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On the Mixed Problem for Wave Equation in a Domain with a Corner

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Introduction

The purpose of this paper is to generalize the results in [5] and to obtain the complete results.

We consider mixed problems

$$
\text{(I)}\quad \begin{cases} L_1[u]=\dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial^2 u}{\partial y^2}+du=f(t, x, y) \\ u(0, x, y)=u_0(x, y)\ , \qquad u_t(0, x, y)=u_1(x, y) \\ B_1[u]|_{x=0}=\left(\dfrac{\partial u}{\partial x}+b\dfrac{\partial u}{\partial y}-c\dfrac{\partial u}{\partial t}+\alpha u\right)\Big|_{x=0}=g_1(t, y) \\ B_2[u]|_{y=0}=\left(\dfrac{\partial u}{\partial y}+\dfrac{1}{b}\dfrac{\partial u}{\partial x}-\dfrac{c}{b}\dfrac{\partial u}{\partial t}+\dfrac{\alpha}{b}u\right)\Big|_{y=0}=g_2(t, x) \\ (t, x, y)\in(\boldsymbol{R}^1_+)^3 \end{cases}
$$

$$
\text{(II)}\quad \begin{cases} L_1[u]=f(t, x, y) \\ u(0, x, y)=u_0(x, y)\ , \qquad u_t(0, x, y)=u_1(x, y) \\ B_3[u]|_{x=0}=\left(\dfrac{\partial u}{\partial x}+\alpha u\right)\Big|_{x=0}=g_1(t, y) \\ B_4[u]|_{y=0}=\left(\dfrac{\partial u}{\partial y}+\beta u\right)\Big|_{y=0}=g_2(t, x) \\ (t, x, y)\in(\boldsymbol{R}^1_+)^3 \end{cases}
$$

and

$$
\text{(III)}\quad \begin{cases} L_2[u]=\dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial^2 u}{\partial y^2}-\dfrac{\partial^2 u}{\partial z^2}+du=f(t, x, y, z) \\ u(0, x, y, z)=u_0(x, y, z)\ , \qquad u_t(0, x, y, z)=u_1(x, y, z) \\ B_5[u]|_{x=0}=\left(\dfrac{\partial u}{\partial x}+\alpha u\right)\Big|_{x=0}=g_1(t, y, z) \end{cases}
$$

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$$\begin{cases} B_6[u]|_{y=0}=\left(\dfrac{\partial u}{\partial y}+\beta u\right)\Big|_{y=0}=g_2(t, x, z) \\ B_7[u]|_{z=0}=\left(\dfrac{\partial u}{\partial z}+\gamma u\right)\Big|_{z=0}=g_3(t, x, y) \\ (t, x, y, z)\in(\boldsymbol{R}^1_+)^4 \end{cases}$$

where b, c, d, α, β and γ are complex constants.

In [5], for the problem (I), we obtained the result that the problem (I) is L^2-well-posed if mixed problems

$$(1)\qquad \begin{cases} L_1[u]=f(t, x, y) \\ u(0, x, y)=u_0(x, y)\ , \qquad u_t(0, x, y)=u_1(x, y) \\ B_1[u]|_{x=0}=g_1(t, y) \\ (t, x, y)\in(\boldsymbol{R}^1_+)^2\times\boldsymbol{R}^1 \end{cases}$$

and

$$(2)\qquad \begin{cases} L_1[u]=f(t, x, y) \\ u(0, x, y)=u_0(x, y)\ , \qquad u_t(0, x, y)=u_1(x, y) \\ B_2[u]|_{y=0}=g_2(t, x) \\ (t, x, y)\in\boldsymbol{R}^1_+\times\boldsymbol{R}^1\times\boldsymbol{R}^1_+ \end{cases}$$

are L^2-well-posed and $b\neq\pm i$. In this paper, we shall show the result that the problem (I) is L^2-well-posed if the problems (1) and (2) are L^2-well-posed and $b=\pm i$. Therefore, we get the complete result that the problem (I) is L^2-well-posed if and only if the problems (1) and (2) are L^2-well-posed. Also, in [5], for the problems (II) and (III), we were concerned with the mixed problems with homogenenous boundary condition and could not obtain the boundary estimate for the solution. In this paper, we treat the mixed problems with non-homogeneous boundary condition and get the similar energy inequality as to the one for the mixed problem with the Neumann boundary condition in a domain with smooth boundary (see [3]).

To obtain the energy inequality, we reduce the mixed problem for wave equation to the one for symmetric hyperbolic system of first order with non-negative boundary condition. This method was used in [1], [4], [5], [6] and [7].

An outline of this paper is as follows. In § 1, we explain the notation. In § 2, we state the results. In § 3, we are concerned with roots of the quadratic equation $(c+1)z^2+2bz+(c-1)=0$ $(b=\pm i)$. In § 4, we treat the mixed problem for symmetric hyperbolic system of first order in a domain with a corner and give the simple proof comparing with the one in [5]. In § 5, we obtain the energy inequality. In § 6 and § 7, we prove the existence of the classical solution.

§ 1. Notation.

$\boldsymbol{R}^n(\boldsymbol{C}^n)$: n-dimensional real (complex) Euclidean space.

$\boldsymbol{R}^n_+$: the set $\{(x, y)\,|\,x>0,\ y\in \boldsymbol{R}^{n-1}\}$.

$$\|u\|^2_{m,\mu,T}=\sum_{\alpha+\beta+\gamma+\delta=m}\int_0^T dt\int_0^\infty dx\int_0^\infty dy\left|e^{-\mu t}\mu^\alpha\left(\frac{\partial}{\partial t}\right)^\beta\left(\frac{\partial}{\partial x}\right)^\gamma\left(\frac{\partial}{\partial y}\right)^\delta u\right|^2 \quad\text{or}$$

$$\sum_{\alpha+\beta+\gamma+\delta+\theta=m}\int_0^T dt\int_0^\infty dx\int_0^\infty dy\int_0^\infty dz\left|e^{-\mu t}\mu^\alpha\left(\frac{\partial}{\partial t}\right)^\beta\left(\frac{\partial}{\partial x}\right)^\gamma\left(\frac{\partial}{\partial y}\right)^\delta\left(\frac{\partial}{\partial z}\right)^\theta u\right|^2 .$$

$$\langle u\rangle^2_{m,\mu,T}=\sum_{\alpha+\beta+\gamma=m}\int_0^T dt\int_0^\infty dy\left|e^{-\mu t}\mu^\alpha\left(\frac{\partial}{\partial t}\right)^\beta\left(\frac{\partial}{\partial y}\right)^\gamma u\right|^2 \quad\text{or}$$

$$\sum_{\alpha+\beta+\gamma+\delta=m}\int_0^T dt\int_0^\infty dy\int_0^\infty dz\left|e^{-\mu t}\mu^\alpha\left(\frac{\partial}{\partial t}\right)^\beta\left(\frac{\partial}{\partial y}\right)^\gamma\left(\frac{\partial}{\partial z}\right)^\delta u\right|^2 .$$

$$\langle\!\langle u\rangle\!\rangle^2_{m,\mu,T}=\sum_{\alpha+\beta+\gamma=m}\int_0^T dt\int_0^\infty dx\left|e^{-\mu t}\mu^\alpha\left(\frac{\partial}{\partial t}\right)^\beta\left(\frac{\partial}{\partial x}\right)^\gamma u\right|^2 \quad\text{or}$$

$$\sum_{\alpha+\beta+\gamma+\delta=m}\int_0^T dt\int_0^\infty dx\int_0^\infty dz\left|e^{-\mu t}\mu^\alpha\left(\frac{\partial}{\partial t}\right)^\beta\left(\frac{\partial}{\partial x}\right)^\gamma\left(\frac{\partial}{\partial z}\right)^\delta u\right|^2 .$$

$$\langle\!\langle\!\langle u\rangle\!\rangle\!\rangle^2_{m,\mu,T}=\sum_{\alpha+\beta+\gamma+\delta=m}\int_0^T dt\int_0^\infty dx\int_0^\infty dy\left|e^{-\mu t}\mu^\alpha\left(\frac{\partial}{\partial t}\right)^\beta\left(\frac{\partial}{dx}\right)^\gamma\left(\frac{\partial}{\partial y}\right)^\delta u\right|^2 .$$

$$|||u(t)|||^2_{m,\mu}=\sum_{\alpha+\beta+\gamma+\delta=m}\int_0^\infty dx\int_0^\infty dy\left|e^{-\mu t}\mu^\alpha\left(\frac{\partial}{\partial t}\right)^\beta\left(\frac{\partial}{\partial x}\right)^\gamma\left(\frac{\partial}{\partial y}\right)^\delta u\right|^2 \quad\text{or}$$

$$\sum_{\alpha+\beta+\gamma+\delta+\theta=m}\int_0^\infty dx\int_0^\infty dy\int_0^\infty dz\left|e^{-\mu t}\mu^\alpha\left(\frac{\partial}{\partial t}\right)^\beta\left(\frac{\partial}{\partial x}\right)^\gamma\left(\frac{\partial}{\partial y}\right)^\delta\left(\frac{\partial}{\partial z}\right)^\theta u\right|^2 .$$

$(\ ,\ )$: the inner product in $L^2[(\boldsymbol{R}^1_+)^2]$ or $L^2[(\boldsymbol{R}^1_+)^3]$.

$((\ ,\ ))$: the inner product in $\boldsymbol{C}^j$.

$$\langle u, v\rangle=\int_0^\infty\int_0^\infty u\bar{v}\,dydz \quad\text{or}\quad \int_0^\infty u\bar{v}\,dy .$$

$$\langle\!\langle u, v\rangle\!\rangle=\int_0^\infty\int_0^\infty u\bar{v}\,dxdz \quad\text{or}\quad \int_0^\infty u\bar{v}\,dx .$$

$$\langle\!\langle\!\langle u, v\rangle\!\rangle\!\rangle=\int_0^\infty\int_0^\infty u\bar{v}\,dxdy .$$

$$[u, v]=\int_{-\infty}^\infty u\bar{v}\,d\eta .$$

$H_m(\Omega)$: the Sobolev space.

$\mathscr{H}_{m,\mu}[(\boldsymbol{R}^1_+)^n]$: the space of functions which are obtained by the completion of $C_0^\infty[(\bar{R}_+^1)^n]$ with the norm $\|u\|_{m,\mu,\infty}$.

$\Lambda_{x,\mu}^{-\theta}=\bar{\mathfrak{F}}_x(\xi^2+\mu^2)^{-\theta/2}\mathfrak{F}_x$, etc. .

$T_{\eta,\mu}^{-\theta}(\cdot)=(\eta^2+\mu^2)^{-\theta/2}\times(\cdot)$, etc. .

$D_x=\dfrac{\partial}{\partial x}$, etc. .

§ 2. Statement of the result.

We consider the mixed problems (I), (II) and (III).

We assume following conditions for the problem (I):

(C.1) $$b=i \quad \text{or} \quad b=-i$$

and

(C.2) The quadratic equation

$$(c+1)z^2+2bz+(c-1)=0 \tag{2.1}$$

has roots in the domain $\bar{D}=\{z\in \boldsymbol{C}\,|\,|z|\leqq 1,\ \mathrm{Re}\, z\leqq 0\}$ if they are different and in $D=\{z\in \boldsymbol{C}\,|\,|z|<1,\ \mathrm{Re}\, z<0\}$ if they are equal.

DEFINITION 1. (i) We say that $\{f, g_1, u_0, u_1\}$ satisfies the compatibility condition of order k in the region Ω_1 (Ω_2) if the following condition (C_{1k}) holds:

(C_{1k}) $$\tilde{B}^{(m)}(f, u_0, u_1)\equiv\sum_{j=0}^{m}\{\tilde{B}_{1j}^{(m)}u_j\}|_{x=0} =(D_t^{m-1}g_1)|_{t=0} \quad (m=1, 2, \cdots, k)$$

where

$$\begin{cases}\sum_{j=0}^{m}\tilde{B}_{1j}^{(m)}D_t^j u\equiv D_t^{m-1}\{\tilde{B}u\}\\ u_{2+i}\equiv\{(D_t^i f)|_{t=0}-(D_t^i\tilde{L}-D_t^{2+i})u\} \quad (i=0, 1, 2, \cdots)\\ \tilde{L}=L_1 \text{ or } L_2\,, \qquad \tilde{B}=B_1 \text{ or } B_3 \text{ or } B_5\end{cases}$$

and

$$\begin{cases}\Omega_1=\{y\,|\,y\geqq 0\} \text{ or } \{(y, z)\,|\,y\geqq 0,\ z\geqq 0\}\\ \Omega_2=\{y\,|\,y\in \boldsymbol{R}^1\}\,.\end{cases}$$

(ii) We say that $\{f, g_2, u_0, u_1\}$ satisfies the compatibility condition of order k in the region Ω_3 (Ω_4) if the following condition (C_{2k}) holds:

(C_{2k})
$$\tilde{B}^{(m)}(f, u_0, u_1) \equiv \sum_{j=0}^{m} \{\tilde{B}_{2j}^{(m)} u_j\}|_{y=0} = (D_t^{m-1} g_2)|_{t=0} \quad (m=1, 2, \cdots k)$$

where

$$\begin{cases} \sum_{j=0}^{m} \tilde{B}_{2j}^{(m)} D_t^j u = D_t^{m-1}(\tilde{B}u) \\ u_{2+i} \equiv \{(D_t^i f)|_{t=0} - (D_t^i \tilde{L} - D_t^{2+i}) u\} \quad (i=0, 1, 2, \cdots) \\ \tilde{L} = L_1 \text{ or } L_2 , \qquad \tilde{B} = B_2 \text{ or } B_4 \text{ or } B_6 \end{cases}$$

and

$$\begin{cases} \Omega_3 = \{x \mid x \geqq 0\} \text{ or } \{(x, z) \mid x \geqq 0, z \geqq 0\} \\ \Omega_4 = \{x \mid x \in \boldsymbol{R}^1\} . \end{cases}$$

(iii) We say that $\{f, g_3, u_0, u_1\}$ satisfies the compatibility condition of order k in the region Ω_5 if the following condition (C_{3k}) holds:

(C_{3k})
$$\tilde{B}^{(m)}(f, u_0, u_1) \equiv \sum_{j=0}^{m} \{\tilde{B}_{3j}^{(m)} u_j\}|_{z=0} = (D_t^{m-1} g_3)|_{t=0} \quad (m=1, 2, \cdots, k)$$

where

$$\begin{cases} \sum_{j=0}^{m} \tilde{B}_{3j}^{(m)} D_t^j u \equiv D_t^{m-1}(\tilde{B}u) \\ u_{2+i} \equiv \{(D_t^i f)|_{t=0} - (D_t^i \tilde{L} - D_t^{2+i}) u\} \quad (i=0, 1, 2, \cdots) \\ \tilde{L} = L_2 , \qquad \tilde{B} = B_7 \end{cases}$$

and

$$\Omega_5 = \{(x, y) \mid x \geqq 0, y \geqq 0\} .$$

DEFINITION 2. (i) We say that $\{g_1, g_2\}$ satisfies the compatibility condition (D_k) $(k=1, 3, 5)$ if the following condition holds:

(D_1) $$g_1(t, 0) = b \cdot g_2(t, 0)$$

(D_3) $$b g_{2xx}(t, 0) = \left[\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} + d\right) g_1\right](t, 0) - (B_1 f)(t, 0, 0)$$

(D_5) $$b g_{2xxxx}(t, 0) = \left[\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial y^2} + d\right)^2 g_1\right](t, 0) - B_1(f_{tt} - f_{yy} + f_{xx} + df)(t, 0, 0) .$$

(ii) We say that $\{g_1, g_2\}$ satisfies the compatibility condition (I_k) $(k=1, 3, 5, 7, 9)$ if the following condition holds:

(I_1) $$\left(\frac{\partial}{\partial y} + \beta\right) g_1 \Big|_{y=0} = \left(\frac{\partial}{\partial x} + \alpha\right) g_2 \Big|_{x=0}$$

$$(\mathrm{I}_3)\quad \left(\frac{\partial}{\partial y}+\beta\right)\left(\frac{\partial}{\partial y}\right)^2 g_1\Big|_{y=0}=\left(\frac{\partial}{\partial x}+\alpha\right)(M+d)g_2\Big|_{x=0}-\left(\frac{\partial}{\partial x}+\alpha\right)\left(\frac{\partial}{\partial y}+\beta\right)f\Big|_{x=y=0}$$

$$(\mathrm{I}_5)\quad \left(\frac{\partial}{\partial y}+\beta\right)\left(\frac{\partial}{\partial y}\right)^4 g_1\Big|_{y=0}=\left(\frac{\partial}{\partial x}+\alpha\right)(M+d)^2 g_2\Big|_{x=0}$$
$$-\left(\frac{\partial}{\partial x}+\alpha\right)\left(\frac{\partial}{\partial y}+\beta\right)\left(M+\frac{\partial^2}{\partial y^2}+d\right)f\Big|_{x=y=0}$$

$$(\mathrm{I}_7)\quad \left(\frac{\partial}{\partial y}+\beta\right)\left(\frac{\partial}{\partial y}\right)^6 g_1\Big|_{y=0}=\left(\frac{\partial}{\partial x}+\alpha\right)(M+d)^3 g_2\Big|_{x=0}-\left(\frac{\partial}{\partial x}+\alpha\right)\left(\frac{\partial}{\partial y}+\beta\right)$$
$$\times\left[(M+d)^2+\frac{\partial^2}{\partial y^2}\left\{M+\frac{\partial^2}{\partial y^2}+d\right\}\right]f\Big|_{x=y=0}$$

$$(\mathrm{I}_9)\quad \left(\frac{\partial}{\partial y}+\beta\right)\left(\frac{\partial}{\partial y}\right)^8 g_1\Big|_{y=0}=\left(\frac{\partial}{\partial x}+\alpha\right)(M+d)^4 g_2\Big|_{x=0}$$
$$-\left(\frac{\partial}{\partial x}+\alpha\right)\left(\frac{\partial}{\partial y}+\beta\right)\left[(M+d)^3+\frac{\partial^2}{\partial y^2}(M+d)^2\right.$$
$$\left.+\frac{\partial^4}{\partial y^4}\left\{M+\frac{\partial^2}{\partial y^2}+d\right\}\right]f\Big|_{x=y=0}$$

where

$$M=\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\quad\text{or}\quad M=\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\,.$$

(iii) We say that $\{g_2, g_3\}$ satisfies the compatibility condition (II_k) $(k=1, 3, 5)$ if the following condition holds:

$$(\mathrm{II}_1)\quad \left(\frac{\partial}{\partial z}+\gamma\right)g_2\Big|_{z=0}=\left(\frac{\partial}{\partial y}+\beta\right)g_3\Big|_{y=0}$$

$$(\mathrm{II}_3)\quad \left(\frac{\partial}{\partial z}+\gamma\right)g_{2zz}\Big|_{z=0}=\left(\frac{\partial}{\partial y}+\beta\right)\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+d\right)g_3\Big|_{y=0}$$
$$-\left(\frac{\partial}{\partial y}+\beta\right)\left(\frac{\partial}{\partial z}+\gamma\right)f\Big|_{y=z=0}$$

$$(\mathrm{II}_5)\quad \left(\frac{\partial}{\partial z}+\gamma\right)g_{2zzzz}\Big|_{z=0}=\left(\frac{\partial}{\partial y}+\beta\right)\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+d\right)^2 g_3\Big|_{y=0}$$
$$-\left(\frac{\partial}{\partial y}+\beta\right)\left(\frac{\partial}{\partial z}+\gamma\right)\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+d\right)f\Big|_{y=z=0}.$$

(iv) We say that $\{g_1, g_3\}$ satisfies the compatibility condition (III_k) $(k=1, 3, 5)$ if the following condition holds:

$$(\mathrm{III}_1)\quad \left(\frac{\partial}{\partial x}+\alpha\right)g_3\Big|_{x=0}=\left(\frac{\partial}{\partial z}+\gamma\right)g_1\Big|_{z=0}$$

(III$_3$) $$\left(\frac{\partial}{\partial x}+\alpha\right)g_{3xx}\Big|_{x=0}=\left(\frac{\partial}{\partial z}+\gamma\right)\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}+d\right)g_1\Big|_{z=0} -\left(\frac{\partial}{\partial x}+\alpha\right)\left(\frac{\partial}{\partial z}+\gamma\right)f\Big|_{x=z=0}$$

(III$_5$) $$\left(\frac{\partial}{\partial x}+\alpha\right)g_{3xxxx}\Big|_{x=0}=\left(\frac{\partial}{\partial z}+\gamma\right)\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}+d\right)^2 g_1\Big|_{z=0} -\left(\frac{\partial}{\partial x}+\alpha\right)\left(\frac{\partial}{\partial z}+\gamma\right)\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial x^2}+d\right)f\Big|_{x=z=0}.$$

DEFINITION 3. (i) We say that $\{f, g_1, u_0, u_1\}$ has the property (E_k):

(E_k) $\{f, g_1, u_0, u_1\}$ satisfies the (C_{1k}) in $\Omega_1=\{y\mid y\geqq 0\}$ and has an extension $\{\tilde{f}, \tilde{g}_1, \tilde{u}_0, \tilde{u}_1\}$ which satisfies the (C_{1k}) in Ω_2 and has the same regularity as $\{f, g_1, u_0, u_1\}$.

(ii) We say that $\{f, g_2, u_0, u_1\}$ has the property (E'_k):

(E'_k) $\{f, g_2, u_0, u_1\}$ satisfies the (C_{2k}) in $\Omega_3=\{x\mid x\geqq 0\}$ and has an extension $\{\tilde{f}, \tilde{g}_2, \tilde{u}_0, \tilde{u}_1\}$ which satisfies the (C_{2k}) in Ω_4 and has the same regularity as $\{f, g_2, u_0, u_1\}$.

(iii) We say that $\{\{f, g_1, u_0, u_1\}, \{f, g_2, u_0, u_1\}\}$ has the property (E''_k):

(E''_k) ① $\{f, g_1, u_0, u_1\}$ satisfies the (C_{1k}) in $\{(y, z)\mid y\geqq 0, z\geqq 0\}$ and has an extension $\{\tilde{f}, \tilde{g}_1, \tilde{u}_0, \tilde{u}_1\}$ which satisfies the (C_{1k}) in $\{(y, z)\mid y\geqq 0, z\in \boldsymbol{R}^1\}$ and has the same regularity as $\{f, g_1, u_0, u_1\}$.

② $\{f, g_2, u_0, u_1\}$ satisfies the (C_{2k}) in $\{(x, z)\mid x\geqq 0, z\geqq 0\}$ and has an extension $\{\tilde{f}, \tilde{g}_2, \tilde{u}_0, \tilde{u}_1\}$ which satisfies the (C_{2k}) in $\{(x, z)\mid (x, z)\in \boldsymbol{R}^2\}$ and has the same regularity as $\{f, g_2, u_0, u_1\}$.

③ $\{\tilde{g}_1, \tilde{g}_2\}$ satisfies the compatibility conditions (I_{2j-1}) $\left(j=1, 2, \cdots, \left[\frac{k-2}{2}\right]\right)$.

We now state our results,

THEOREM 1. *Assume the conditions* (C.1) *and* (C.2). *Let u be the solusion of the problem* (I) *which belongs to $\mathscr{H}_{2,\mu}[(\boldsymbol{R}^1_+)^3]$. Then, there exist positive constants C and μ_0 such that the following inequality holds for any $t\in \boldsymbol{R}^1_+$ and any $\mu\geqq\mu_0$*

(2.2) $$|||u(t)|||^2_{1,\mu}+\mu\|u\|^2_{1,\mu,t} +\mu\sum_{k=0}^{1}\left\{\left\langle \Lambda^{-1/2}_{y,\mu}\left(\frac{\partial}{\partial x}\right)^k u\right\rangle^2_{1-k,\mu,t}+\left\langle\!\left\langle \Lambda^{-1/2}_{x,\mu}\left(\frac{\partial}{\partial y}\right)^k u\right\rangle\!\right\rangle^2_{1-k,\mu,t}\right\}$$

$$\leqq C\left\{|||u(0)|||_{1,\mu}^2+\frac{1}{\mu}\|f\|_{0,\mu,t}^2+\frac{1}{\mu}\langle \Lambda_{y,\mu}^{1/2}g_1\rangle_{0,\mu,t}^2+\frac{1}{\mu}\langle\!\langle \Lambda_{x,\mu}^{1/2}g_2\rangle\!\rangle_{0,\mu,t}^2\right\}.$$

THEOREM 2. *Assume the conditions* (C.1) *and* (C.2). *Let* (f, g_1, g_2, u_0, u_1) *belongs to* $C_0^\infty[(\bar{R}_+^1)^3]\times[C_0^\infty[(\bar{R}_+^1)^2]]^4$ *and suppose that the conditions* (E_6'), (C_{16}) *in* $\{y|y\geqq 0\}$, (D_1), (D_3) *and* (D_5) *hold.*

Then, there exists a unique classical solution $u\in\mathscr{H}_{5,\mu}[(R_+^1)^3]$ *of the problem* (I) *which satisfies* (2.2).

REMARK 1. We have Theorem 2 by the assumption that the conditions (E_6), (C_{26}) in $\{x|x\geqq 0\}$, (D_1), (D_3) and (D_5) hold.

THEOREM 3. *Let* u *be the solution of the problem* (II) *which belongs to* $\mathscr{H}_{2,\mu}[(R_+^1)^3]$.

Then, there exist positive constants C *and* μ_0 *such that the following inequality holds for any* $t\in R_+^1$ *and any* $\mu\geqq\mu_0$

$$(2.3)\qquad |||u(t)|||_{1,\mu}^2+\mu\|u\|_{1,\mu,t}^2 +\mu\sum_{k=0}^{1}\left\{\left\langle \Lambda_{y,\mu}^{-1/2}\left(\frac{\partial}{\partial x}\right)^k u\right\rangle_{1-k,\mu,t}^2+\left\langle\!\!\left\langle \Lambda_{x,\mu}^{-1/2}\left(\frac{\partial}{\partial y}\right)^k u\right\rangle\!\!\right\rangle_{1-k,\mu,t}^2\right\}$$
$$\leqq C\left\{|||u(0)|||_{1,\mu}^2+\frac{1}{\mu}\|f\|_{0,\mu,t}^2+\frac{1}{\mu}\langle \Lambda_{y,\mu}^{1/2}g_1\rangle_{0,\mu,t}^2+\frac{1}{\mu}\langle\!\langle \Lambda_{x,\mu}^{1/2}g_2\rangle\!\rangle_{0,\mu,t}^2\right\}.$$

THEOREM 4. *Let* (f, g_1, g_2, u_0, u_1) *belongs* $C_0^\infty[(\bar{R}_+^1)^3]\times[C_0^\infty[(\bar{R}_+^1)^2]]^4$ *and suppose that the conditions* (E_8'), (C_{18}) *in* $\{y|y\geqq 0\}$ (I_1), (I_3) *and* (I_5) *hold.*

Then, there exists a unique classical solution $u\in\mathscr{H}_{5,\mu}[(R_+^1)^3]$ *of the problem* (II) *which satisfies* (2.3).

REMARK 2. We have Theorem 4 by the assumption that the conditions (E_8), (C_{28}) in $\{x|x\geqq 0\}$, (I_1), (I_3) and (I_5) hold.

THEOREM 5. *Let* u *be the solution of the problem* (III) *which belongs to* $\mathscr{H}_{2,\mu}[(R_+^1)^4]$.

Then, there exist positive constants C *and* μ_0 *such that the following inequality holds for any* $t\in R_+^1$ *and any* $\mu\geqq\mu_0$

$$(2.4)\qquad |||u(t)|||_{1,\mu}^2+\mu\|u\|_{1,\mu,t}^2+\mu\sum_{k=0}^{1}\left\{\left\langle \Lambda_{y,z,\mu}^{-1/2}\left(\frac{\partial}{\partial x}\right)^k u\right\rangle_{1-k,\mu,t}^2+\left\langle\!\!\left\langle \Lambda_{x,z,\mu}^{-1/2}\left(\frac{\partial}{\partial y}\right)^k u\right\rangle\!\!\right\rangle_{1-k,\mu,t}^2\right.$$
$$\left.+\left\langle\!\!\left\langle\!\!\left\langle \Lambda_{x,y,\mu}^{-1/2}\left(\frac{\partial}{\partial z}\right)^k u\right\rangle\!\!\right\rangle\!\!\right\rangle_{1-k,\mu,t}^2\right\}$$
$$\leqq C\left\{|||u(0)|||_{1,\mu}^2+\frac{1}{\mu}\|f\|_{0,\mu,t}^2+\frac{1}{\mu}\langle \Lambda_{y,z,\mu}^{1/2}g_1\rangle_{0,\mu,t}^2+\frac{1}{\mu}\langle\!\langle \Lambda_{x,z,\mu}^{1/2}g_2\rangle\!\rangle_{0,\mu,t}^2\right.$$
$$\left.+\frac{1}{\mu}\langle\!\langle\!\langle \Lambda_{x,y,\mu}^{1/2}g_3\rangle\!\rangle\!\rangle_{0,\mu,t}^2\right\}.$$

THEOREM 6. *Let* $(f, g_1, g_2, g_3, u_0, u_1)$ *belongs to* $C_0^\infty[(\bar{R}_+^1)^4]\times[C_0^\infty[(\bar{R}_+^1)^3]]^5$ *and suppose that the conditions* (E_{12}''), (C_{35}), (II_1), (II_3), (II_5), (III_1), (III_3) *and* (III_5) *hold.*

Then, there exists a unique classical solution $u\in\mathscr{H}_{5,\mu}^1[(R_+^1)^4]$ *of the problem* (III) *which satisfies* (2.4).

REMARK 3. We have Theorem 6 by the similar assumption in (y, z) or (x, z).

§ 3. The root of the quadratic equation (2.1).

We are concerned with the root of the quadratic equation (2.1) where $b=\pm i$.

Firstly, we treat the case where $b=i$. Then, the roots of (2.1) are $-i$ and $i((c-1)/(c+1))$. By (C.2) and the simple calculation, we have the following two cases for a root $i((c-1)/(c+1))$:

(i) $$\operatorname{Re} c>0$$

and

$$\left|i\left(\frac{c-1}{c+1}\right)\right|<1 . \tag{3.1}$$

(ii) $c=ic_1$ (c_1 is a positive number)

and

$$\operatorname{Re}\left\{i\left(\frac{c-1}{c+1}\right)\right\}<0 . \tag{3.2}$$

Secondly, we treat the case where $b=-i$. By the same arguments, we have the following two cases for a root $-i((c-1)/(c+1))$:

(i) $$\operatorname{Re} c>0$$

and

$$\left|-i\left(\frac{c-1}{c+1}\right)\right|<1 . \tag{3.3}$$

(ii) $c=ic_2$ (c_2 is a negative number)

and

$$\operatorname{Re}\left\{-i\left(\frac{c-1}{c+1}\right)\right\}<0 . \tag{3.4}$$

The above analysis is used to obtain the energy inequality in § 5.

§ 4. Mixed problem for symmetric hyperbolic system of first order.

We consider the mixed problem

$$(4.1)\qquad \begin{cases} \dfrac{\partial U}{\partial t}=A\dfrac{\partial U}{\partial x}+B\dfrac{\partial U}{\partial y}+K(t, x, y)U+F(t, x, y) \\ U(0, x, y)=U_0(x, y) \\ PU|_{x=0}=G_1(t, y) \\ QU|_{y=0}=G_2(t, x) \\ (t, x, y)\in(\boldsymbol{R}^1_+)^3 \end{cases}$$

where $U={}^t(U_1, \cdots, U_N)$, A and B are $N\times N$ constant Hermite matrices, $\det(AB)\neq 0$, K, P and Q are respectively $N\times N$, $p\times N$ and $q\times N$ smooth complex matrices, and are constant outside a compact set in $(\bar{\boldsymbol{R}}^1_+)^3$, $\bar{\boldsymbol{R}}^1_+\times\boldsymbol{R}^1$ and $(\bar{\boldsymbol{R}}^1_+)^2$.

We assume the following condition for the problem (4.1):

$$(\mathrm{C}.3)\qquad \begin{cases} ((AU, U))\geqq 0 \quad \text{for any } U\in\operatorname{Ker}P(t, y) \quad ((t, y)\in\bar{\boldsymbol{R}}^1_+\times\boldsymbol{R}^1) \\ ((BU, U))\geqq C((U, U)) \quad \text{for any } U\in\operatorname{Ker}Q(t, x) \quad ((t, x)\in(\bar{\boldsymbol{R}}^1_+)^2) \end{cases}$$

where C is a positive constant.

We extend K to the region $\{(t, x, y)\,|\,t\geqq 0, x\geqq 0, y<0\}$ as smooth functions and set $U(t, x, y)=0$ $(y<0)$. Then, by the Fourier transform of (4.1) with respect to y, we have

$$(4.2)\qquad \begin{cases} \hat{U}_t=A\hat{U}_x+i\eta B\hat{U}-B\cdot U(t, x, 0)+\widehat{KU}+\hat{F} \\ \hat{U}(0, x, \eta)=\hat{U}_0(x, \eta) \\ \widehat{PU}|_{x=0}=\hat{G}_1(t, \eta) \\ (t, x, \eta)\in(\boldsymbol{R}^1_+)^2\times\boldsymbol{R}^1 \end{cases}$$

where η is the dual variable of y. We set

$$(4.3)\qquad e^{-\mu t}T^{-1/2}_{\eta,\mu}\hat{U}=W\,.$$

Then, by (4.2) and (4.3), we obtain

$$W_x=A^{-1}W_t+\mu A^{-1}W-e^{-\mu t}T^{-1/2}_{\eta,\mu}(i\eta)A^{-1}B\hat{U}+e^{-\mu t}A^{-1}BT^{-1/2}_{\eta,\mu}\cdot U(t, x, 0)$$
$$-A^{-1}e^{-\mu t}T^{-1/2}_{\eta,\mu}\widehat{KU}-e^{-\mu t}A^{-1}T^{-1/2}_{\eta,\mu}\hat{F}$$

and A^{-1} is a Hermite matrix.

$$
\begin{aligned}
-\frac{d}{dx}[W, W] &= -[W_x, W]-[W, W_x] \\
&= -[A^{-1}W_t+\mu A^{-1}W-e^{-\mu t}T_{\eta,\mu}^{-1/2}(i\eta)A^{-1}B\hat{U} \\
&\quad +e^{-\mu t}A^{-1}BT_{\eta,\mu}^{-1/2}U(t, x, 0)-A^{-1}e^{-\mu t}T_{\eta,\mu}^{-1/2}\widehat{KU} \\
&\quad -e^{-\mu t}A^{-1}T_{\eta,\mu}^{-1/2}\hat{F}, W]-[W, A^{-1}W_t+\mu A^{-1}W \\
&\quad -e^{-\mu t}T_{\eta,\mu}^{-1/2}(i\eta)A^{-1}B\hat{U}+e^{-\mu t}A^{-1}BT_{\eta,\mu}^{-1/2}U(t, x, 0) \\
&\quad -e^{-\mu t}A^{-1}T_{\eta,\mu}^{-1/2}\widehat{KU}-e^{-\mu t}A^{-1}T_{\eta,\mu}^{-1/2}\hat{F}] \\
&= -\frac{d}{dt}[A^{-1}W, W]-2\mu[A^{-1}W, W] \\
&\quad +\{[e^{-\mu t}A^{-1}BT_{\eta,\mu}^{-1/2}(i\eta)\hat{U}, W]+[W, e^{-\mu t}A^{-1}BT_{\eta,\mu}^{-1/2}(i\eta)\hat{U}]\} \\
&\quad -[A^{-1}Be^{-\mu t}T_{\eta,\mu}^{-1/2}U(t, x, 0), W]-[W, A^{-1}Be^{-\mu t}T_{\eta,\mu}^{-1/2}U(t, x, 0)] \\
&\quad +[A^{-1}e^{-\mu t}T_{\eta,\mu}^{-1/2}\widehat{KU}, W]+[W, A^{-1}e^{-\mu t}T_{\eta,\mu}^{-1/2}\widehat{KU}] \\
&\quad +[A^{-1}e^{-\mu t}T_{\eta,\mu}^{-1/2}\hat{F}, W]+[W, A^{-1}e^{-\mu t}T_{\eta,\mu}^{-1/2}\hat{F}]\,.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
(4.4)\qquad \langle\Lambda_{\gamma,\mu}^{-1/2}U\rangle_{0,\mu,t}^2 &= \int_0^t\int_{-\infty}^{\infty}\int_0^{\infty}\Big\{-\frac{d}{dx}((W, W))\Big\}dx\,d\eta\,dt \\
&\leqq \frac{C_1}{\mu}\{|||U(t)|||_{0,\mu}^2+|||U(0)|||_{0,\mu}^2\}+C_2\|U\|_{0,\mu,t}^2+C_3\|U\|_{0,\mu,t}^2 \\
&\quad +\frac{C_4}{\mu}\langle\!\langle U\rangle\!\rangle_{0,\mu,t}^2+C_5\|U\|_{0,\mu,t}^2+\frac{C_6}{\mu}\|U\|_{0,\mu,t}^2 \\
&\quad +\frac{C_7}{\mu^2}\|F\|_{0,\mu,t}^2+C_8\|U\|_{0,\mu,t}^2\,.
\end{aligned}
$$

By (4.4), we obtain

Lemma 4.1. *Assume the condition* (C.3). *Let U be the solution of the problem* (4.1) *which belongs to* $\mathscr{H}_{1,\mu}[(\boldsymbol{R}_+^1)^8]$.

Then, there exist positive constants C and μ_0 such that

$$
\begin{aligned}
(4.5)\qquad \langle\Lambda_{\gamma,\mu}^{-1/2}U\rangle_{0,\mu,t}^2 \leqq C\Big\{&\frac{1}{\mu}|||U(t)|||_{0,\mu}^2+\frac{1}{\mu}|||U(0)|||_{0,\mu}^2+\frac{1}{\mu}\langle\!\langle U\rangle\!\rangle_{0,\mu,t}^2+\|U\|_{0,\mu,t}^2 \\
&+\frac{1}{\mu^2}\|F\|_{0,\mu,t}^2\Big\}
\end{aligned}
$$

for any $t\in\boldsymbol{R}_+^1$ and any $\mu\geqq\mu_0$.

Theorem 4.2. *Assume the condition* (C.3). *Let U be the solution of the problem* (4.1) *which belongs to* $\mathscr{H}_{1,\mu}[(\boldsymbol{R}_+^1)^8]$.

Then, there exist positive constants C and μ_0 such that the energy inequality holds for any $t\in R^1_+$ and any $\mu\geqq\mu_0$

$$\text{(4.6)}\qquad |||U(t)|||^2_{0,\mu}+\mu\|U\|^2_{0,\mu,t}+\mu\langle\Lambda^{-1/2}_{y,\mu}U\rangle^2_{0,\mu,t}+\langle\!\langle U\rangle\!\rangle^2_{0,\mu,t} \leqq C\Big\{|||U(0)|||^2_{0,\mu}+\frac{1}{\mu}\|F\|^2_{0,\mu,t}+\frac{1}{\mu}\langle\Lambda^{1/2}_{y,\mu}G_1\rangle^2_{0,\mu,t}+\langle\!\langle G_2\rangle\!\rangle^2_{0,\mu,t}\Big\}.$$

PROOF.

$$\text{(4.7)}\qquad \frac{d}{dt}(e^{-\mu t}U(t),\ e^{-\mu t}U(t)) = -2\mu(e^{-\mu t}U,\ e^{-\mu t}U)+(e^{-\mu t}(AU_x+BU_y+KU+F),\ e^{-\mu t}U) +(e^{-\mu t}U,\ e^{-\mu t}(AU_x+BU_y+KU+F)) \leqq -C_1\mu(e^{-\mu t}U,\ e^{-\mu t}U)+\frac{C_2}{\mu}(e^{-\mu t}F,\ e^{-\mu t}F)-\langle Ae^{-\mu t}U,\ e^{-\mu t}U\rangle -\langle\!\langle Be^{-\mu t}U,\ e^{-\mu t}U\rangle\!\rangle$$

where C_1 and C_2 are positive constants. By the condition (C.3), we obtain

$$\text{(4.8)}\qquad \langle Ae^{-\mu t}U,\ e^{-\mu t}U\rangle\geqq-\delta\mu\langle\Lambda^{-1/2}_{y,\mu}e^{-\mu t}U,\ \Lambda^{-1/2}_{y,\mu}e^{-\mu t}U\rangle -\frac{C_3}{\mu}\langle\Lambda^{1/2}_{y,\mu}e^{-\mu t}G_1,\ \Lambda^{1/2}_{y,\mu}e^{-\mu t}G_1\rangle$$

and

$$\text{(4.9)}\qquad \langle\!\langle Be^{-\mu t}U,\ e^{-\mu t}U\rangle\!\rangle\geqq C_4\langle\!\langle e^{-\mu t}U,\ e^{-\mu t}U\rangle\!\rangle-C_5\langle\!\langle e^{-\mu t}G_2,\ e^{-\mu t}G_2\rangle\!\rangle$$

where δ is a sufficiently positive constants, C_3, C_4 and C_5 are positive constants. By (4.6), (4.7), (4.8) and (4.9), we get Theorem 4.2. Q.E.D.

§ 5. Energy inequalities.

Firstly, we transform the mixed problems (I), (II) and (III) for wave equation into the ones for symmetric hyperbolic system of first order.

We set respectively

$$\text{(5.1)}\qquad \begin{cases} z_1=-i \\ z_2=i\left(\dfrac{c-1}{c+1}\right)\end{cases} \quad\text{or}\quad \begin{cases} z_1=i \\ z_2=-i\left(\dfrac{c-1}{c+1}\right)\end{cases}$$

for $b=i$ or $b=-i$, and use (5.1) for the problem (I).

LEMMA 5.1. *Assume the conditions* (C.1) *and* (C.2). *Then, the problem* (I) *is transformed into the following problem:*

$$
(5.2)\qquad \begin{cases} \dfrac{\partial U}{\partial t}=A_1\dfrac{\partial U}{\partial x}+B_1\dfrac{\partial U}{\partial y}+D_1U+F_1(t,\, x,\, y) \\ U(0,\, x,\, y)=U_0(x,\, y) \\ P_1U|_{x=0}=G_1(t,\, y) \\ Q_1U|_{y=0}=G_2(t,\, x) \\ (t,\, x,\, y)\in(\boldsymbol{R}^1_+)^3 \end{cases}
$$

where

$$
A_1=\begin{pmatrix} -1 & & & & 0 \\ & 1 & & & \\ & & -1 & & \\ & & & 1 & \\ 0 & & & & 1 \end{pmatrix}, \qquad B_1=\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & 1 \end{pmatrix}.
$$

D_1 *is a* 5×5 *constant matrix,* $F_1=(f,\, z_1f,\, f,\, z_2f,\, 0)$

$$
P_1=Q_1=\begin{pmatrix} 1 & z_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & z_1 & 0 \end{pmatrix}
$$

$$
G_1={}^t\left(-\frac{2}{c+1}g_1,\; -\frac{2}{c+1}g_1\right), \qquad G_2={}^t\left(-\frac{2b}{c+1}g_2,\; -\frac{2b}{c+1}g_2\right)
$$

and

$$
(5.3)\qquad \begin{cases} ((A_1U,\, U))\geqq 0 & \text{for any } U\in\operatorname{Ker} P_1 \\ ((B_1U,\, U))\geqq 0 & \text{for any } U\in\operatorname{Ker} Q_1\,, \end{cases}
$$

PROOF. We set

$$
(5.4)\qquad U=\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix}=\begin{pmatrix} u_t-(u_x+\alpha u)+z_1u_y \\ z_1\{u_t+(u_x+\alpha u)\}+u_y \\ u_t-(u_x+\alpha u)+z_2u_y \\ z_2\{u_t+(u_x+\alpha u)\}+u_y \\ u \end{pmatrix}.
$$

Then, by direct calculations, we have Lemma 5.1. Q.E.D

We treat the case where $\operatorname{Re} c>0$ in (I). We set

$$
(5.5)\qquad V=\begin{pmatrix} U_1 \\ U_2 \\ U_5 \end{pmatrix}
$$

for U in (5.4). Then, by (3.1) and (3.3), we have

LEMMA 5.2. *The following fact holds*:

$$
(5.6)\quad \begin{cases}
\dfrac{\partial V}{\partial t}=\begin{pmatrix}-1 & & 0\\ & 1 & \\ 0 & & 1\end{pmatrix}\dfrac{\partial V}{\partial x}+\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\dfrac{\partial V}{\partial y}+D_{11}V+E_{11}W+H_1\\
\qquad =A_{11}V_x+B_{11}V_y+D_{11}V+E_{11}W+H_1\\
V(0, x, y)=V_0(x, y)\\
P_{11}V|_{x=0}=-\dfrac{2}{c+1}g_1\\
Q_{11}V|_{y=0}=-\dfrac{2b}{c+1}g_2\\
(t, x, y)\in(\boldsymbol{R}^1_+)^3
\end{cases}
$$

where D_{11} *and* E_{11} *are respectively* 3×3 *and* 3×2 *constant matrices*, $W={}^t(U_3, U_4)$, $H_1={}^t(f, z_1f, 0)$, $P_{11}=Q_{11}=(1, z_2, 0)$ *and for a positive constant* C

$$
(5.7)\quad \begin{cases}
((A_{11}V, V))\geqq C((V, V)) & \text{for any } V\in\operatorname{Ker}P_{11}\\
((B_{11}V, V))\geqq 0 & \text{for any } V\in\operatorname{Ker}Q_{11}\,.
\end{cases}
$$

Next, we treat the case where $\operatorname{Re}c=0$ in (I). We set

$$
(5.8)\quad V=\begin{pmatrix}U_1\\ U_2\\ U_5\end{pmatrix}
$$

for U in (5.4). Then, by (3.2) and (3.4), we have

LEMMA 5.3. *The following fact holds*:

$$
(5.9)\quad \begin{cases}
\dfrac{\partial V}{\partial t}=\begin{pmatrix}-1 & & 0\\ & 1 & \\ 0 & & 1\end{pmatrix}\dfrac{\partial V}{\partial x}+\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\dfrac{\partial V}{\partial y}+D_{12}V+E_{12}W+H_2\\
\qquad =A_{11}V_x+B_{11}V_y+D_{12}V+E_{12}W+H_2\\
V(0, x, y)=V_0(x, y)\\
P_{12}V|_{x=0}=-\dfrac{2}{c+1}g_1\\
Q_{12}V|_{y=0}=-\dfrac{2b}{c+1}g_2\\
(t, x, y)\in(\boldsymbol{R}^1_+)^3
\end{cases}
$$

where D_{12} *and* E_{12} *are respectively* 3×3 *and* 3×2 *constant matrices*, $W=$

${}^t(U_3, U_4)$, $H_2=(f, z_1 f, 0)$, $P_{12}=Q_{12}=(1, z_2, 0)$ *and for a positive constant* C

$$(5.10)\qquad \begin{cases} ((A_{11}V, V))\geqq 0 & \textit{for any } V\in \operatorname{Ker} P_{12} \\ ((B_{11}V, V))\geqq C((V, V)) & \textit{for any } V\in \operatorname{Ker} Q_{12}\,. \end{cases}$$

Now, we consider the problems (II) and (III).

LEMMA 5.4. *The problem* (II) *is transformed into the following problem:*

$$(5.11)\qquad \begin{cases} \dfrac{\partial U}{\partial t}=A_2\dfrac{\partial U}{\partial x}+B_2\dfrac{\partial U}{\partial y}+D_2U+F_2 \\ U(0, x, y)=U_0(x, y) \\ P_2U|_{x=0}=-2g_1 \\ Q_2U|_{y=0}=\sqrt{2}\,g_2 \\ (t, x, y)\in (\boldsymbol{R}^1_+)^3 \end{cases}$$

where

$$A_2=\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 0 & \\ 0 & & & 1 \end{pmatrix},\qquad B_2=\begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

D_2 *is a* 4×4 *constant matrix,* $F_2={}^t(f, f, 0, 0)$

$$P_2=(1, -1, 0, 0)\,,\qquad Q_2=(0, 0, 1, 0)$$

and

$$(5.12)\qquad \begin{cases} ((A_2U, U))\geqq 0 & \textit{for any } U\in \operatorname{Ker} P_2 \\ ((B_2U, U))\geqq 0 & \textit{for any } U\in \operatorname{Ker} Q_2\,. \end{cases}$$

PROOF. We set

$$(5.13)\qquad U=\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix}=\begin{pmatrix} u_t-(u_x+\alpha u) \\ u_t+(u_x+\alpha u) \\ \sqrt{2}(u_y+\beta u) \\ u \end{pmatrix}.$$

Then, by direct calculations, we have Lemma 5.4. Q.E.D.

LEMMA 5.5. *The problem* (III) *is transformed into the following problem:*

$$(5.14)\qquad \begin{cases} \dfrac{\partial U}{\partial t}=A_3\dfrac{\partial U}{\partial x}+B_3\dfrac{\partial U}{\partial y}+E_3\dfrac{\partial U}{\partial z}+D_3U+F_3 \\ U(0,x,y,z)=U_0(x,y,z) \\ P_3U|_{x=0}=G_1 \\ Q_3U|_{y=0}=G_2 \\ R_3U|_{z=0}=G_3 \\ (t,x,y,z)\in(\boldsymbol{R}^1_+)^4 \end{cases}$$

where

$$A_3=\begin{pmatrix} -1 & & & & \\ & 1 & & 0 & \\ & & 0 & & \\ & 0 & & 0 & \\ & & & & 1 \end{pmatrix},\qquad B_3=\begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & & \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & & \\ & 0 & & 0 & \\ & & & & 0 \end{pmatrix}$$

$$E_3=\begin{pmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$F={}^t(f,f,0,0,0)\,,\qquad P_3=(-1,1,0,0,0)$$
$$Q_3=(0,0,1,0,0)\,,\qquad R_3=(0,0,0,1,0)$$
$$G_1=-2g_1\,,\qquad G_2=\sqrt{2}\,g_2\,,\qquad G_3=\sqrt{2}\,g_3$$

and

$$(5.15)\qquad \begin{cases} ((A_3U,U))\geqq 0 & \textit{for any } U\in\operatorname{Ker}P_3 \\ ((B_3U,U))\geqq 0 & \textit{for any } U\in\operatorname{Ker}Q_3 \\ ((E_3U,U))\geqq 0 & \textit{for any } U\in\operatorname{Ker}R_3\,. \end{cases}$$

PROOF. We set

$$(5.16)\qquad U=\begin{pmatrix}U_1\\U_2\\U_3\\U_4\\U_5\end{pmatrix}=\begin{pmatrix}u_t-(u_x+\alpha u)\\u_t+(u_x+\alpha u)\\\sqrt{2}(u_y+\beta u)\\\sqrt{2}(u_z+\gamma u)\\u\end{pmatrix}.$$

Then, by direct calculations, we have Lemma 5.5. Q.E.D.

Secondly, we shall get the energy inequalities for the problems (I), (II) and (III).

LEMMA 5.6. (1) *Let u belong to $\mathscr{H}_{2,\mu}[(R_+^1)^3]$. Then, we have*

$$(5.17)\qquad \begin{cases}\text{(i)}\ \int_0^\infty e^{-2\mu t}|(\Lambda_{y,\mu}^{-1/2}u_y)(t,0,y)|^2dy\leqq C|||u(t)|||_{1,\mu}^2\\ \text{(ii)}\ \int_0^\infty e^{-2\mu t}|(\Lambda_{x,\mu}^{-1/2}u_x)(t,x,0)|^2dx\leqq C|||u(t)|||_{1,\mu}^2\end{cases}$$

where C is a positive constant, any $t\in R_+^1$ and any $\mu\geqq\mu_0$ (μ_0 is a positive constant).

(2) *Let u belong to $\mathscr{H}_{2,\mu}[(R_+^1)^4]$. Then, we have*

$$(5.18)\qquad \begin{cases}\text{(i)}\ \int_0^\infty\int_0^\infty e^{-2\mu t}\{|(\Lambda_{y,z,\mu}^{-1/2}u_y)(t,0,y,z)|^2+|(\Lambda_{y,z,\mu}^{-1/2}u_z)(t,0,y,z)|^2\}dydz\\ \qquad\leqq C|||u(t)|||_{1,\mu}^2\\ \text{(ii)}\ \int_0^\infty\int_0^\infty e^{-2\mu t}\{|(\Lambda_{x,z,\mu}^{-1/2}u_x)(t,x,0,z)|^2+|(\Lambda_{x,z,\mu}^{-1/2}u_z)(t,x,0,z)|^2\}dxdz\\ \qquad\leqq C|||u(t)|||_{1,\mu}^2\\ \text{(iii)}\ \int_0^\infty\int_0^\infty e^{-2\mu t}\{|(\Lambda_{x,y,\mu}^{-1/2}u_x)(t,x,y,0)|^2+|(\Lambda_{x,y,\mu}^{-1/2}u_y)(t,x,y,0)|^2\}dxdy\\ \qquad\leqq C|||u(t)|||_{1,\mu}^2\end{cases}$$

where C is a positive constant, any $t\in R_+^1$ and any $\mu\geqq\mu_0$ (μ_0 is a positive constant).

The proof of this lemma is not given here, because it is popular.

LEMMA 5.7. (1) *Let u belong to $\mathscr{H}_{2,\mu}[(R_+^1)^3]$. Then, there exist positive constants C and μ_0 such that for any $t\in R_+^1$ and any $\mu\geqq\mu_0$*

$$(5.19)\qquad |||\mu u(t)|||_{0,\mu}^2+\mu||\mu u||_{0,\mu,t}^2+\mu\langle\Lambda_{y,\mu}^{-1/2}\mu u\rangle_{0,\mu,t}^2+\mu\langle\!\langle\Lambda_{x,\mu}^{-1/2}\mu u\rangle\!\rangle_{0,\mu,t}^2$$
$$\leqq C\{|||u(0)|||_{1,\mu}^2+\mu||u_t||_{0,\mu,t}^2+\mu||u_x||_{0,\mu,t}^2+\mu||u_y||_{0,\mu,t}^2\}.$$

(2) *Let u belong to $\mathscr{H}_{2,\mu}[(R_+^1)^4]$. Then, there exist positive constants*

C and μ_0 such that for any $t \in \boldsymbol{R}^1_+$ and any $\mu \geqq \mu_0$

$$
\begin{aligned}
(5.20) \qquad & |||\mu u(t)|||^2_{0,\mu} + \mu\|\mu u\|^2_{0,\mu,t} + \mu\langle \Lambda^{-1/2}_{y,z,\mu}\mu u\rangle^2_{0,\mu,t} \\
& + \mu\langle\!\langle \Lambda^{-1/2}_{x,z,\mu}\mu u\rangle\!\rangle^2_{0,\mu,t} + \mu \langle\!\langle\!\langle \Lambda^{-1/2}_{x,y}\mu u\rangle\!\rangle\!\rangle^2_{0,\mu,t} \\
& \leqq C\{|||u(0)|||^2_{1,\mu} + \mu\|u_t\|^2_{0,\mu,t} + \mu\|u_x\|^2_{0,\mu,t} + \mu\|u_y\|^2_{0,\mu,t} + \mu\|u_z\|^2_{0,\mu,t}\} .
\end{aligned}
$$

PROOF. It follows easily

$$
\begin{aligned}
\frac{d}{dt}(e^{-\mu t}u(t), e^{-\mu t}u(t)) &= -2\mu(e^{-\mu t}u, e^{-\mu t}u) + 2\,\mathrm{Re}(e^{-\mu t}u_t, e^{-\mu t}u) \\
&\leqq -C_1\mu(e^{-\mu t}u, e^{-\mu t}u) + \frac{C_2}{\mu}(e^{-\mu t}u_t, e^{-\mu t}u_t)
\end{aligned}
$$

where C_1 and C_2 are positive constants. Then, we get

$$
(5.21) \qquad |||\mu u(t)|||^2_{0,\mu} + C_1\mu\|\mu u\|^2_{0,\mu} \leqq |||u(0)|||^2_{1,\mu} + C_2\mu\|u_t\|^2_{0,\mu,t} .
$$

Also, we obtain

$$
\begin{aligned}
(5.22) \qquad \langle \mu u\rangle^2_{0,\mu,t} &= \int_0^t \langle e^{-\mu t}\mu u, e^{-\mu t}\mu u\rangle dt \\
&= -\mu\int_0^t \{(e^{-\mu t}u_x, e^{-\mu t}\mu u) + (e^{-\mu t}\mu u, e^{-\mu t}u_x)\}dt \\
&\leqq C\cdot\mu(\|\mu u\|^2_{0,\mu,t} + \|u_x\|^2_{0,\mu,t})
\end{aligned}
$$

and similarly, we have

$$
(5.23) \qquad \langle\!\langle \mu u\rangle\!\rangle^2_{0,\mu,t} \leqq C\cdot\mu(\|\mu u\|^2_{0,\mu,t} + \|u_y\|^2_{0,\mu,t})
$$

where C is a positive constant. Also, we have

$$
(5.24) \qquad \begin{cases} \langle \mu u\rangle^2_{0,\mu,t} \geqq \mu\langle \Lambda^{-1/2}_{y,\mu}\mu u\rangle^2_{0,\mu,t} \\ \langle\!\langle \mu u\rangle\!\rangle^2_{0,\mu,t} \geqq \mu\langle\!\langle \Lambda^{-1/2}_{x,\mu}\mu u\rangle\!\rangle^2_{0,\mu,t} . \end{cases}
$$

By (5.21), (5.22), (5.23) and (5.24), we have (5.19). By the same method, we have (5.20). Q.E.D.

PROOF OF THEOREM 1. We treat the case where $\mathrm{Re}\, c > 0$ in (I) because the similar method and Lemma 5.3 are applied to the case where $\mathrm{Re}\, c = 0$ in (I).

By results in § 4, Lemma 5.2 and Lemma 5.6, we have

$$
(5.25) \qquad \mu\sum_{k=0}^{1}\left\{\left\langle \Lambda^{-1/2}_{y,\mu}\left(\frac{\partial}{\partial x}\right)^k u\right\rangle^2_{1-k,\mu,t} + \left\langle\!\!\left\langle \Lambda^{-1/2}_{x,\mu}\left(\frac{\partial}{\partial y}\right)^k u\right\rangle\!\!\right\rangle^2_{1-k,\mu,t}\right\}
$$

$$\leqq C\Big\{|||u(0)|||^2_{1,\mu}+\frac{1}{\mu}||f||^2_{0,\mu,t}+\langle g_1\rangle^2_{0,\mu,t}+\frac{1}{\mu}\langle\!\langle \Lambda^{1/2}_{x,\mu}g_2\rangle\!\rangle^2_{0,\mu,t}$$
$$+\mu||u||^2_{1,\mu,t}+\frac{1}{\mu}||U_3||^2_{0,\mu,t}+\frac{1}{\mu}||U_4||^2_{0,\mu,t}\Big\}$$

where U_3 and U_4 in (5.4). For U in (5.4), we obtain

(5.26) $$\frac{d}{dt}(e^{-\mu t}U,\ e^{-\mu t}U)$$
$$=-2\mu(e^{-\mu t}U,\ e^{-\mu t}U)+(e^{-\mu t}U_t,\ e^{-\mu t}U)+(e^{-\mu t}U,\ e^{-\mu t}U_t)$$
$$=-2\mu(e^{-\mu t}U,\ e^{-\mu t}U)+(e^{-\mu t}(A_1U_x+B_1U_y+D_1U+F_1),\ e^{-\mu t}U)$$
$$+(e^{-\mu t}U,\ e^{-\mu t}(A_1U_x+B_1U_y+D_1U+F_1))$$
$$\leqq -C_1\mu(e^{-\mu t}U,\ e^{-\mu t}U)+\frac{C_2}{\mu}(e^{-\mu t}F_1,\ e^{-\mu t}F_1)$$
$$-\langle A_1e^{-\mu t}U,\ e^{-\mu t}U\rangle-\langle\!\langle B_1e^{-\mu t}U,\ e^{-\mu t}U\rangle\!\rangle$$

and by (5.3), we have

(5.27) $$\begin{cases}\langle A_1e^{-\mu t}U,\ e^{-\mu t}U\rangle\geqq -\delta\mu\langle\Lambda^{-1/2}_{y,\mu}e^{-\mu t}U,\ \Lambda^{-1/2}_{y,\mu}e^{-\mu t}U\rangle \\ \qquad\qquad -\dfrac{C_3}{\mu}\langle\Lambda^{1/2}_{y,\mu}e^{-\mu t}G_1,\ \Lambda^{1/2}_{y,\mu}e^{-\mu t}G_1\rangle \\ \langle\!\langle B_1e^{-\mu t}U,\ e^{-\mu t}U\rangle\!\rangle\geqq -\delta\mu\langle\!\langle\Lambda^{-1/2}_{x,\mu}e^{-\mu t}U,\ \Lambda^{-1/2}_{x,\mu}e^{-\mu t}U\rangle\!\rangle \\ \qquad\qquad -\dfrac{C_4}{\mu}\langle\!\langle\Lambda^{1/2}_{x,\mu}e^{-\mu t}G_2,\ \Lambda^{1/2}_{x,\mu}e^{-\mu t}G_2\rangle\!\rangle\end{cases}$$

where δ is a sufficiently small positive constant. By (5.4), (5.25), (5.26) and (5.27), we obtain Theorem 1. Q.E.D.

PROOF OF THEOREM 3. For U in (5.13), we set $U(t, x, y)=0$ $(y<0)$. By the Fourier transform of (5.11) with respect to y, we have

(5.28) $$\begin{cases}\hat{U}_t=A_2\hat{U}_x+i\eta B_2\hat{U}-B_2\cdot U(t,x,0)+D_2\hat{U}+\hat{F}_2 \\ \hat{U}(0,x,\eta)=\hat{U}_0(x,\eta) \\ P\hat{U}|_{x=0}=\hat{G}_1 \\ (t,x,\eta)\in(\boldsymbol{R}^1_+)^2\times\boldsymbol{R}^1\end{cases}$$

where $\hat{U}=\int_{-\infty}^{\infty}e^{-iy\cdot\eta}U(t,x,y)dy$. We set

(5.29) $$\begin{cases}e^{-\mu t}T^{-1}_{\eta,\mu}\hat{U}=V \\ e^{-\mu t}T^{-1/2}_{\eta,\mu}\hat{U}=W\,.\end{cases}$$

By $Q_2U|_{y=0}=\sqrt{2}\,g_2$, (5.28) and (5.29), we obtain

$$V_t=A_2V_x+e^{-\mu t}T_{\eta,\mu}^{-1}(i\eta)B_2\hat{U}-e^{-\mu t}T_{\eta,\mu}^{-1}B_2\cdot U(t,x,0)-\mu V+D_2V+e^{-\mu t}T_{\eta,\mu}^{-1}\hat{F}_2$$

and

$$B_2\cdot U(t,x,0)={}^t\left(g_2,\ g_2,\ \frac{U_1+U_2}{\sqrt{2}},\ 0\right). \tag{5.30}$$

Therefore, we obtain

$$\begin{aligned}-A_4V_x=&-A_2V_t+A_2B_2e^{-\mu t}T_{\eta,\mu}^{-1}(i\eta)\hat{U}-e^{-\mu t}T_{\eta,\mu}^{-1}\cdot H\\&+A_2(-\mu V+D_2V+e^{-\mu t}T_{\eta,\mu}^{-1}\hat{F}_2)\end{aligned} \tag{5.31}$$

where

$$A_4=\begin{pmatrix}1&&&0\\&1&&\\&&0&\\0&&&1\end{pmatrix} \tag{5.32}$$

and

$$H={}^t(-g_2,\ g_2,\ 0,\ 0)\ .$$

By $(V,\ e^{-\mu t}\hat{U})=(W,\ W)$ and (5.31), we get

$$\begin{aligned}-\frac{d}{dx}&[A_4W,\ W]\\&=-[A_4W_x,\ W]-[W,\ A_4W_x]\\&=-[A_4V_x,\ e^{-\mu t}\hat{U}]-[e^{-\mu t}\hat{U},\ A_4V_x]\\&=[-A_2V_t+A_2e^{-\mu t}T_{\eta,\mu}^{-1}(i\eta)B_2\hat{U}-e^{-\mu t}T_{\eta,\mu}^{-1}\cdot H\\&\quad+A_2(-\mu V+D_2V+e^{-\mu t}T_{\eta,\mu}^{-1}\hat{F}_2),\ e^{-\mu t}\hat{U}]\\&\quad+[e^{-\mu t}\hat{U},\ -A_2V_t+A_2e^{-\mu t}T_{\eta,\mu}^{-1}(i\eta)B_2\hat{U}-e^{-\mu t}T_{\eta,\mu}^{-1}\cdot H\\&\quad+A_2(-\mu V+D_2V+e^{-\mu t}T_{\eta,\mu}^{-1}\hat{F}_2)]\\&=-\frac{d}{dt}[A_2W,\ W]+[A_2B_2e^{-\mu t}T_{\eta,\mu}^{-1}(i\eta)\hat{U},\ e^{-\mu t}\hat{U}]\\&\quad+[e^{-\mu t}\hat{U},\ A_2B_2e^{-\mu t}T_{\eta,\mu}^{-1}(i\eta)\hat{U}]-[e^{-\mu t}T_{\eta,\mu}^{-1}\cdot H,\ e^{-\mu t}\hat{U}]\\&\quad-[e^{-\mu t}\hat{U},\ e^{-\mu t}T_{\eta,\mu}^{-1}\cdot H]+[A_2(-\mu V+D_2V+e^{-\mu t}T_{\eta,\mu}^{-1}\hat{F}_2),\ e^{-\mu t}\hat{U}]\\&\quad+[e^{-\mu t}\hat{U},\ A_2(-\mu V+D_2V+e^{-\mu t}T_{\eta,\mu}^{-1}\hat{F})]\ .\end{aligned} \tag{5.33}$$

By Lemma 5.6 and (5.33), we have

$$(5.34)\qquad \langle \Lambda_{y,\mu}^{-1/2}U\rangle_{0,\mu,t}^2 \leqq \frac{C_1}{\mu}(|||U(t)|||_{0,\mu}^2+|||U(0)|||_{0,\mu}^2)+C_2\|U\|_{0,\mu,t}^2 + \frac{C_3}{\mu^2}\|F_2\|_{0,\mu,t}^2+\frac{C_4}{\mu^2}\langle\!\langle \Lambda_{x,\mu}^{1/2}G_2\rangle\!\rangle_{0,\mu,t}^2$$

where C_1, C_2, C_3 and C_4 are positive constants. By the symmetricity of the condition in x and y, we get

$$(5.35)\qquad \langle\!\langle \Lambda_{x,\mu}^{-1/2}U\rangle\!\rangle_{0,\mu,t}^2 \leqq \frac{C_1'}{\mu}(|||U(t)|||_{0,\mu}^2+|||U(0)|||_{0,\mu}^2)+C_2'\|U\|_{0,\mu,t}^2 + \frac{C_3}{\mu^2}\|F_2\|_{0,\mu,t}^2+\frac{C_4'}{\mu^2}\langle \Lambda_{y,\mu}^{1/2}G_1\rangle_{0,\mu,t}^2$$

where C_1', C_2', C_3' and C_4' are positive constants. For U in (5.13), we obtain

$$(5.36)\qquad \frac{d}{dt}(e^{-\mu t}U,\ e^{-\mu t}U)\leqq -C_1\mu(e^{-\mu t}U,\ e^{-\mu t}U)+\frac{C_2}{\mu}(e^{-\mu t}F_2,\ e^{-\mu t}F_2) -\langle A_2e^{-\mu t}U,\ e^{-\mu t}U\rangle-\langle\!\langle B_2e^{-\mu t}U,\ e^{-\mu t}U\rangle\!\rangle$$

and by (5.12), we have

$$(5.37)\qquad \begin{cases} \langle A_2e^{-\mu t}U,\ e^{-\mu t}U\rangle \geqq -\delta\mu\langle \Lambda_{y,\mu}^{-1/2}e^{-\mu t}U,\ \Lambda_{y,\mu}^{-1/2}e^{-\mu t}U\rangle \\ \qquad\qquad -C_3\langle \Lambda_{y,\mu}^{1/2}e^{-\mu t}G_1,\ \Lambda_{y,\mu}^{1/2}e^{-\mu t}G_1\rangle \\ \langle\!\langle B_2e^{-\mu t}U,\ e^{-\mu t}U\rangle\!\rangle \geqq -\delta\mu\langle\!\langle \Lambda_{x,\mu}^{-1/2}e^{-\mu t}U,\ \Lambda_{x,\mu}^{-1/2}e^{-\mu t}U\rangle\!\rangle \\ \qquad\qquad -C_4\langle\!\langle \Lambda_{x,\mu}^{1/2}e^{-\mu t}G_2,\ \Lambda_{x,\mu}^{1/2}e^{-\mu t}G_2\rangle\!\rangle \end{cases}$$

where δ is a sufficiently small positive constant, C_1, C_2, C_3 and C_4 are positive constants. By (5.34), (5.35), (5.36) and (5.37), we get Theorem 3. Q.E.D.

PROOF OF THEOREM 5. For U in (5.16), we set $U(t, x, y, z)=0$ ($y<0$ or $z<0$). By the Fourier transform of (5.14) with respect to (y, z), we have

$$(5.38)\qquad \hat{U}_t=A_3\hat{U}_x+i\eta B_3\hat{U}+i\zeta E_3\hat{U}-B_3\tilde{U}(t, x, 0, \zeta)-E_3\tilde{\tilde{U}}(t, x, \eta, 0)+D_3\hat{U}+\hat{F}_3$$

where

$$\begin{cases} \hat{U}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(y\cdot\eta+z\cdot\zeta)}U(t, x, y, z)dydz \\ \tilde{U}=\int_{-\infty}^{\infty} e^{-iz\cdot\zeta}U(t, x, 0, z)dz \\ \tilde{\tilde{U}}=\int_{-\infty}^{\infty} e^{-iy\cdot\eta}U(t, x, y, 0)dy\ . \end{cases}$$

We set

$$(5.39)\qquad \begin{cases} e^{-\mu t}T^{-1}_{\eta,\zeta,\mu}\hat{U}=V \\ e^{-\mu t}T^{-1/2}_{\eta,\zeta,\mu}\hat{U}=W\,. \end{cases}$$

By $Q_3U|_{y=0}=G_2$, $R_3U|_{z=0}=G_3$, (5.38) and (5.39), we obtain

$$(5.40)\qquad \begin{aligned} V_t=&A_3V_x+e^{-\mu t}T^{-1}_{\eta,\zeta,\mu}\{(i\eta)B_3\hat{U}+(i\zeta)E_3\hat{U}\} \\ &-e^{-\mu t}T^{-1}_{\eta,\zeta,\mu}\{B_2\tilde{U}(t,x,0,\zeta)+E_3\tilde{\tilde{U}}(t,x,\eta,0)\} \\ &-\mu V+D_3V+e^{-\mu t}T^{-1}_{\eta,\zeta,\mu}\hat{F}_3 \end{aligned}$$

and

$$(5.41)\qquad \begin{cases} B_3\cdot\tilde{U}(t,x,0,\zeta)={}^t\left(\dfrac{\tilde{G}_2}{\sqrt{2}},\ \dfrac{\tilde{G}_2}{\sqrt{2}},\ \dfrac{\tilde{U}_1+\tilde{U}_2}{\sqrt{2}},\ 0,\ 0\right) \\ E_3\cdot\tilde{\tilde{U}}(t,x,\eta,0)={}^t\left(\dfrac{\tilde{\tilde{G}}_3}{\sqrt{2}},\ \dfrac{\tilde{\tilde{G}}_3}{\sqrt{2}},\ 0,\ \dfrac{\tilde{\tilde{U}}_1+\tilde{\tilde{U}}_2}{\sqrt{2}},\ 0\right)\,. \end{cases}$$

Therefore, we have

$$(5.42)\qquad \begin{aligned} -A_5V_x=&A_3V_t-A_3[e^{-\mu t}T^{-1}_{\eta,\zeta,\mu}(i\eta B_3\hat{U}+i\zeta E_3\hat{U})] \\ &-e^{\mu t}T^{-1}_{\eta,\zeta,\mu}(H_1\tilde{U}+H_2\tilde{\tilde{U}})+A_3[-\mu V+D_3V+e^{-\mu t}T^{-1}_{\eta,\zeta\mu}\hat{F}_3] \end{aligned}$$

where

$$(5.43)\qquad A_5=\begin{pmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & 0 & & \\ & 0 & & 0 & \\ & & & & 1 \end{pmatrix}$$

and

$$(5.44)\qquad \begin{cases} H_1={}^t\left(-\dfrac{\tilde{G}_2}{\sqrt{2}},\ \dfrac{\tilde{G}_2}{\sqrt{2}},\ 0,0,0\right) \\ H_2={}^t\left(-\dfrac{\tilde{\tilde{G}}_3}{\sqrt{2}},\ \dfrac{\tilde{\tilde{G}}_3}{\sqrt{2}},\ 0,0,0\right)\,. \end{cases}$$

By the same arguments as the one for the proof of Theorem 3, Lemma 5.5, Lemma 5.6 and (5.42), we have Theorem 5. Q.E.D.

§ 6. The existence of the solution (I).

In this section, we shall prove Theorems 2 and 4.

LEMMA 6.1. *Let u be the solution of the problem* (I) *which belongs to* $\mathscr{H}_{6,\mu}[(\boldsymbol{R}^1_+)^3]$.

Then, there exist positive constants C and μ_0 such that the following inequality holds for any $t \in \boldsymbol{R}^1_+$ and any $\mu \geqq \mu_0$

$$\text{(6.1)}\quad |||u(t)|||^2_{5,\mu}+\mu\|u\|^2_{5,\mu,t}+\mu\sum_{k=0}^{1}\left\{\left\langle \Lambda^{-1/2}_{y,\mu}\left(\frac{\partial}{\partial x}\right)^k u\right\rangle^2_{5-k,\mu,t}+\left\langle\!\left\langle \Lambda^{-1/2}_{x,\mu}\left(\frac{\partial}{\partial y}\right)^k u\right\rangle\!\right\rangle^2_{5-k,\mu,t}\right\}$$
$$\leqq C\left\{|||u(0)|||^2_{5,\mu}+\frac{1}{\mu}\|f\|^2_{4,\mu,t}+\frac{1}{\mu}\langle \Lambda^{1/2}_{y,\mu}g_1\rangle^2_{4,\mu,t}+\frac{1}{\mu}\langle\!\langle \Lambda^{1/2}_{x,\mu}g_2\rangle\!\rangle^2_{4,\mu,t}\right\}.$$

PROOF. By the same method in [1: § 5], we have Lemma 6.1. Q.E.D.

PROOF OF THEOREM 2. By Lemma 6.1 and the same arguments in [5: § 5], we have Theorem 2. Q.E.D.

PROOF OF THEOREM 4. We consider the mixed problem

$$\text{(6.2)}\quad \begin{cases} L_1[w_1]=\tilde{f}(t, x, y) \\ w_1(0, x, y)=\tilde{u}_0(x, y)\ , \qquad w_{1t}(0, x, y)=\tilde{u}_1(x, y) \\ B_4[w_1]|_{y=0}=\tilde{g}_2(t, x) \\ (t, x, y)\in \boldsymbol{R}^1_+\times\boldsymbol{R}^1\times\boldsymbol{R}^1_+ \end{cases}$$

where $\tilde{l}$ is an extended function in the domain $\{(t, x, y)\,|\,t\geqq 0,\ x<0,\ y\geqq 0\}$ or $\{(x, y)\,|\,x<0,\ y\geqq 0\}$. Then, we have the solution $w_1\in\mathscr{H}_{8,\mu}[\boldsymbol{R}^1_+\times\boldsymbol{R}^1\times\boldsymbol{R}^1_+]$ of the problem (6.3) and w_1 has a compact support in the domain $\boldsymbol{R}^1_x\times\bar{\boldsymbol{R}}^1_{+y}$ for fixed t ($\geqq 0$). We set

$$\text{(6.3)}\quad n(t, y)=\left(\frac{\partial}{\partial y}+\beta\right)g_1(t, y)-\left(\frac{\partial}{\partial y}+\beta\right)\left[\left(\frac{\partial}{\partial x}+\alpha\right)w_1\right]\Big|_{x=0}.$$

Then, we obtain, by (I_1),

$$\text{(6.4)}\quad n(t, 0)=\left(\frac{\partial}{\partial y}+\beta\right)g_1(t, 0)-\left(\frac{\partial}{\partial x}+\alpha\right)\left[\left(\frac{\partial}{\partial y}+\beta\right)w_1\Big|_{y=0}\right]\Big|_{x=0}$$
$$=\left(\frac{\partial}{\partial y}+\beta\right)g_1(t, 0)-\left(\frac{\partial}{\partial x}+\alpha\right)g_2(t, 0)=0\ .$$

Also, by (I_3) and (I_5), we have

$$\text{(6.5)}\quad \begin{cases} n_{yy}(t, 0)=0 \\ n_{yyyy}(t, 0)=0\ . \end{cases}$$

We extend $n(t, y)$ to the region $\{y\,|\,y<0\}$ by the following

$$\text{(6.6)}\quad \tilde{n}(t, y)=\begin{cases} n(t, y) & (y\geqq 0) \\ -n(t, -y) & (y<0)\ . \end{cases}$$

Then, we have $\Lambda_{y,\mu}^{1/2}\tilde{n}\in\mathscr{H}_{5,\mu}[(R_+^1\times R^1)]$ and $\tilde{n}$ has a compact support in R_y^1 for fixed t ($\geqq 0$). Here, we consider the problem

$$(6.7)\qquad \begin{cases} L_1[w_2]=0 \\ w_2(0, x, y)=0\ , \qquad w_{2t}(0, x, y)=0 \\ B_3[w_2]|_{x=0}=\tilde{n} \\ (t, x, y)\in(R_+^1)^2\times R^1\ . \end{cases}$$

Then, we have the solution w_2 of the problem (6.7) which belongs to $\mathscr{H}_{6,\mu}[(R_+^1)^2\times R^1]$ and has a compact support in the region $\bar{R}_{+x}^1\times R_y^1$ for fixed $t(\geqq 0)$. Also, we have $w_2(t, x, 0)=0$. Next, we solve the equation

$$(6.8)\qquad \frac{\partial w_3}{\partial y}+\beta w_3=w_2$$

for $L(R_{+y}^1)$ space. Then, we have the solution

$$w_3=e^{-\beta y}\int_\infty^y e^{\beta s}w_2(t, x, s)ds\ .$$

We set

$$u=w_1+w_3\ .$$

By the above construction, we obtain the solution u of the problem (II) which satisfies Theorem 4. Q.E.D.

§ 7. The existence of the solution (II).

In this section, we shall prove Theorem 6.

By the assumption, we extend u_0, u_1, f, g_1 and g_2 to the region $\{z\,|\,z<0\}$. We consider the problem

$$(7.1)\qquad \begin{cases} L_2[w_1]=\tilde{f}(t, x, y, z) \\ w_1(0, x, y, z)=\tilde{u}_0(x, y, z)\ , \qquad w_{1t}(0, x, y, z)=\tilde{u}_1(x, y, z) \\ B_5[w_1]|_{x=0}=\tilde{g}_1(t, y, z) \\ B_6[w_1]|_{y=0}=\tilde{g}_2(t, x, z) \\ (t, x, y, z)\in(R_+^1)^3\times R^1\ . \end{cases}$$

By the assumption and the result in § 6, we have the solution $w_1\in\mathscr{H}_{10,\mu}[(R_+^1)^3\times R^1]$ of the problem (7.1) and w_1 has a compact support in the domain $\bar{R}_{+x}^1\times\bar{R}_{+y}^1\times R_z^1$ for fixed t ($\geqq 0$). We set

$$(7.2)\qquad n(t, x, y)=\Big(\frac{\partial}{\partial x}+\alpha\Big)\Big(\frac{\partial}{\partial y}+\beta\Big)g_3-\Big(\frac{\partial}{\partial x}+\alpha\Big)\Big(\frac{\partial}{\partial y}+\beta\Big)\Big(\frac{\partial}{\partial z}+\gamma\Big)w_1\Big|_{z=0}\ .$$

Then, we have, by (III_1) and (III_3),

$$n(t,0,y)=\left(\frac{\partial}{\partial y}+\beta\right)\left\{\left[\left(\frac{\partial}{\partial x}+\alpha\right)g_3\Big|_{x=0}\right]-\left(\frac{\partial}{\partial z}+\gamma\right)\left[\left(\frac{\partial}{\partial x}+\alpha\right)w_1\Big|_{x=0}\right]\Big|_{z=0}\right\}$$
$$=\left(\frac{\partial}{\partial y}+\beta\right)\left\{\left(\frac{\partial}{\partial x}+\alpha\right)g_3\Big|_{x=0}-\left(\frac{\partial}{\partial z}+\gamma\right)g_1\Big|_{z=0}\right\}=0 \tag{7.3}$$

and

$$n_{xx}(t,0,y)=\left(\frac{\partial}{\partial y}+\beta\right)\left\{\left[\left(\frac{\partial}{\partial x}+\alpha\right)g_{3xx}\Big|_{x=0}\right]-\left(\frac{\partial}{\partial z}+\gamma\right)\left[\left(\frac{\partial}{\partial x}+\alpha\right)w_{1xx}\Big|_{x=0}\right]\Big|_{z=0}\right\}$$
$$=\left(\frac{\partial}{\partial y}+\beta\right)\left\{\left(\frac{\partial}{\partial x}+\alpha\right)g_{3xx}\Big|_{x=0}-\left(\frac{\partial}{\partial z}+\gamma\right)\left[\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}+d\right)g_1\right.\right.$$
$$\left.\left.-\left(\frac{\partial}{\partial x}+\alpha\right)f\Big|_{x=0}\right]\Big|_{z=0}\right\}$$
$$=0\,. \tag{7.4}$$

Similarly, we get, by (III_5),

$$n_{xxxx}(t,0,y)=0\,. \tag{7.5}$$

Also, by (II_1), (II_3) and (II_5), we obtain

$$n(t,x,0)=n_{yy}(t,x,0)=n_{yyyy}(t,x,0)=0\,. \tag{7.6}$$

Now, we consider the mixed problem

$$\begin{cases} L_2[w_2]=0 \\ w_2(0,x,y,z)=0\,, \qquad w_{2t}(0,x,y,z)=0 \\ B_7[w_2]|_{z=0}=\tilde{n}(t,x,y) \\ (t,x,y,z)\in \boldsymbol{R}^1_+\times(\boldsymbol{R}^1)^2\times\boldsymbol{R}^1_+ \end{cases} \tag{7.7}$$

where

$$\tilde{n}(t,x,y)=\begin{cases} n(t,x,y) & (x\geqq 0,\ y\geqq 0) \\ -n(t,-x,y) & (x<0,\ y\geqq 0) \\ -n(t,x,-y) & (x\geqq 0,\ y<0) \\ n(t,-x,-y) & (x<0,\ y<0)\,. \end{cases} \tag{7.8}$$

Then, we have the solution $w_2\in\mathscr{H}_{5,\mu}[\boldsymbol{R}^1_+\times(\boldsymbol{R}^1)^2\times\boldsymbol{R}^1_+]$ of the problem (7.7) which satisfies

$$w_2(t,0,y,z)=w_2(t,x,0,z)=0 \tag{7.9}$$

and has a compact support in (x, y, z) for fixed t ($\geqq 0$). We solve the equation

$$\left(\frac{\partial}{\partial x}+\alpha\right)\left(\frac{\partial}{\partial y}+\beta\right)w_3=w_2 \tag{7.10}$$

for $L^2(R^1_{+x}\times R^1_{+y})$ space. Then, we get the solution

$$w_3=e^{-\alpha x-\beta y}\int_{\infty}^{x}\int_{\infty}^{y} e^{\alpha r+\beta s}w_2(t, r, s, z)drds\ . \tag{7.11}$$

The function w_3 satisfies

$$\begin{cases} w_3(0, x, y, z)=w_{3t}(0, x, y, z)=0 \\ \left(\frac{\partial}{\partial x}+\alpha\right)w_3\Big|_{x=0}=0 \\ \left(\frac{\partial}{\partial y}+\beta\right)w_3\Big|_{y=0}=0 \\ \left(\frac{\partial}{\partial z}+\gamma\right)w_3\Big|_{z=0}=-\left(\frac{\partial}{\partial z}+\gamma\right)w_1\Big|_{z=0}+g_3(t, x, y) \end{cases} \tag{7.12}$$

and

$$L_2[w_3]=0\ . \tag{7.13}$$

We set $u=w_1+w_3$. By the above construction, we obtain the solution u of the problem (III) which satisfies Theorem 6. Q.E.D.

References

[1] K. Kojima and M. Taniguchi, Mixed problem for hyperbolic equations in a domain with a corner, Funkcial. Ekvac., **23** (1980), 171–195.

[2] S. Miyatake, Mixed problem for hyperbolic equations of second order with first order complex boundary operators, Japan. J. Math., **1** (1975), 111–158.

[3] S. Miyatake, A sharp form of the existence theorem for hyperbolic mixed problems of second order, J. Math. Kyoto Univ., **17** (1977), 199–223.

[4] M. Taniguchi, Mixed problem for wave equation in the domain with a corner, Funkcial. Ekvac., **21** (1978), 249–259.

[5] M. Taniguchi, Mixed problem for hyperbolic equations of second order in a domain with a corner, Tokyo J. Math., **5** (1982), 183–211.

[6] M. Taniguchi, Mixed problem for weakly hyperbolic equations of second order with degenerate first order boundary condition, Tokyo J. Math., **7** (1984), 61–98.

[7] M. Taniguchi, Mixed problem for weakly hyperbolic equations of second order with degenerate Neumann boundary condition, Funkcial. Ekvac., **27** (1984), 331–366.

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