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On the Convolution of Functions of Two Variables and Generalized Harmonic Analysis

Dedicated to Professor Hisaharu Umegaki on his sixtieth birthday

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Introduction

In the linear filter theory, Wiener considered especially a weighting K in the time domain, i.e. the filters $K*$ for which the response g to an input signal f is given by

$$g(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}K(t-\tau)f(\tau)d\tau=(K*f)(t)\ ,\quad t\in(-\infty,\infty)\ .$$

Also, he indicated the importance of admitting as inputs arbitrary signals of the class S. His main theorem in [9] is: If

$$(1+|t|)K(t)\in L^1\cap L^2(-\infty,\infty)\ ,$$

then the responce of the filter K to a signal f in S is a signal $g\in S'$, by using the generalized harmonic analysis (cf. Masani [5]).

In this paper, we shall extend this result to the case of functions of two variables under a restricted rectangular mean concerning the double limit process, using the generalized harmonic analysis of functions of two variables in Matsuoka [6].

Wiener has proved a Tauberian theorem in a generalized sense, with respect to a weighted moving average of a function which is bounded on the average. On the other hand, in Anzai, Koizumi and Matsuoka [1], we have considered the form of general Tauberian theorems about a weighted moving average $K*f$ of f. We shall also extend the above theorem of Wiener to the case of functions of two variables under a restricted rectangular mean concerning the double limit process in consideration of the modified form.

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The proofs can be done along the similar lines as in Wiener [9].

§ 1. The spectral analysis of convolution.

Throughout this paper, all functions are assumed to be complex valued and (Borel) measurable and we use the following notation (see Matsuoka [6]):

(a) $W(R^2)=\left\{f(x_1, x_2)\in L^2_{\mathrm{loc}}(R^2)\colon \sup_{0<S,T<\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|f(s,t)|^2 ds dt<\infty\right\}$;

(b) The double generalized Fourier transform of f:

$$\begin{aligned} s(u, v; f) &= \underset{A\to\infty}{\mathrm{l.i.m.}}\ \frac{1}{2\pi}\left[\int_1^A+\int_{-A}^{-1}\right]\left[\int_1^A+\int_{-A}^{-1}\right] f(s,t)\frac{e^{-ius}}{-is}\frac{e^{-ivt}}{-it}dsdt \\ &+\underset{A\to\infty}{\mathrm{l.i.m.}}\ \frac{1}{2\pi}\left[\int_1^A+\int_{-A}^{-1}\right]\int_{-1}^{1} f(s,t)\frac{e^{-ius}-1}{-is}\frac{e^{-ivt}}{-it}dsdt \\ &+\underset{A\to\infty}{\mathrm{l.i.m.}}\ \frac{1}{2\pi}\int_{-1}^{1}\left[\int_1^A+\int_{-A}^{-1}\right] f(s,t)\frac{e^{-ius}}{-is}\frac{e^{-ivt}-1}{-it}dsdt \\ &+\frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1} f(s,t)\frac{e^{-ius}-1}{-is}\frac{e^{-ivt}-1}{-it}dsdt\ ; \end{aligned}$$

(c) $\Delta_{\varepsilon,\eta}s(u, v; f)=s(u+\varepsilon, v+\eta; f)-s(u-\varepsilon, v+\eta; f)-s(u+\varepsilon, v-\eta; f)$
$+s(u-\varepsilon, v-\eta; f)$;

(d) The notations $\mathscr{R}_1$-$\lim_{S,T\to\infty}$ and $\mathscr{R}_2$-$\lim_{\varepsilon,\eta\to 0}$ mean that in each of them a limit exists and has the same limit for every positive constant C whenever S and T tend to infinity or ε and η tend to zero in such a way that $S=CT$ or $\eta=C\varepsilon$ respectively;

(e) $\phi(x_1, x_2; f)=\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S} f(x_1+s, x_2+t)\overline{f(s,t)}dsdt$;

(f) $S(R^2)=\{f(x_1, x_2)\in W(R^2)\colon \phi(x_1, x_2; f)$ exists for all $(x_1, x_2)\in R^2\}$;

(g) $S'(R^2)=\{f(x_1, x_2)\in S(R^2)\colon \phi(x_1, x_2; f)$ is continuous on $R^2\}$;

(h) The Fourier transform of f:

$$\hat{f}(u, v)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s,t)e^{-i(us+vt)}dsdt\ ;$$

(i) The convolution of f and g:

$$(f*g)(x_1, x_2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x_1-s, x_2-t)g(s,t)dsdt\ ;$$

(j) $\mathscr{K}(R^2)=\{K(x_1, x_2): (1+|x_1|)(1+|x_2|)K(x_1, x_2)\in L^1\cap L^2(R^2)\}$;

(k) By $\mathscr{K}_0$, we denote a subclass of $\mathscr{K}(R^2)$, such that for each $(u, v)\in R^2$, there is in $\mathscr{K}_0$ a function $K(x_1, x_2)=K_{u,v}(x_1, x_2)$ depending on (u, v) with $\hat{K}(u, v)\neq 0$.

In this section, we shall show a multiplier property, which is analogous to that in the harmonic analysis, concerning the convolution $K*f$ of $f\in W(R^2)$ and $K\in\mathscr{K}(R^2)$. And also we shall establish the spectral relation between a given function and its convolution.

First, we state the following multiplier property.

Theorem 1. *If $f\in W(R^2)$ and $K\in\mathscr{K}(R^2)$, then*

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varDelta_{\varepsilon,\eta}s(u, v; K*f) -\hat{K}(u, v)\varDelta_{\varepsilon,\eta}s(u, v; f)|^2dudv=0\,. \tag{1.1}$$

Note that this theorem is thought of as an analogue of the multiplier property $(K*f)^\wedge=\hat{K}\hat{f}$.

We shall prove Theorem 1, after showing the following two lemmas and a proposition.

Lemma 2. *Suppose $f((x_1, x_2), (u_1, u_2); \lambda)=f((x_1, x_2), \cdot; \lambda)\in L^2(R^2)$ for every $(x_1, x_2)\in R^2$ and every $\lambda\in(-\infty, \infty)$,*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f((x_1, x_2), (u_1, u_2); \lambda)|^2du_1du_2$$

is bounded in $(x_1, x_2)\in R^2$ and $\lambda\in(-\infty, \infty)$, and there exists a function $f((x_1, x_2), (u_1, u_2))=f((x_1, x_2), \cdot)\in L^2(R^2)$ for every $(x_1, x_2)\in R^2$ such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f((x_1, x_2), (u_1, u_2); \lambda)-f((x_1, x_2), (u_1, u_2))|^2du_1du_2\longrightarrow 0\,, \tag{1.2}$$

for every $(x_1, x_2)\in R^2$, as $\lambda\to\infty$. If $K(x_1, x_2)\in L^1(R^2)$, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K(x_1, x_2)f((x_1, x_2), (u_1, u_2); \lambda)dx_1dx_2 -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K(x_1, x_2)f((x_1, x_2), (u_1, u_2))dx_1dx_2\right|^2du_1du_2\longrightarrow 0 \tag{1.3}$$

as $\lambda\to\infty$.

Proof. We first note that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f((x_1, x_2), (u_1, u_2))|^2du_1du_2$ is uniformly bounded on $(x_1, x_2)\in R^2$. In fact,

$$\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f((x_1, x_2), (u_1, u_2))|^2 du_1 du_2\right\}^{1/2}$$
$$\leqq\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f((x_1, x_2), (u_1, u_2); \lambda)-f((x_1, x_2), (u_1, u_2))|^2 du_1 du_2\right\}^{1/2}$$
$$+\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f((x_1, x_2), (u_1, u_2); \lambda)|^2 du_1 du_2\right\}^{1/2} .$$

Letting $\lambda\to\infty$ on the right hand side, we have, from (1.2),

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f((x_1, x_2), (u_1, u_2))|^2 du_1 du_2$$
$$\leqq\limsup_{\lambda\to\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f((x_1, x_2), (u_1, u_2); \lambda)|^2 du_1 du_2 ,$$

which is uniformly bounded on $(x_1, x_2)\in R^2$ in view of the assumption of the lemma. Therefore, there is a constant C independent of (x_1, x_2) such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f((x_1, x_2), (u_1, u_2))|^2 du_1 du_2\leqq C .$$

Hence, we see that (1.2) holds boundedly on $(x_1, x_2)\in R^2$.

Now, the left hand side of (1.3) is not greater than

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|K(x_1, x_2)|\,|f((x_1, x_2), (u_1, u_2); \lambda)$$
$$-f((x_1, x_2), (u_1, u_2))|dx_1 dx_2\right\}^2 du_1 du_2 ,$$

which is, by the Schwarz inequality,

$$\leqq\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|K(x_1, x_2)|dx_1 dx_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|K(x_1, x_2)|\,|f((x_1, x_2), (u_1, u_2); \lambda)$$
$$-f((x_1, x_2), (u_1, u_2))|^2 dx_1 dx_2\right\} du_1 du_2$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|K(x_1, x_2)|dx_1 dx_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|K(x_1, x_2)|dx_1 dx_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f((x_1, x_2), (u_1, u_2); \lambda)$$
$$-f((x_1, x_2), (u_1, u_2))|^2 du_1 du_2 .$$

This converges to 0 as $\lambda\to\infty$ by the dominated convergence theorem, since (1.2) holds boundedly as we have seen above. Thus, (1.3) holds.

PROPOSITION 3. *If $f\in W(R^2)$ and K is a function on R^2 such that*

$$(1+|x_1|)(1+|x_2|)K(x_1, x_2)\in L^2(R^2) , \tag{1.4}$$

then $K*f\in W(R^2)$.

PROOF. Since, by the Schwarz inequality and (1.4),

$$(1.5)\quad |(K*f)(x_1, x_2)|^2$$
$$\leqq\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|f(\xi_1, \xi_2)|^2}{\{1+(x_1-\xi_1)^2\}\{1+(x_2-\xi_2)^2\}}d\xi_1 d\xi_2$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|K(x_1-\xi_1, x_2-\xi_2)|^2\{1+(x_1-\xi_1)^2\}\{1+(x_2-\xi_2)^2\}d\xi_1 d\xi_2$$
$$\leqq \text{const.}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|f(\xi_1, \xi_2)|^2}{\{1+(x_1-\xi_1)^2\}\{1+(x_2-\xi_2)^2\}}d\xi_1 d\xi_2\ ,$$
$$\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|(K*f)(s, t)|^2 ds dt$$
$$\leqq \text{const.}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(\xi_1, \xi_2)|^2\left\{\frac{1}{2S}\int_{-S}^{S}\frac{ds}{1+(s-\xi_1)^2}\right\}$$
$$\times\left\{\frac{1}{2T}\int_{-T}^{T}\frac{dt}{1+(t-\xi_2)^2}\right\}d\xi_1 d\xi_2$$
$$=\text{const.}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(\xi_1, \xi_2)|^2\left[\frac{1}{2S}\operatorname{Tan}^{-1}\frac{2S}{1-S^2+\xi_1^2}\right]$$
$$\times\left[\frac{1}{2T}\operatorname{Tan}^{-1}\frac{2T}{1-T^2+\xi_2^2}\right]d\xi_1 d\xi_2\ .$$

Here we write Tan^{-1} for the principal value of arctan. Now, let us notice that

$$(1.6)\qquad \frac{1}{2T}\operatorname{Tan}^{-1}\frac{2T}{1-T^2+\xi^2}\leqq\begin{cases}\dfrac{\pi}{2T} & (\xi\in(-2T, 2T))\\[2mm] \dfrac{4}{1+\xi^2} & (\text{elsewhere})\ .\end{cases}$$

As for the detailed calculations, refer to Wiener [9]. Therefore,

$$\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|(K*f)(s, t)|^2 ds dt$$
$$\leqq\text{const.}\left\{\frac{\pi^2}{4ST}\int_{-2T}^{2T}\int_{-2S}^{2S}|f(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2+\frac{2\pi}{T}\int_{-2T}^{2T}\int_{-\infty}^{\infty}\frac{|f(\xi_1, \xi_2)|^2}{1+\xi_1^2}d\xi_1 d\xi_2\right.$$
$$\left.+\frac{2\pi}{S}\int_{-\infty}^{\infty}\int_{-2S}^{2S}\frac{|f(\xi_1, \xi_2)|^2}{1+\xi_2^2}d\xi_1 d\xi_2+16\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|f(\xi_1, \xi_2)|^2}{(1+\xi_1^2)(1+\xi_2^2)}d\xi_1 d\xi_2\right\}\ .$$

Thus, using Theorem 1 and the inequality preceding (3.10) of Matsuoka [6],

$$\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|(K*f)(s, t)|^2 ds dt\leqq\text{const.}\sup_{0<U,V<\infty}\frac{1}{4UV}\int_{-V}^{V}\int_{-U}^{U}|f(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2\ ,$$

which implies $K*f\in W(R^2)$.

LEMMA 4. *Under the hypotheses of Proposition* 3,

$$(1.7)\quad \Delta_{\varepsilon,\eta}s(u, v; K*f)-\hat{K}(u, v)\Delta_{\varepsilon,\eta}s(u, v; f)$$
$$=\underset{A\to\infty}{\text{l.i.m.}}\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K(\xi_1, \xi_2)\int_{-A}^{A}\int_{-A}^{A}4f(s-\xi_1, t-\xi_2)$$
$$\times\left[\frac{\sin\varepsilon s}{s}\frac{\sin\eta t}{t}-\frac{\sin\varepsilon(s-\xi_1)}{s-\xi_1}\frac{\sin\eta(t-\xi_2)}{t-\xi_2}\right]e^{-i(us+vt)}dsdtd\xi_1d\xi_2\,.$$

PROOF. First, by (2.4) of Matsuoka [6], we have

$$(1.8)\quad \hat{K}(u, v)\Delta_{\varepsilon,\eta}s(u, v; f)$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K(\xi_1, \xi_2)\ \underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A}f(s-\xi_1, t-\xi_2)$$
$$\times\frac{2\sin\varepsilon(s-\xi_1)}{s-\xi_1}\frac{2\sin\eta(t-\xi_2)}{t-\xi_2}e^{-i(us+vt)}dsdtd\xi_1d\xi_2\,.$$

Now, let us put

$$F_1((\xi_1, \xi_2), (u, v); A)$$
$$=\frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A}f(s-\xi_1, t-\xi_2)\frac{2\sin\varepsilon(s-\xi_1)}{s-\xi_1}\frac{2\sin\eta(t-\xi_2)}{t-\xi_2}e^{-i(us+vt)}dsdt\,.$$

Then, $F_1((\xi_1, \xi_2), (u, v); A)$ belongs to $L^2(R^2)$ in (u, v) for every (ξ_1, ξ_2) and A,

$$(1.9)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F_1((\xi_1, \xi_2), (u, v); A)|^2dudv$$
$$\leqq\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(s, t)|^2\frac{4\sin^2\varepsilon s}{s^2}\frac{4\sin^2\eta t}{t^2}dsdt<\infty\,,$$

and $\text{l.i.m.}_{A\to\infty}F_1((\xi_1, \xi_2), (u, v); A)=F_1((\xi_1, \xi_2), (u, v))$ exists for every (ξ_1, ξ_2). Moreover, we have

$$\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F_1((\xi_1, \xi_2), (u, v); A)-F_1((\xi_1, \xi_2), (u, v))|^2dudv\right\}^{1/2}$$
$$\leqq\left\{\int_{-\infty}^{\infty}\left[\int_{A-\xi_1}^{\infty}+\int_{-\infty}^{-A-\xi_1}\right]|f(s, t)|^2\frac{4\sin^2\varepsilon s}{s^2}\frac{4\sin^2\eta t}{t^2}dsdt\right\}^{1/2}$$
$$+\left\{\left[\int_{A-\xi_2}^{\infty}+\int_{-\infty}^{-A-\xi_2}\right]\int_{-\infty}^{\infty}|f(s, t)|^2\frac{4\sin^2\varepsilon s}{s^2}\frac{4\sin^2\eta t}{t^2}dsdt\right\}^{1/2},$$

which tends to 0 for every (ξ_1, ξ_2) (actually uniformly on any bounded region of (ξ_1, ξ_2)) as $A\to\infty$. Therefore, since $K(\xi_1, \xi_2)\in L^1\cap L^2(R^2)$ by (1.4), and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F_1((\xi_1, \xi_2), (u, v); A)|^2 dudv$ is bounded in (ξ_1, ξ_2) and A, because of (1.9), it follows from Lemma 2 that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\xi_1, \xi_2)\ \underset{A\to\infty}{\text{l.i.m.}}\ F_1((\xi_1, \xi_2), (u, v); A)d\xi_1 d\xi_2$$
$$=\underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\xi_1, \xi_2)F_1((\xi_1, \xi_2), (u, v); A)d\xi_1 d\xi_2 .$$

Consequently, combining this with (1.8), we get

(1.10) $$\hat{K}(u, v)\Delta_{\varepsilon,\eta}s(u, v; f)$$
$$=\underset{A\to\infty}{\text{l.i.m.}}\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\xi_1, \xi_2)\int_{-A}^{A}\int_{-A}^{A} f(s-\xi_1, t-\xi_2)$$
$$\times\frac{2\sin\varepsilon(s-\xi_1)}{s-\xi_1}\ \frac{2\sin\eta(t-\xi_2)}{t-\xi_2}e^{-i(us+vt)}dsdtd\xi_1 d\xi_2 .$$

Next, by (1.5),

$$|(K*f)(x_1, x_2)|^2 \leqq \text{const.}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1+\xi_1^2}{1+(x_1-\xi_1)^2}\ \frac{1+\xi_2^2}{1+(x_2-\xi_2)^2}\ \frac{|f(\xi_1, \xi_2)|^2}{(1+\xi_1^2)(1+\xi_2^2)}d\xi_1 d\xi_2 .$$

Now, let us notice that for all real ξ,

(1.11) $$\frac{1+\xi^2}{1+(x-\xi)^2}\leqq\frac{\sqrt{x^2+4}+|x|}{\sqrt{x^2+4}-|x|}=\left\{\frac{\sqrt{x^2+4}+|x|}{2}\right\}^2 ,$$

which is due to Wiener [9]. Therefore, by Theorem 1 of Matsuoka [6],

$$|(K*f)(x_1, x_2)|\leqq(\text{const.}\ |x_1|+\text{const.})(\text{const.}\ |x_2|+\text{const.}) ,$$

and so, applying Proposition 3 and the Fubini theorem, we have

(1.12) $$\Delta_{\varepsilon,\eta}s(u, v; K*f)=\underset{A\to\infty}{\text{l.i.m.}}\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\xi_1, \xi_2)\int_{-A}^{A}\int_{-A}^{A} f(s-\xi_1, t-\xi_2)$$
$$\times\frac{2\sin\varepsilon s}{s}\ \frac{2\sin\eta t}{t}e^{-i(us+vt)}dsdtd\xi_1 d\xi_2 .$$

Thus, by (1.10) and (1.12), (1.7) is proved.

PROOF OF THEOREM 1. First, by letting

$$F_2((x_1, x_2), (u, v); A)$$
$$=\frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A} 4f(s-x_1, t-x_2)$$

$$\times\left[\frac{\sin\varepsilon s}{s}\frac{\sin\eta t}{t}-\frac{\sin\varepsilon(s-x_1)}{s-x_1}\frac{\sin\eta(t-x_2)}{t-x_2}\right]e^{-i(us+vt)}dsdt$$

and

$$F_2((x_1, x_2), (u, v))=\underset{A\to\infty}{\text{l.i.m.}}\, F_2((x_1, x_2), (u, v); A) \quad \text{in} \quad (u, v)\ ,$$

we shall show that

$$(1.13)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F_2((x_1, x_2), (u, v); A)-F_2((x_1, x_2), (u, v))|^2 dudv \longrightarrow 0$$

uniformly on any bounded region of (x_1, x_2) as $A\to\infty$. In order to do this we observe

$$\begin{aligned}
&\frac{\sin\varepsilon s}{s}\frac{\sin\eta t}{t}-\frac{\sin\varepsilon(s-x_1)}{s-x_1}\frac{\sin\eta(t-x_2)}{t-x_2}\\
&\qquad=\left[\frac{\sin\varepsilon s}{s}-\frac{\sin\varepsilon(s-x_1)}{s-x_1}\right]\left[\frac{\sin\eta t}{t}-\frac{\sin\eta(t-x_2)}{t-x_2}\right]\\
&\qquad\quad+\left[\frac{\sin\varepsilon s}{s}-\frac{\sin\varepsilon(s-x_1)}{s-x_1}\right]\frac{\sin\eta(t-x_2)}{t-x_2}\\
&\qquad\quad+\frac{\sin\varepsilon(s-x_1)}{s-x_1}\left[\frac{\sin\eta t}{t}-\frac{\sin\eta(t-x_2)}{t-x_2}\right]\\
&\qquad=D_1(s, t)+D_2(s, t)+D_3(s, t)\ , \quad \text{say}\ .
\end{aligned}$$

Now, it follows from (3.12) of Matsuoka [6] that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left[\int_A^{\infty}+\int_{-\infty}^{-A}\right]|4f(s-x_1, t-x_2)D_1(s, t)|^2 dsdt\\
&\quad\leqq\int_{|t|>2|x_2|}\left[\int_A^{\infty}+\int_{-\infty}^{-A}\right]\left|f(s-x_1, t-x_2)\frac{16\varepsilon|x_1|}{|s|+|x_1|}\frac{16\eta|x_2|}{|t|+|x_2|}\right|^2 dsdt\\
&\qquad+\int_{|t|\leqq 2|x_2|}\left[\int_A^{\infty}+\int_{-\infty}^{-A}\right]\left|f(s-x_1, t-x_2)\frac{16\varepsilon|x_1|}{|s|+|x_1|}4\eta\right|^2 dsdt\\
&\quad\leqq\int_{|t|>|x_2|}\left[\int_{A-|x_1|}^{\infty}+\int_{-\infty}^{-A+|x_1|}\right](256\varepsilon^2x_1^2)(256\eta^2x_2^2)\frac{|f(s, t)|^2}{s^2t^2}dsdt\\
&\qquad+\int_{|t|\leqq|x_2|}\left[\int_{A-|x_1|}^{\infty}+\int_{-\infty}^{-A+|x_1|}\right](256\varepsilon^2x_1^2)(16\eta^2)\frac{|f(s, t)|^2}{s^2}dsdt
\end{aligned}$$

and hence

$$\begin{aligned}
&\left[\int_A^{\infty}+\int_{-\infty}^{-A}\right]\int_{-\infty}^{\infty}|4f(s-x_1, t-x_2)D_1(s, t)|^2 dsdt\\
&\quad\leqq\left[\int_{A-|x_2|}^{\infty}+\int_{-\infty}^{-A+|x_2|}\right]\int_{|s|>|x_1|}(256\varepsilon^2x_1^2)(256\eta^2x_2^2)\frac{|f(s, t)|^2}{s^2t^2}dsdt
\end{aligned}$$

$$+\left[\int_{A-|x_2|}^{\infty}+\int_{-\infty}^{-A+|x_2|}\right]\int_{|s|\leq|x_1|}(16\varepsilon^2)(256\eta^2x_2^2)\frac{|f(s,t)|^2}{t^2}dsdt\,.$$

Similarly,

$$\int_{-\infty}^{\infty}\left[\int_{A}^{\infty}+\int_{-\infty}^{-A}\right]|4f(s-x_1,\,t-x_2)D_2(s,\,t)|^2dsdt$$
$$\leqq\int_{-\infty}^{\infty}\frac{4\sin^2\eta t}{t^2}\left[\int_{A-|x_1|}^{\infty}+\int_{-\infty}^{-A+|x_1|}\right]256\varepsilon^2x_1^2\frac{|f(s,t)|^2}{s^2}dsdt\,,$$

$$\left[\int_{A}^{\infty}+\int_{-\infty}^{-A}\right]\int_{-\infty}^{\infty}|4f(s-x_1,\,t-x_2)D_2(s,\,t)|^2dsdt$$
$$\leqq\left[\int_{A-|x_2|}^{\infty}+\int_{-\infty}^{-A+|x_2|}\right]\frac{4\sin^2\eta t}{t^2}\int_{|s|>|x_1|}256\varepsilon^2x_1^2\frac{|f(s,t)|^2}{s^2}dsdt$$
$$+\left[\int_{A-|x_2|}^{\infty}+\int_{-\infty}^{-A+|x_2|}\right]\frac{4\sin^2\eta t}{t^2}\int_{|s|\leq|x_1|}16\varepsilon^2|f(s,t)|^2dsdt\,,$$

and this argument is applied also to $D_3(s, t)$. Consequently, combining these with the one-sided Wiener formula of Koizumi [2],

$$(1.14)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Bigg|\frac{1}{(\varepsilon x_1)(\eta x_2)}\cdot\frac{1}{2\pi}\,\underset{B\to\infty}{\text{l.i.m.}}\int_{-B}^{B}\int_{-B}^{B}4f(s-x_1,\,t-x_2)$$
$$\times\left[\frac{\sin\varepsilon s}{s}\,\frac{\sin\eta t}{t}-\frac{\sin\varepsilon(s-x_1)}{s-x_1}\,\frac{\sin\eta(t-x_2)}{t-x_2}\right]e^{-i(us+vt)}dsdt$$
$$-\frac{1}{(\varepsilon x_1)(\eta x_2)}\cdot\frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A}4f(s-x_1,\,t-x_2)$$
$$\times\left[\frac{\sin\varepsilon s}{s}\,\frac{\sin\eta t}{t}-\frac{\sin\varepsilon(s-x_1)}{s-x_1}\,\frac{\sin\eta(t-x_2)}{t-x_2}\right]e^{-i(us+vt)}dsdt\Bigg|^2dudv$$

tends to 0 uniformly for all ε and η and any bounded region of (x_1, x_2) as $A\to\infty$, which implies (1.13). And also, by applying the same argument as the proof of Lemma 5 of Matsuoka [6], it follows that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F_2((x_1, x_2), (u, v); A)|^2dudv$ is bounded in (x_1, x_2) and A. Moreover, other conditions of Lemma 2 are satisfied and, therefore, by Lemma 4, we obtain

$$(1.15)\quad \Delta_{\varepsilon,\eta}s(u,\,v;\,K*f)-\hat{K}(u,\,v)\Delta_{\varepsilon,\eta}s(u,\,v;\,f)$$
$$=\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K(\xi_1,\,\xi_2)\,\underset{A\to\infty}{\text{l.i.m.}}\int_{-A}^{A}\int_{-A}^{A}4f(s-\xi_1,\,t-\xi_2)$$
$$\times\left[\frac{\sin\varepsilon s}{s}\,\frac{\sin\eta t}{t}-\frac{\sin\varepsilon(s-\xi_1)}{s-\xi_1}\,\frac{\sin\eta(t-\xi_2)}{t-\xi_2}\right]e^{-i(us+vt)}dsdtd\xi_1d\xi_2\,.$$

Next, it is immediately clear that $\frac{1}{(\varepsilon x_1)(\eta x_2)}F_2((x_1, x_2), (u, v); A)$ belongs to $L^2(R^2)$ in (u, v), and by (1.14), so does $\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{(\varepsilon x_1)(\eta x_2)}F_2((x_1, x_2), (u, v); A)$. It also is easy to show that on any bounded region of (x_1, x_2), $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{1}{(\varepsilon x_1)(\eta x_2)}F_2((x_1, x_2), (u, v); A)\right|^2 dudv$ is uniformly bounded in ε, η and (x_1, x_2) (refer to the argument that was used in order to show (1.13)), and by (1.14), so is $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{1}{(\varepsilon x_1)(\eta x_2)}\underset{A\to\infty}{\text{l.i.m.}}\, F_2((x_1, x_2), (u, v); A)\right|^2 dudv$. Therefore, since

$$\mathscr{R}_2\text{-}\underset{\varepsilon,\eta\to 0}{\text{l.i.m.}}\ \frac{1}{\varepsilon^{1/2}\eta^{1/2}x_1x_2}\ \underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi}\int_{-A}^{A}\int_{-A}^{A} 4f(s-x_1, t-x_2)$$
$$\times\left[\frac{\sin\varepsilon s}{s}\frac{\sin\eta t}{t}-\frac{\sin\varepsilon(s-x_1)}{s-x_1}\frac{\sin\eta(t-x_2)}{t-x_2}\right]e^{-i(us+vt)}dsdt=0$$

for every (x_1, x_2), an application of Lemma 2, in which $1/\varepsilon$ is taken in place of λ, and $\xi_1\xi_2K(\xi_1, \xi_2)$ in place of $K(\xi_1, \xi_2)$, gives us

$$\mathscr{R}_2\text{-}\underset{\varepsilon,\eta\to 0}{\text{l.i.m.}}\ \frac{1}{\varepsilon^{1/2}\eta^{1/2}}\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K(\xi_1, \xi_2)\ \underset{A\to\infty}{\text{l.i.m.}}\int_{-A}^{A}\int_{-A}^{A} 4f(s-\xi_1, t-\xi_2)$$
$$\times\left[\frac{\sin\varepsilon s}{s}\frac{\sin\eta t}{t}-\frac{\sin\varepsilon(s-\xi_1)}{s-\xi_1}\frac{\sin\eta(t-\xi_2)}{t-\xi_2}\right]e^{-i(us+vt)}dsdtd\xi_1d\xi_2=0\ .$$

Thus, the theorem now follows immediately from (1.15).

The combination of Theorem 1 above and Theorem 3 of Matsuoka [6] gives us

COROLLARY 5. *Suppose $f\in W(R^2)$ and $K\in\mathscr{K}(R^2)$. Then*

$$\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|(K*f)(s, t)|^2dsdt=0$$

if and only if

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\hat{K}(u, v)\Delta_{\varepsilon,\eta}s(u, v; f)|^2dudv=0\ .$$

In the following, we shall determine the spectral relation between f and $K*f$. It is to be noted that whenever $f\in W(R^2)$ and K satisfies (1.4), $K*f$ is also defined (see Proposition 3).

THEOREM 6. *If $f\in S(R^2)$ and $K\in\mathscr{K}(R^2)$, then*
(a) *$K*f\in S(R^2)$ and*

$$\phi(x_1, x_2; K*f)=\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K(\xi_1, \xi_2)\overline{K(\eta_1, \eta_2)}$$
$$\times\phi(x_1+\eta_1-\xi_1, x_2+\eta_2-\xi_2; f)d\xi_1 d\xi_2 d\eta_1 d\eta_2 ; \tag{1.16}$$

(b) $K*f\in S'(R^2)$.

PROOF. (a): We see

$$\begin{aligned}&\phi(x_1, x_2; K*f)\\&=\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K(\xi_1, \xi_2)\overline{K(\eta_1, \eta_2)}\\&\quad\times\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}f(x_1+s-\xi_1, x_2+t-\xi_2)\overline{f(s-\eta_1, t-\eta_2)}dsdtd\xi_1 d\xi_2 d\eta_1 d\eta_2 .\end{aligned}$$

Now, by applying the Schwarz inequality and the same argument as in the proof of Lemma 1 of Matsuoka [6], we have for all large values of S, T,

$$\begin{aligned}&\left|\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}f(x_1+s-\xi_1, x_2+t-\xi_2)\overline{f(s-\eta_1, t-\eta_2)}dsdt\right|\\&\leqq\left\{\frac{1}{4ST}\int_{-T-|x_2-\xi_2|}^{T+|x_2-\xi_2|}\int_{-S-|x_1-\xi_1|}^{S+|x_1-\xi_1|}|f(s, t)|^2dsdt\right\}^{1/2}\\&\quad\times\left\{\frac{1}{4ST}\int_{-T-|\eta_2|}^{T+|\eta_2|}\int_{-S-|\eta_1|}^{S+|\eta_1|}|f(s, t)|^2dsdt\right\}^{1/2}\\&\leqq\left\{\left(1+\frac{|x_1-\xi_1|}{S}\right)\left(1+\frac{|x_2-\xi_2|}{T}\right)\left(1+\frac{|\eta_1|}{S}\right)\left(1+\frac{|\eta_2|}{T}\right)\right\}^{1/2}\\&\quad\times\sup_{0<U,V<\infty}\frac{1}{4UV}\int_{-V}^{V}\int_{-U}^{U}|f(s, t)|^2dsdt\\&\leqq\text{const.}\ (1+|\xi_1|)(1+|\xi_2|)(1+|\eta_1|)(1+|\eta_2|) .\end{aligned}$$

Thus, since $K(x_1, x_2)(1+|x_1|)(1+|x_2|)\in L^1(R^2)$, by Lemma 2 of Matsuoka [6],

$$\begin{aligned}&\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}f(x_1+s-\xi_1, x_2+t-\xi_2)\overline{f(s-\eta_1, t-\eta_2)}dsdt\\&\quad=\phi(x_1+\eta_1-\xi_1, x_2+\eta_2-\xi_2; f) ,\end{aligned}$$

and (1.16) immediately follows from the dominated convergence theorem.

(b): Let us put

$$G(x_1, x_2)=\frac{1}{\xi}\int_0^{\xi}K(x_1+s, x_2)ds \quad (0<\xi<1) .$$

Then, it follows that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x_1x_2G(x_1, x_2)|dx_1dx_2 \leqq \sup_{0<s<\xi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(|x_1+s|+\xi)|x_2|\,|K(x_1+s, x_2)|dx_1dx_2$$
$$\leqq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+|x_1|)(1+|x_2|)|K(x_1, x_2)|dx_1dx_2 < \infty ,$$

and the Schwarz inequality applied to $G(x_1, x_2)$ gives us

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|(1+|x_1|)(1+|x_2|)G(x_1, x_2)|^2dx_1dx_2$$
$$\leqq \sup_{0<s<\xi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+|x_1|)^2(1+|x_2|)^2|K(x_1+s, x_2)|^2dx_1dx_2$$
$$\leqq (1+\xi)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1+|x_1|)^2(1+|x_2|)^2|K(x_1, x_2)|^2dx_1dx_2 < \infty ,$$

that is $G \in \mathscr{K}(R^2)$. Therefore, by Theorem 1,

$$\text{(1.17)} \qquad \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}s(u, v; G*f)-\hat{G}(u, v)\Delta_{\varepsilon,\eta}s(u, v; f)|^2dudv$$
$$=0 ,$$

and since

$$\hat{G}(u, v)=\frac{e^{iu\xi}-1}{iu\xi}\hat{K}(u, v) ,$$

(1.17) becomes

$$\text{(1.18)} \qquad \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big|\Delta_{\varepsilon,\eta}s(u, v; G*f)$$
$$-\frac{e^{iu\xi}-1}{iu\xi}\hat{K}(u, v)\Delta_{\varepsilon,\eta}s(u, v; f)\Big|^2dudv=0 .$$

Also, by Theorem 1,

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big|\frac{e^{iu\xi}-1}{iu\xi}\Delta_{\varepsilon,\eta}s(u, v; K*f)$$
$$-\frac{e^{iu\xi}-1}{iu\xi}\hat{K}(u, v)\Delta_{\varepsilon,\eta}s(u, v; f)\Big|^2dudv=0 .$$

Hence, combining this with (1.18), we have, by means of the Minkowski inequality,

$$\text{(1.19)} \qquad \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big|\frac{e^{iu\xi}-1}{iu\xi}\Delta_{\varepsilon,\eta}s(u, v; K*f)$$
$$-\Delta_{\varepsilon,\eta}s(u, v; G*f)\Big|^2dudv=0 .$$

On the other hand, applying the same argument that was used in the proof of Proposition 3 and using the Schwarz inequality, we have

$$\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|[(K-G)*f](s,t)|^2dsdt$$
$$\leqq \text{const.} \sup_{0<U,V<\infty}\frac{1}{4UV}\int_{-V}^{V}\int_{-U}^{U}|f(\xi_1,\xi_2)|^2d\xi_1d\xi_2$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|K(x_1,x_2)-G(x_1,x_2)|^2(1+x_1^2)(1+x_2^2)dx_1dx_2$$
$$\leqq \text{const.} \sup_{0<s<\xi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|K(x_1,x_2)-K(x_1+s,x_2)|^2(1+x_1^2)(1+x_2^2)dx_1dx_2$$
$$\leqq \text{const.} \sup_{0<s<\xi}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|K(x_1,x_2)(1+|x_1|)(1+|x_2|)\right.$$
$$-K(x_1+s,x_2)(1+|x_1+s|)(1+|x_2|)|^2dx_1dx_2$$
$$\left.+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|K(x_1+s,x_2)(|x_1+s|-|x_1|)(1+|x_2|)|^2dx_1dx_2\right\}.$$

Since sup of the first term in { } of the last expression obviously vanishes as $\xi\to 0$, and that of the second term in { } is dominated by $s^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|K(x_1,x_2)|^2(1+|x_2|)^2dx_1dx_2$, and likewise vanishes, we have

$$\lim_{\xi\to 0}\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|[(K-G)*f](s,t)|^2dsdt=0\,.$$

Because of Theorem 3 of Matsuoka [6], we therefore have

$$\text{(1.20)}\qquad \lim_{\xi\to 0}\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}s(u,v;K*f)-\Delta_{\varepsilon,\eta}s(u,v;G*f)|^2dudv$$
$$=0\,.$$

Now, if we combine (1.20) with (1.19) and make use of the Minkowski inequality, we obtain

$$\lim_{\xi\to 0}\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0}\frac{1}{16\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|1-\frac{e^{iu\xi}-1}{iu\xi}\right|^2|\Delta_{\varepsilon,\eta}s(u,v;K*f)|^2dudv=0\,.$$

Since, when $|u\xi|>4$,

$$\left|1-\frac{e^{iu\xi}-1}{iu\xi}\right|\geqq 1-\frac{2}{4}=\frac{1}{2}\,,$$

it follows at once that

$$\lim_{\varepsilon\to 0} \mathscr{R}_2\text{-}\limsup_{\varepsilon,\eta\to 0} \frac{1}{64\pi^2\varepsilon\eta}\int_{-\infty}^{\infty}\left[\int_{4/\varepsilon}^{\infty}+\int_{-\infty}^{-4/\varepsilon}\right]|\Delta_{\varepsilon,\eta}s(u,\ v;\ K*f)|^2 dudv=0\ .$$

Similarly,

$$\lim_{\varepsilon\to 0} \mathscr{R}_2\text{-}\limsup_{\varepsilon,\eta\to 0} \frac{1}{64\pi^2\varepsilon\eta}\left[\int_{4/\varepsilon}^{\infty}+\int_{-\infty}^{-4/\varepsilon}\right]\int_{-\infty}^{\infty}|\Delta_{\varepsilon,\eta}s(u,\ v;\ K*f)|^2 dudv=0\ .$$

Thus, by Theorem 7 of Matsuoka [6] and (a), we have $K*f\in S'(R^2)$. This completes the proof of Theorem 2.

§ 2. Some Tauberian theorems.

In this section, we shall extend a Tauberian theorem which is due to Wiener [9, Theorem 29] to the case of functions of two variables.

THEOREM 7. *Suppose* $f\in W(R^2)$, $K_1\in\mathscr{K}(R^2)$, $\hat{K}_1(u,v)\neq 0$ *for all* $(u,v)\in R^2$, *and*

$$\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|(K_1*f)(s,\ t)|^2 dsdt=0\ . \tag{2.1}$$

Then the limit relation

$$\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|(K_2*f)(s,\ t)|^2 dsdt=0 \tag{2.2}$$

holds for every $K_2\in\mathscr{K}(R^2)$.

THEOREM 8. *Suppose* $f\in W(R^2)$, *and for all* $K_1\in\mathscr{K}_0$,

$$\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|(K_1*f)(s,\ t)|^2 dsdt=0\ . \tag{2.3}$$

Then the limit relation

$$\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|(K_2*f)(s,\ t)|^2 dsdt=0 \tag{2.4}$$

holds for every $K_2\in\mathscr{K}(R^2)$.

Before proving the theorems, we show the following two lemmas.

LEMMA 9. *Suppose* $f\in W(R^2)$, $K\in\mathscr{K}(R^2)$, $\hat{K}(u,v)\neq 0$ *for all* $(u,v)\in R^2$, *and*

$$\mathscr{R}_1\text{-}\lim_{S,T\to\infty}\frac{1}{4ST}\int_{-T}^{T}\int_{-S}^{S}|(K*f)(s,\ t)|^2 dsdt=0\ .$$

Then, for any bounded rectangle ρ *with sides parallel to the axes,*

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{16\pi^2\varepsilon\eta} \iint_\rho |\Delta_{\varepsilon,\eta} s(u, v; f)|^2 du dv = 0 . \tag{2.5}$$

PROOF. Since $\hat{K}(u, v)$ is continuous and never 0, there exists some constant $C>0$ such that $|\hat{K}(u, v)|>C$ on any bounded region of (u, v), by the hypothesis. Thus, by Corollary 5,

$$\begin{aligned} &\mathscr{R}_2\text{-}\limsup_{\varepsilon,\eta\to 0} \frac{C^2}{16\pi^2\varepsilon\eta} \iint_\rho |\Delta_{\varepsilon,\eta} s(u, v; f)|^2 du dv \\ &\qquad \leqq \mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{16\pi^2\varepsilon\eta} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{K}(u, v)\Delta_{\varepsilon,\eta} s(u, v; f)|^2 du dv = 0 , \end{aligned}$$

which implies (2.5).

LEMMA 10. *Suppose* $f \in W(R^2)$, *and all* $K \in \mathscr{K}_0$,

$$\mathscr{R}_1\text{-}\lim_{S,T\to\infty} \frac{1}{4ST} \int_{-T}^{T}\int_{-S}^{S} |(K * f)(s, t)|^2 ds dt = 0 .$$

Then, for any bounded rectangle ρ *with sides parallel to the axes,*

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{16\pi^2\varepsilon\eta} \iint_\rho |\Delta_{\varepsilon,\eta} s(u, v; f)|^2 du dv = 0 . \tag{2.6}$$

PROOF. There exists a decomposition of ρ such that

(i) ρ is the union of rectangles, $\rho = \cup_{i=1}^n \rho_i$, whose sides are parallel to the axes;

(ii) $^\forall \rho_i$, $^\exists K \in \mathscr{K}_0$: $\hat{K} \neq 0$ on ρ_i.

Thus, by Lemma 9, for any rectangle ρ_i,

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{16\pi^2\varepsilon\eta} \iint_{\rho_i} |\Delta_{\varepsilon,\eta} s(u, v; f)|^2 du dv = 0 ,$$

which gives us (2.6).

PROOF OF THEOREM 7. By Lemma 9 and Theorem 1, we have respectively

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{16\pi^2\varepsilon\eta} \int_{-C}^{C}\int_{-C}^{C} |\hat{K}_2(u, v)\Delta_{\varepsilon,\eta} s(u, v; f)|^2 du dv = 0$$

and

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{16\pi^2\varepsilon\eta} \int_{-C}^{C}\int_{-C}^{C} |\Delta_{\varepsilon,\eta} s(u, v; K_2 * f) - \hat{K}_2(u, v)\Delta_{\varepsilon,\eta} s(u, v; f)|^2 du dv = 0 ,$$

for any $C>0$. It follows, therefore, that

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{16\pi^2\varepsilon\eta} \int_{-C}^{C}\int_{-C}^{C} |\Delta_{\varepsilon,\eta} s(u, v; K_2 * f)|^2 du dv = 0 . \tag{2.7}$$

While, from the proof of part (b) of Theorem 6,

$$\lim_{C\to\infty} \mathscr{R}_2\text{-}\limsup_{\varepsilon,\eta\to 0} \frac{1}{16\pi^2\varepsilon\eta} \int_{-\infty}^{\infty} \left[\int_{C}^{\infty} + \int_{-\infty}^{-C}\right] |\Delta_{\varepsilon,\eta} s(u, v; K_2 * f)|^2 du dv = 0$$

and

$$\lim_{C\to\infty} \mathscr{R}_2\text{-}\limsup_{\varepsilon,\eta\to 0} \frac{1}{16\pi^2\varepsilon\eta} \left[\int_{C}^{\infty} + \int_{-\infty}^{-C}\right] \int_{-\infty}^{\infty} |\Delta_{\varepsilon,\eta} s(u, v; K_2 * f)|^2 du dv = 0 .$$

Consequently, combining these with (2.7), we easily obtain

$$\mathscr{R}_2\text{-}\lim_{\varepsilon,\eta\to 0} \frac{1}{16\pi^2\varepsilon\eta} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Delta_{\varepsilon,\eta} s(u, v; K_2 * f)|^2 du dv = 0 .$$

Thus, by Theorem 3 of Matsuoka [6], (2.2) is proved. This concludes the proof of Theorem 7.

By using Lemma 10 and Theorem 7, we can also prove Theorem 8.

REMARK 1. In order to study the spectral analysis and the Tauberian theorems about the convolution of functions of two variables, we use, in this paper, the generalized harmonic analysis of functions of two variables which was established by Matsuoka [6]. Their limit processes, therefore, also depend on the limit process involved in the above generalized harmonic analysis. On the other hand, the generalized harmonic analysis of functions of two variables is also obtained under the unrestricted rectangular mean concerning the double limit process. Thus, if we use this generalized harmonic analysis, then the spectral analysis and the Tauberian theorems about the convolution of functions of two variables are also obtained under the above limit process instead of the restricted limit process.

REMARK 2. In the course of preparation of this paper, the author found the papers of Lau [3, 4], in which the extended theorems of Wiener [9] are shown in some more simplified method.

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