $H_{2,K}$. Using the Schwarz inequality and Theorem 6.2, we have $|\langle \phi, \psi \rangle| \leqq (d_{\pi_1} d_{\pi_2})^{-1} |\phi_0| |\psi_0| |\langle \phi, \psi \rangle|$ for all $\phi \in H_{1,K}$ and ψ in $H_{2,K}$.

Consequently the bilinear form can be extended to a continuous linear form on $H_1\times H_2$. Let S be the linear operator of $H_{1,K}$ to $H_{2,K}$ defined by $(S\phi, \psi)=\langle\phi, \psi\rangle$. Then S is continuous and $S\circ\pi_1(x)=\pi_2(x)\circ S$ for all x in G. Since π_i's are irreducible and inequivalent to each other, we get $S=0$. Thus the conclusion follows.

Let (π, H) be an irreducible unitary representation of G. We define a linear form θ_π on $C_c^\infty(G)$ by

$$\theta_\pi(f)=\mathrm{trace}\left(\int_G f(x)\pi(x)dx\right). \tag{6.4}$$

It is known that θ_π is a distribution on G (see, for instance, Theorem 4.5.7.6 in [8]).

THEOREM 4. *Let (π, H) be an irreducible unitary representation of G. Assume that there exists a K-finite vector ϕ_0 such that*

$$0<\lim_{\varepsilon\to+0}\varepsilon\int_G|(\pi(x)\phi_0, \phi_0)|^2e^{-\varepsilon d(x)}dx<\infty .$$

Then θ_π is tempered.

PROOF. Choose orthogonal base $\phi_1, \phi_2, \cdots, \phi_n, \cdots$ in H such that $E(\tau_i)\phi_i=\phi_i$ for some τ_i in $\mathscr{E}(K)$. We put $N(\tau)=$ the number of elements in the set $\{i: E(\tau)\phi_i=\phi_i\}$. Then we have $N(\tau)\leqq$ a const.$(\deg\tau)^2$ (cf. Theorem 4.5.2.10 in [8]). Let Ω_K be the Casimir operator on K and $\tau(\Omega_K)$ the positive constant defined by $\Omega_K\chi_\tau=\tau(\Omega_K)\chi_\tau$. Choosing a suitable positive number m, we have $\sum_{\tau\in\mathscr{E}(K)}N(\tau)\tau(\Omega_K)^{-m}<\infty$. Let h be an element in the Schwarz space $\mathfrak{C}(G)$ of G (see (5.11) for definition). We have immediately

$$|\theta_\pi(h)|\leqq\sum_{\tau\in\mathscr{E}(K)}\tau(\Omega_K)^{-m}\left(\sum_{\tau_i=\tau}\int_G|h(x;\Omega_K^m)|\,|(\pi(x)\phi_i, \phi_i)|\,dx\right). \tag{6.5}$$

On the other hand,

$$\begin{aligned}\int_G|h(x;\Omega_K^m)|\,|(\pi(x)\phi_i, \phi_i)|\,dx&\leqq\lim_{T\to\infty}\int_{T\geqq d(x)}|h(x;\Omega_K^m)|\,|(\pi(x)\phi_i, \phi_i)|\,dx\\&\leqq\lim_{T\to\infty}\left(\int_{T\geqq d(x)}|h(x;\Omega_K^m)|^2T^2dx\right)^{1/2}\left((1/T)^2\int_{T\geqq d(x)}|(\pi(x)\phi_i, \phi_i)|^2dx\right)^{1/2}.\end{aligned}$$

Bearing in mind Lemma 1.4 and (5.10), we find a positive number p so that $\lim_{T\to\infty}\int_{T\geqq d(x)}T\Xi(x)^2(1+d(x))^{-2p}dx<\infty$. Hence we obtain

$$\lim_{T\to\infty}T\int_{T\geqq d(x)}|h(x;\Omega_K^m)|^2dx\leqq\nu(h)^2\lim_{T\to\infty}T\int_{T\geqq d(x)}\Xi(x)^2(1+d(x))^{-2p} \tag{6.6}$$

where $\nu(h)=\sup_{x\in G}|h(x;\Omega_K^m)|\Xi(x)^{-1}(1+d(x))^{-p}$. Consequently our proof of the theorem is reduced the following.

LEMMA 2. *Let $H(G,\chi)$ be the same as in (2.2) and $H_K(G,\chi)$ the set of all K-finite functions in $H(G,\chi)$. Then there exists a positive real constant C_0 such that $\lim_{T\to\infty}(1/T)\int_{T\geqq d(x)}|f(x)|^2dx\leqq C_0|f|^2$ for all f in $H_K(G,\chi)$.*

PROOF. Let β be a fixed element in the simple root system Ψ_0 of $\Phi(A_0)$ and $A(\beta, r_0)$ the same as in Lemma 4.1. Then, by the integral formula in Lemma 1.4, we have

$$|f|^2\geqq\lim_{\varepsilon\to+0}\varepsilon\int_{A(\beta,r_0)}h(a)e^{-\varepsilon d(x)}dx \quad \text{where} \quad h(a)=\int_{K\times K}|f(kak')|^2D(a)dkdk'\ ,$$

hence by Lemma 4.1

$$\geqq\lim_{\varepsilon\to+0}\varepsilon\int_0^\infty\int_{B_{r_0}(t)}h(a_ta^*)e^{-\varepsilon d(a_ta^*)}da^*dt\ .$$

We choose an element H in $\mathfrak{a}$ $(=\{X\in\mathfrak{a}_0;\ \alpha(X)=0$ for all α in $\Psi_0-\{\beta\}\})$ satisfying $\beta(H)=1$. By (4.8) $r_0d(a_ta^*)\leqq c_2\rho(\log a_ta^*)\leqq c_2\beta(\log a_ta^*)\leqq c_2\bar{\beta}(\log a_ta^*)=c_2t$. This implies that

$$|f|^2\geqq\lim_{\varepsilon\to+0}\varepsilon\int_0^\infty\int_{B_{r_0}(t)}h(a_ta^*)e^{-\varepsilon c^{-1}t}da^*dt \tag{6.7}$$

$$\geqq\lim_{\varepsilon\to+0}\int_0^\infty\int_{B_{r_0}(t)}h(a_ta^*)e^{-\varepsilon c^{-1}t}da^*dt \tag{6.8}$$

where $c=r_0c_2^{-1}$.

Applying Tauberian theorem of Hardy-Littlewood to the integral (6.8), we have

$$|f|^2\geqq c\lim_{T\to+\infty}(1/T)\int_{\substack{T\geqq t\\ a_ta^*\in A(\beta,r_0)}}h(a_ta^*)da^*dt\ .$$

Again by (4.8), $d(a_ta^*)\geqq c_1\rho(\log a_ta^*)\geqq c_1\beta(\log a_t)=c_1t$. Therefore

$$\int_{\substack{T\geqq t\\ a_ta^*\in A(\beta,r_0)}}h(a_ta^*)da^*dt\geqq\int_{\substack{c_1T\geqq d(x)\\ a\in A(\beta,r_0)}}h(a)da\ .$$

Consequently we have $|f|^2\geqq c_1^{-1}c\lim_{T\to+\infty}(1/T)\int_{\substack{T\geqq d(x)\\ x\in KA(\beta,r_0)K}}|f(x)|^2dx$ for all f in $H_K(G,\chi)$. Let l be the number of elements in Ψ_0. By the above inequality, we have immediately $lc_1c^{-1}|f|^2\geqq\lim_{T\to+\infty}(1/T)\int_{T\geqq d(x)}|f(x)|^2dx$ for all f in $H(G, \chi)$. This completes our proof.

§7. Appendix.

Let χ be a character of $\mathfrak{z}$, and define the space $H_\tau(G, \chi)$ by (3.2) for a finite dimensional double unitary representation τ of K. In this section we shall calculate the constant term f_P of f in $H_\tau(G, \chi)$ more explicitely after the following preparations. Let $\tilde{\mathfrak{a}}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ and $\mu=\mu_{\mathfrak{g}/\tilde{\mathfrak{a}}}$ the Harish-Chandra isomorphism of $\mathfrak{z}$ onto $I(\tilde{\mathfrak{a}})$. For each character χ of $\mathfrak{z}$, there is a linear form Λ on $\tilde{\mathfrak{a}}_c$ such that $\mu(z)(\Lambda)=\chi(z)$ for all z in $\mathfrak{z}$. We shall denote $\chi=\chi_\Lambda$. Let $\mathfrak{a}_R$ and $\mathfrak{a}_I$ be the subspaces of $\tilde{\mathfrak{a}}$ defined by $\mathfrak{a}_R=\{H\in\tilde{\mathfrak{a}};\ \theta(H)=-H\}$ and $\mathfrak{a}_I=\{H\in\tilde{\mathfrak{a}};\ \theta(H)=H\}$ respectively. Then we have $\tilde{\mathfrak{a}}=\mathfrak{a}_R\oplus\mathfrak{a}_I$.

DEFINITION. A character $\chi=\chi_\Lambda$ of $\mathfrak{z}$ is called real if Λ is real on $\sqrt{-1}\,\mathfrak{a}_I\oplus\mathfrak{a}_R$, and χ is regular if Λ satisfies $(\Lambda, \alpha)\neq 0$ for all roots α in $\Phi(\tilde{\mathfrak{a}}_c)$.

Let τ be a finite dimensional double representation of K and $P=MAN$ the p.s.g.r. of G corresponding to a simple root β in $\Phi(A_0)$. We shall use the following notations;

χ^*: a character of the center $\mathfrak{z}(\mathfrak{m})$ of $\mathfrak{u}(\mathfrak{m})$,
$L^2_\tau(M;\chi^*)$: the set of all square integrable τ_{K_M}-spherical functions f which satisfy $zf=\chi^*(z)f$ for all z in $\mathfrak{z}(\mathfrak{m})$.

Let f be an element in $H_\tau(G, \chi)$. Then by Theorem 5.1, f_P is of the form

$$f_P(a_tm)=\sum_{i=1}^{q} e^{\lambda_i t}f_i(m) \tag{7.1}$$

for some f_i in $L_\tau^2(M;\chi_i^*)$ and λ_i in $\sqrt{-1}\boldsymbol{R}$ ($\lambda_i\neq\lambda_j$ for $i\neq j$) where χ_i^* is a character of $\mathfrak{z}(\mathfrak{m})$, $a_t=\exp(tH)$ and H is an element in $\mathfrak{a}$ satisfying $\beta(H)=1$. We now assume $f_P\neq 0$. By the results of Harish-Chandra (cf. [7], Proposition 15.7 and Proposition 15.13), we have $\operatorname{rank}(M)=\operatorname{rank}(K\cap M)$, and all χ_i^*'s are real regular. Let $\tilde{\mathfrak{a}}^*$ be a compact Cartan subalgebra of $\mathfrak{m}$. Then $\tilde{\mathfrak{a}}=\mathfrak{a}\oplus\tilde{\mathfrak{a}}^*$ is a Cartan subalgebra of $\mathfrak{g}$. We define a linear form $\bar{\beta}$ on $\tilde{\mathfrak{a}}$ by

$$\bar{\beta}=0 \text{ on } \tilde{\mathfrak{a}}^* \quad\text{and}\quad \bar{\beta}=\beta \text{ on } \mathfrak{a}\,. \tag{7.2}$$

Since $\mu_P(z)(e^{\lambda_i}f_i)=\chi(z)(e^{\lambda_i}f_i)$ for z in $\mathfrak{z}$, χ is given by $\chi=\chi_{(\lambda_i\bar{\beta}+\Lambda_i)}$ for a suitable real regular linear form Λ_i on $\tilde{\mathfrak{a}}^*$ satisfying $\chi_i^*=\chi_{\Lambda_i}^*$. This implies that there exists s_i in $W(\tilde{\mathfrak{a}})$ such that $s_i(\sqrt{-1}\lambda_i\bar{\beta}+\Lambda_i)=\sqrt{-1}\lambda_1\bar{\beta}+\Lambda_i$ for

$i=2, 3, \cdots, q$. Therefore $s_i(\lambda_i\bar{\beta})=\lambda_1\bar{\beta}$ for all i. This conclude that $i\leqq 2$ and $\lambda_i=\pm\lambda_1$, and hence we have the following.

LEMMA 1. *Let f be an element in $H_\tau(G, \chi)$. Assume that $f_{P_\beta}\neq 0$. Then P_β is cuspidal and f_{P_β} is of the form*

$$f_{P_\beta}(a_t m)=e^{\sqrt{-1}\lambda t}f_+(m)+e^{-\sqrt{-1}\lambda t}f_-(m)$$

where $a_t=\exp(tH)$, H is an element in $\mathfrak{a}$ satisfying $\beta(H)=1$, λ is a real number, f_+ and f_- belong to $L^2_\tau(M; \chi^*)$, χ^* is a real regular character of $\mathfrak{z}(\mathfrak{m})$.

LEMMA 2. *Let $P=P_\beta$ and $P'=P_{\beta'}$ be two maximal p.s.g.r. of G. Assume that there are two functions f in $H_\tau(G, \chi)$ and f' in $H_{\tau'}(G, \chi)$ such that $f_P\neq 0$ and $f'_{P'}\neq 0$ (τ and τ' are some finite dimensional double representations of K). Then we have $P=P'$.*

PROOF. Let $\tilde{\mathfrak{a}}_0$ be Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_0$ and $\bar{\beta}$ (resp. $\bar{\beta}'$) be the same as in (7.2) coresponding to β (resp. β'). Choosing two elements y_β in M_c and $y_{\beta'}$ in $M'_{c'}$ we have

(7.3) $\quad {}^t\mathrm{Ad}(y_\beta)\bar{\beta}=\beta$ on $\mathfrak{a}$, ${}^t\mathrm{Ad}(y_\beta)\bar{\beta}=0$ on $\mathfrak{a}^\perp$, $\mathrm{Ad}(y_\beta)(\tilde{\mathfrak{a}}_0)_c=\tilde{\mathfrak{a}}_{c'}$
$\quad {}^t\mathrm{Ad}(y_\beta)\bar{\beta}'=\beta'$ on $\mathfrak{a}'$, ${}^t\mathrm{Ad}(y_{\beta'})\bar{\beta}'=0$ on $(\mathfrak{a}')^\perp$, $\mathrm{Ad}(y_{\beta'})(\mathfrak{a}_0)_c=\mathfrak{a}'_c$
$\quad$ where $\mathfrak{a}^\perp$ is the orthogonal complement of $\mathfrak{a}$ in $\tilde{\mathfrak{a}}_0$.

We see that ${}^t\mathrm{Ad}(y_\beta)\bar{\beta}$ and ${}^t\mathrm{Ad}(y_{\beta'})\bar{\beta}'$ are dominant on $\mathfrak{a}_0$. Let χ^* and $(\chi')^*$ be the same as in Lemma 7.1 corresponding to P and P' respectively. Then $\chi^*=\chi^*_\Lambda$ and $(\chi')^*=(\chi')^*_{\Lambda}$ for two linear forms Λ and Λ'. Consequently we have $\chi=\chi_{{}^t\mathrm{Ad}(y_\beta)(\sqrt{-1}\lambda\beta+\Lambda)}=\chi_{{}^t\mathrm{Ad}(y_{\beta'})(\sqrt{-1}\lambda'\beta'+\Lambda')}$ for λ, λ' in $\sqrt{-1}\boldsymbol{R}$. Therefore $s\ {}^t\mathrm{Ad}(y_\beta)(\sqrt{-1}\lambda\beta+\Lambda)={}^t\mathrm{Ad}(y_{\beta'})(\sqrt{-1}\lambda'\beta'+\Lambda')$ for a suitable element s in $W(\tilde{\mathfrak{a}}_0)$, and hence $s{}^t\mathrm{Ad}(y_\beta)\bar{\beta}={}^t\mathrm{Ad}(y_{\beta'})\bar{\beta}'$. Since ${}^t\mathrm{Ad}(y_\beta)\bar{\beta}$ and ${}^t\mathrm{Ad}(y_{\beta'})\bar{\beta}'$ are dominant, we have $\beta=\beta'$. This completes our proof.

Finally we shall state an example G having a maximal cuspidal p.s.g.r. of G.

LEMMA 3. *Let G be a connected noncompact real semisimple Lie group with finite center. Assume that G has a compact Cartan subgroup. Then there exists a maximal p.s.g.r. of G which is cuspidal.*

PROOF. We choose, for each α in $\Phi(\tilde{\mathfrak{a}}_0)$, X_α in $\mathfrak{g}_\alpha$ satisfying $B(X_\alpha, X_{-\alpha})=1$. Then we have $B(H, H_\alpha)=\alpha(H)$ for all H in $\tilde{\mathfrak{a}}_0$. By our assumption for G, there exists a strongly orthogonal system $\{\alpha_1, \alpha_2, \cdots, \alpha_l\}$ in $\Phi(\tilde{\mathfrak{a}}_0)$ such that $\{H_{\alpha_i}; 1\leqq i\leqq l\}$ generates $\mathfrak{a}_0$ over $\boldsymbol{R}$ (see, for instance, Lemma 3 in [5]). We may assme all α_i's are positive. We put $\Phi^*=\{\alpha\in\Phi(A_0); (\alpha, \alpha_1)=0\}$.

Then Φ^* is an abstruct root system, and $\Psi_0-\Phi^*$ consists of exactly one root β. Let $P=P_\beta$ be the p.s.g.r. of G corresponding to β. We shall prove P is cuspidal. Let α be one of each element in the strongly orthogonal system. Since α is real on $\tilde{\mathfrak{a}}_0$, we can assume $X_\alpha \in \mathfrak{g}_\alpha$ and $\theta(X_\alpha)=-X_{-\alpha}$. Put $Y_{\pm\alpha}=\sqrt{2}|\alpha|^{-2}X_{\pm\alpha}$, $Z_\alpha=2|\alpha|^{-2}H_\alpha$. Then we have $[Y_\alpha, Y_{-\alpha}]=Z_\alpha$ and $[Z_\alpha, Y_{\pm\alpha}]=2Y_{\pm\alpha}$. Let Y_α be the element in G_c defined by $y_\alpha=\exp(\sqrt{-1}\pi/4)(Y_\alpha+Y_{-\alpha})$. Using the above relations between Z_α and $Y_{\pm\alpha}$, we can calculate that $\mathrm{Ad}(y_\alpha)\sqrt{-1}Z_\alpha=Y_\alpha-Y_{-\alpha}$ and $\mathrm{Ad}(y_\alpha)H=H$ for all H in the set $\{X_\alpha \in \tilde{\mathfrak{a}}_0;\ \alpha(X)=0\}$. From these facts it follows that $\tilde{\mathfrak{a}}^*=\mathrm{Ad}(y)(\mathfrak{a}_0\cap\mathfrak{k}+\sum_{i=2}^{l}\boldsymbol{R}Z_{\alpha_i})\subseteqq\mathfrak{k}\cap\mathfrak{m}$ where $y=\sum_{i=2}^{l}y_{\alpha_i}$. Furthermore, since $\tilde{\mathfrak{a}}^*\oplus\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$, we conclude $\tilde{\mathfrak{a}}^*$ is a Cartan subalgebra of $\mathfrak{m}$. This conclude that P is a maximal cuspidal p.s.g.r. of G.

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Present Address:
Department of Mathematics
Tsuda College
Tsuda-machi, Kodaira, Tokyo 187