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On the Decay of Correlation for Piecewise Monotonic Mappings II

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Introduction

In this paper, we will consider a class of piecewise linear transformations defined on the unit interval [0, 1]. We will show that under some suitable conditions the transformations belonging to this class exhibit mixing properties, and we derive estimates for the decay rate of correlation for them. Specifically, we will prove:

THEOREM 0-1. *Let F be a transformation on the unit interval* [0, 1] *satisfying conditions* i), ii), iii) *given in* §1. *Suppose that the infimum of the lower Lyapunov number is positive and the second Fredholm eigenvalue η is less than* 1. *Then F has a unique invariant probability measure μ absolutely continuous with respect to the Lebesgue measure on* [0, 1] *and the dynamical system* ([0, 1], μ, F) *is mixing, and the following estimate for the decay rate of correlation holds for any pair of functions $f\in BV$ and $g\in L^1$*:

$$\lim_{n\to\infty}(\eta+\varepsilon)^{-n}\left\{\int f(x)g(F^{(n)}(x))d\mu-\int fd\mu\int gd\mu\right\}=0\ , \tag{0.1}$$

for any $\varepsilon>0$.

This result extends the results obtained by the author in [8], and we will discuss in [9] some further results for more general cases. Some related topics have appeared in [2], [6], [10], [13] and [14]. Precise definitions of the lower Lyapunov number ξ and the second Fredholm eigenvalue η will be stated in §1.

Certain critical phenomena appear as $\xi\downarrow 0$, which indicates that the state of the system approaches the so-called window state. Concerning window states, we refer the readers to [3]. For the case where $\xi<0$,

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we also prove the following:

THEOREM 0-2. *Assume that $\xi<0$. Then, there exists an attractive periodic orbit* (cf. [1]).

We summarize notations in §1. In §2, we will derive a renewal equation for admissible words, which is one of our main tools for proving the theorems stated above. In §3, we will prove Theorem 0-1 and Theorem 0-2.

§1. Notations.

In this paper, we will treat piecewise linear mappings F which satisfy the following conditions:

There exists a partition (finite or countable) $\{I_a\}_{a\in A}$ of the unit interval into subintervals (we call each $a\in A$ an alphabet) and

i) F is linear on each subinterval I_a,

ii) there exists a special alphabet $0\in A$ such that

$$F(I_a)\supset(0, 1) \quad \text{for} \quad a\neq 0\ ,$$

iii) we treat the following four cases,

type 1) $\inf_{x\in I_0} F(x)=0$ and $F'(x)>0$ on $x\in I_0$,

type 2) $\sup_{x\in I_0} F(x)=1$ and $F'(x)>0$ on $x\in I_0$,

type 3) $\sup_{x\in I_0} F(x)=1$ and $F'(x)<0$ on $x\in I_0$,

type 4) $\inf_{x\in I_0} F(x)=0$ and $F'(x)<0$ on $x\in I_0$.

REMARK 1. A mapping F of type 2 or of type 4 is conjugate to a mapping $G=I\cdot F\cdot I$ which is of type 1 or of type 3, respectively, with the conjugacy given by $I(x)=1-x$. Thus, hereafter, we only treat mappings of type 1 or of type 3.

REMARK 2. The examples of type 1 are β-transformations (cf. [4], [12] for constant slope and [5], [11] for more general cases) and the examples of type 2 are unimodal linear transformations (cf. [3]).

We denote by ξ the infimum of the lower Lyapunov number, that is,

$$\xi=\operatorname*{ess\,inf}_{x\in[0,1]}\ \varliminf_{n\to\infty}\frac{1}{n}\log|(F^{(n)})'(x)|\ , \tag{1.1}$$

where $F^{(n)}$ is the n-th iterate of the mapping F.

1.1. Slope, signature and subinterval.

In the following, to simplify notations, we denote a subinterval $I_a(a\in A)$

by (a). For an alphabet $a \in A$, we define

$$\operatorname{sgn} a = \begin{cases} +1 & \text{if } F \text{ is monotone increasing on } (a) , \\ -1 & \text{if } F \text{ is monotone decreasing on } (a) , \end{cases} \tag{1.2}$$

$$\lambda^a = |F'(x)| \quad x \in (a) , \tag{1.3}$$

and we call λ^a the slope of F on (a). We call a finite sequence of alphabets $w = a_1 \cdots a_n$ $(a_i \in A)$ a word and define for each w its slope, interval, length and signature formally as follows:

$$\lambda^w = \lambda^{a_1} \cdots \lambda^{a_n} \quad (\lambda^{-w} = (\lambda^w)^{-1}) , \tag{1.4}$$

$$(w) = \bigcap_{i=1}^{n} F^{(i-1)}((a_i)) , \tag{1.5}$$

$$|w| = n \tag{1.6}$$

and

$$\operatorname{sgn} w = \prod_{i=1}^{n} \operatorname{sgn} a_i . \tag{1.7}$$

We consider a formal symbol ϕ which we call an empty word and define

$$\lambda^\phi = 1 , \tag{1.8}$$

$$(\phi) = [0, 1] , \tag{1.9}$$

$$|\phi| = 0 \tag{1.10}$$

and

$$\operatorname{sgn} \phi = +1 . \tag{1.11}$$

For words $w = a_1 \cdots a_n$ and $w' = b_1 \cdots b_m$, we define

$$ww' = a_1 \cdots a_n b_1 \cdots b_m \tag{1.12}$$

$$w\phi = \phi w = w . \tag{1.13}$$

By $\langle w \rangle$, we denote the indicator function of the set (w).

1.2. Admissibility.

A word $w \in W$ is called admissible if

$$(w) \neq \phi . \tag{1.14}$$

We denote

(1.15) $$W(F)=\{w \in W\colon w \text{ is admissible}\}\,.$$

To classify the set $W(F)$, we need the following sets of words. Let

(1.16) $$K=\{a_1\cdots a_n\colon a_i \in A \text{ and there exists } j\ (1\leqq j\leqq i-1) \text{ such that } a_1=\cdots=a_j=0 \text{ and } a_{j+1},\cdots,a_i\neq 0\}\cup\{\phi\}\,,$$

and we call each element of the set K a 0-word (which was called k-word in [8]). Let

(1.17) $$S=\{w_1\cdots w_n\colon w_i\in K\ (1\leqq i\leqq n-1),\ w_n\in K \text{ or } w_n=0\cdots 0\}\,,$$

and call each element of the set S a sentence. Of course, we can regard a sentence as a word. We denote by $K(F)$ the set of 0-words which are admissible and by $S(F)$ the set of sentences which are admissible. An infinite sequence of alphabets $a_1a_2\cdots$ is called admissible if $a_1\cdots a_n\in W(F)$ for any n.

1.3. Order.

We define orders on the set A, W and K. For alphabets $a, b\in A$, $a<b$ if $x<y$ for $x\in(a)$ and $y\in(b)$. For words $w=a_1\cdots a_n$ and $w'=b_1\cdots b_m$, $w<w'$ if there exists i $(i\geqq 0)$ such that $a_1\cdots a_i=b_1\cdots b_i$ and one of the following holds:

(11.8) $$\begin{aligned}&\text{i)}\quad a_{i+1}<b_{i+1} \quad\text{and}\quad \operatorname{sgn} a_1\cdots a_i=+1\,,\\ &\text{ii)}\quad a_{i+1}>b_{i+1} \quad\text{and}\quad \operatorname{sgn} a_1\cdots a_i=-1\,.\end{aligned}$$

We can introduce the total order $\ll$ on the class K of 0-words by defining $w\ll w'$ if $w0<w'0$. For infinite sequences of alphabets, we also define order as above. For admissible words $w, w'\in W(F)$, $w<w'$ means $x<y$ for any $x\in(w)$ and $y\in(w')$.

1.4. Expansion of x.

For $x\in[0, 1]$ and i $(i\geqq 1)$, we define $a_i^x\in A$ by

(1.19) $$F^{(i-1)}(x)\in(a_i^x)$$

and the infinite sequence of alphabets $a_1^xa_2^x\cdots$ is called the expansion of x. We usually identify $x\in[0, 1]$ and its expansion. For a word w and a point $x\in[0, 1]$, we call wx admissible if there exists a point y whose expansion is $wa_1^xa_2^x\cdots$ and we identify wx with y. Let

(1.20) $$c=\begin{cases}\sup\{x\in(0)\} & \text{if } F \text{ is of type } 1\,,\\ \inf\{x\in(0)\} & \text{if } F \text{ is of type } 3\,.\end{cases}$$

We define the expansion of c by the limit of $a_1^x a_2^x \cdots$ as $x \to c$ $(x \in (0))$. This sequence plays an essential role in this article. Let

$$(1.21)\qquad a(i, j)=\begin{cases} a_i^c a_{i+1}^c \cdots a_j^c & \text{if } i \leqq j \\ \phi & \text{otherwise}, \end{cases}$$

and

$$(1.22)\qquad a(i, \infty)=a_i^c a_{i+1}^c \cdots .$$

We also denote the expansion of c by 0-words $w_i^c = a_{i,1} \cdots a_{i,j(i)} \in K$ $(a_{i,j} \in A)$, that is,

$$(1.23)\qquad a_1^c a_2^c \cdots = w_1^c w_2^c \cdots = a_{1,1} \cdots a_{1,j(1)} a_{2,1} \cdots .$$

We define

$$(1.24)\qquad \theta_p = \begin{cases} w_1^c \cdots w_p^c & p \geqq 1 \\ \phi & p=0 . \end{cases}$$

$$(1.25)\qquad a_{i,n} = a^c_{|\theta_{i-1}|+n} \qquad \text{even if } n > |w_i^c| ,$$

and

$$(1.26)\qquad a_i(m, n) = \begin{cases} a_{i,m} \cdots a_{i,n} & \text{if } m \leqq n , \\ \phi & \text{otherwise} . \end{cases}$$

1.5. Type of words.

We will define the type of words which will be one of our main tool for describing the symbolic structure of F. Let for a word w and $m \geqq 0$,

$$(1.27)\qquad w_m^* = \begin{cases} a(1, m) & \text{if } w = a(m+1, m+|w|) , \\ \phi & \text{otherwise} , \end{cases}$$

and we define $q(m)=q(m, w)$ and $r(m)=r(m, w)$ by

$$(1.28)\qquad q(m) = p - r(m) ,$$

$$(1.29)\qquad r(m) = \begin{cases} \max\{n \leqq |w| : a(p-n+1, p) = a(1, n) \text{ and } \operatorname{sgn} a(1, p-n) = -1\} , \\ 0 \quad \text{if there exists no such } n , \end{cases}$$

where

$$(1.30)\qquad p = \max\{n : w_m^* w = w' a(1, n) \text{ for some word } w'\} .$$

We call a word w of the type $(q(0), r(0))$, and a word which is of the type (0, 0) will be called complete. We denote the type of $w_m^* w$ by $(q^*(m), r^*(m))$ and we define $p(m)$ by

$$p(m)=\begin{cases} r^*(m) & \text{if } r^*(m)>r(m)=0 \\ q(m) & \text{otherwise .} \end{cases} \tag{1.31}$$

For $p\geqq 0$, we define an integer $t(p)$ by

$$\begin{aligned} &a_i^c=a_{i-q}^c \qquad \text{for } q\leqq i\leqq t(p) , \\ &a_{t(p)+1}^c \neq a_{t(p)+1-q}^c , \end{aligned} \tag{1.32}$$

where θ_p is of the type (q, r).

1.6. Generating functions.
Let

$$s(n, x)=\sum_{s\in S(n,x)} \lambda^{-s} , \tag{1.33}$$

$$w(n, x)=\sum_{w\in W(n,x)} \lambda^{-w} , \tag{1.34}$$

$$w(m, n, x)=\sum_{w\in W(m,n,x)} \lambda^{-w} \tag{1.35}$$

and for a word $w \in W(F)$,

$$w(w, m, n, x)=\sum_{u\in W(w,m,n,x)} \lambda^{-u} , \tag{1.36}$$

where

$$S(n, x)=\{s \in S\colon |s|=n \text{ and } sx \text{ is admissible}\} , \tag{1.37}$$

$$W(n, x)=\{w \in W\colon |w|=n \text{ and } wx \text{ is admissible}\} , \tag{1.38}$$

$$W(m, n, x)=\{a_1\cdots a_n \in W(n, x)\colon a_1\cdots a_m=a(1, m)\} , \tag{1.39}$$

$$W(w, m, n, x)=\left\{\begin{matrix} a_1\cdots a_n \in W(n, x)\colon a_1\cdots a_{|w|}=w \text{ and} \\ wx\leqq a(m+1, \infty) \quad \text{if} \quad s(a_{m+1}^c)=-1 \\ wx> a(m+1, \infty) \quad \text{if} \quad s(a_{m+1}^c)=+1 \end{matrix}\right\} , \tag{1.40}$$

and

$$s(a)=\begin{cases} -1 & \text{if } a\leqq 0 \\ +1 & \text{if } a>0 . \end{cases} \tag{1.41}$$

Let

$$(1.42)\qquad \chi(n, x)=\begin{cases}\lambda^{-a(1,n)} & \text{if } \operatorname{sgn} a(1, n)s(a_{m+1}^c)=-1 \text{ and } a(1, n)x \\ & \quad\text{is admissible ,} \\ -\lambda^{-a(1,n)} & \text{if } \operatorname{sgn} a(1, n)s(a_{m+1}^c)=+1 \text{ and } a(1, n)x \\ & \quad\text{is not admissible ,} \\ 0 & \text{otherwise .}\end{cases}$$

The generating functions of $w(n, x)$ and $\chi(n, x)$ are denoted by $w(z; x)$ and $\chi(z; x)$, respectively, that is,

$$(1.43)\qquad w(z; x)=\sum_{n\geqq 0} z^n w(n, x) ,$$

and

$$(1.44)\qquad \chi(z; x)=\begin{cases}\sum_{n\geqq 0} z^n \chi(n, x) & \text{if } F \text{ is aperiodic ,} \\ \sum_{n=0}^{N-1} z^n \chi(n, x) & \text{if } F \text{ is } N\text{-periodic ,}\end{cases}$$

where we call the mapping F N-periodic if there exists an integer N which satisfies

$$(1.45)\qquad N=\min\{n\colon \lim_{x\to c} F^{(n)}(x)=c, \text{ where } x\in(0) \text{ and } F^{(n)} \text{ is monotone increasing in the neighborhood of } c \text{ in } (0)\} ,$$

if there exists no such N, we call F aperiodic.

We will write down a renewal equation of symbolic structure in terms of $w(z; x)$ and $\chi(z; x)$.

1.7. Fredholm determinant.

Let

$$(1.46)\qquad b_1=\begin{cases}\sum_{a\neq 0} \lambda^{-a} & \text{if } s(a_2^c)\operatorname{sgn} a_1^c=-1 , \\ \sum_{a\in A} \lambda^{-a} & \text{if } s(a_2^c)\operatorname{sgn} a_1^c=+1 ,\end{cases}$$

and for $j\geqq 2$

$$(1.47)\qquad b_j=\operatorname{sgn} a(1, j-1)(-s(a_j^c))(\textstyle\sum^* \lambda^{-a})\lambda^{-a(1,j-1)} ,$$

where $\sum^*$ is the sum over all $a\in A$ such that

$$(1.48)\qquad \begin{array}{llll} a<a_j^c & \text{if } s(a_j^c)=-1 & \text{and} & \operatorname{sgn} a_j^c s(a_j^c)s(a_{j+1}^c)=+1 , \\ a\leqq a_j^c & \text{if } s(a_j^c)=-1 & \text{and} & \operatorname{sgn} a_j^c s(a_j^c)s(a_{j+1}^c)=-1 , \\ a>a_j^c & \text{if } s(a_j^c)=+1 & \text{and} & \operatorname{sgn} a_j^c s(a_j^c)s(a_{j+1}^c)=+1 , \\ a\geqq a_j^c & \text{if } s(a_j^c)=+1 & \text{and} & \operatorname{sgn} a_j^c s(a_j^c)s(a_{j+1}^c)=-1 .\end{array}$$

Then the Fredholm determinant of the mapping F is defined by

$$(1.49)\qquad \Phi(z)=\begin{cases}1-\sum_{j\geq 1} b_j z^j & \text{if } F \text{ is aperiodic ,}\\ 1-\sum_{j=1}^{N} b_j z^j-\lambda^{-a(1,N)}z^N & \text{if } F \text{ is } N\text{-periodic .}\end{cases}$$

We call z which satisfies $\Phi(1/z)=0$ a Fredholm eigenvalue. We will show that 1 is one of the Fredholm eigenvalues in §2 and the derivative $\Phi'(1)$ defines the invariant measure in the following sense. Let

$$(1.50)\qquad \rho(x)=-\Phi'(1)^{-1}\chi(1;x)\ ,$$

and by μ we denote a measure with its density ρ. In §3, we will prove that μ is the invariant probability measure with respect to the mapping F. Let

$$(1.51)\qquad \eta=\begin{cases}1 & \text{if } \Phi'(1)=0\ ,\\ e^{-\xi} & \text{if there exists no Fredholm eigenvalue except 1 ,}\\ \min\{|\gamma|:\ \gamma \text{ is a Fredholm eigenvalue which does not equal } 1\}\ ,\end{cases}$$

and we call η the second Fredholm eigenvalue. As stated in Theorem 0-1, we will show that η is the decay rate of correlation.

1.8. Perron-Frobenius operator.

For a dynamical system ([0, 1], μ, F), we denote the Perron-Frobenius operator by P, that is, for f, g which are integrable with respect to μ,

$$(1.52)\qquad \int Pf(x)g(x)d\mu=\int f(x)g(F(x))d\mu\ .$$

By Q^n, we denote the n-th correlation, that is, for f which is integrable with respect to μ,

$$(1.53)\qquad Q^n f(x)=P^n f(x)-\int f d\mu\ .$$

We denote

L^1=the set of integrable functions on the unit interval with respect to the Lebesgue measure,

BV=the set of bounded variation functions on the unit interval,

and by $\|\cdot\|$ and $V(\cdot)$, we denote the L^1-norm and the total variation, respectively. We need a 'quasi measure' ν^ε to estimate the decay rate of the correlation. For a monotone function f, we define

(1.54) $$(f)_0(x)=\min f(x)\ ,$$

and for $n\geqq 1$, we inductively define functions

(1.55) $$(f)_n(x)=\sum_{|w|=n} B_w\langle w\rangle(x)+(f)_{n-1}(x)\ ,$$

where

(1.56) $$B_w=\min_{x\in(w)}\{f(x)-(f)_{n-1}(x)\}\ .$$

We define ν^ε in the following way:

1) For a monotone function f, we define

(1.57) $$\nu_n^\varepsilon(f)=\sum_{k=0}^{n}\sum_{|w|=k}|B_w|(\eta+\varepsilon)^{-k}\lambda^{-w}\ .$$

For a function $f\in BV$, we define

(1.58) $$\nu_n^\varepsilon(f)=\inf\{\nu_n^\varepsilon(f_1)+\nu_n^\varepsilon(f_2)\}\ ,$$

where infimum is taken over all f_1 and f_2 which are monotone and $f_1+f_2=f$. Finally, for a function $f\in BV$, we define

(1.59) $$\nu^\varepsilon(f)=\overline{\lim_{n\to\infty}}\,\nu_n^\varepsilon(f)\ .$$

1.9. Some technical notations.

We need several notations to construct the renewal equation.

(1.60) $$\bar r(p,\ n,\ x)=\sum_{s\in R(p,n,x)}\lambda^{-s}\ ,$$

(1.61) $$k(p,\ j)=\sum_{w\in K(p,j)}\lambda^{-w}\ ,$$

(1.62) $$r(p,\ n,\ x)=\begin{cases}\lambda^{-\theta_p}\sum_{j\geqq 1}k(p,\ j)s(n-|\theta_p|-j,\ x) & \text{if}\quad \operatorname{sgn}\theta_p=+1\ ,\\ \lambda^{-\theta_p}s(n-|\theta_p|,\ x)-\lambda^{\theta_p}\sum_{j\geqq 1}k(p,\ j)s(n-|\theta_p|-j,\ x) & \\ \qquad\lambda^{-\theta_{p+1}}s(n-|\theta_{p+1}|,\ x) & \text{if}\quad \operatorname{sgn}\theta_p=-1\ ,\end{cases}$$

and for $|\theta_p|\leqq m\leqq|\theta_{p+1}|$,

(1.63) $$\bar r(p,\ m,\ n,\ x)=\sum_{w\in R(p,m,n,x)}\lambda^{-w}\ ,$$

(1.64) $$k(p,\ m,\ j)=\sum_{w\in K(p,m,j)}\lambda^{-w}\ ,$$

where

(1.65) $$R(p, n, x)=\{s \in S(n, x): s=w_1 \cdots w_q (q>p),\ w_i=w_i^c \text{ for } 1 \leqq i \leqq p \text{ and } w_{p+1} \neq w_{p+1}^c\},$$

(1.66) $$R(p, m, n, x)=\{w \in W(m, n, x): w \neq \theta_{p+1}\},$$

(1.67) $$K(p, j)=\{w \in K: |w|=j \text{ and } w \ll w_{p+1}^c\},$$

and

(1.68) $$K(p, m, j)=\{a_1 \cdots a_j \in K(p, j): a_1 \cdots a_m = a_{p+1}(1, m)\}.$$

Note that

(1.69) $$s(n, x)=\sum_p \bar{r}(p, n, x)$$

and

(1.70) $$w(m, n, x)=\sum_p \bar{r}(p, m, n, x).$$

We will define $b_j(w, m)$ and $\chi(w, m, n, x)$ $(w \in W(F), m \geqq 0, j \geqq 1)$. Let
condition i) $s(a_{m+1}^c)=-1$ and $w>a(m+1, m+|w|)$,
condition ii) $s(a_{m+1}^c)=+1$ and $w<a(m+1, m+|w|)$,
condition iii) $r(m)=0$ and $\operatorname{sgn} w_m^* s(a_{|w_m^*|+1}^c) \operatorname{sgn} a(1, p(m)) s(a_{p(m)+1}^c)=-1$,
condition iv) $r^*(m)>r(m)>0$ and $s(a_{r^*(m)+1}^c) s(a_{r(m)+1}^c)=-1$.
Then we define:

case 1) When the condition i) or ii) holds, then for any j, we define

(1.71) $$b_j(w, m)=\chi(w, m, j, x)=0.$$

case 2) When the conditions i) and ii) do not hold but the condition iii) or iv) holds, then we define

(1.72) $$b_{|w|}(w, m)=\lambda^{-w},$$

and

(1.73) $$\chi(w, m, |w|, x)=0.$$

case 3) When the conditions i) to iv) do not hold, then we define

(1.74) $$b_{|w|}(w, m)=\chi(w, m, |w|, x)=0.$$

case 4) When the conditions i) and ii) do not hold and $j \geqq |w|+1$, then we define

(1.75) $$b_j(w, m)=-s(a_{|w_m^*|+1}^c) \operatorname{sgn} w_w^* \lambda^{-w} \lambda^{a(1, q(m)+r(m))} b_{j+q(m)+r(m)-|w|}+b_j^*(w, m),$$

where

$$b_j^*(w, m)=\begin{cases} -s(a^c_{|w^*_m|+1})\operatorname{sgn} w^*_m\lambda^{-w}\lambda^{a(1,r(m))}b_{j+r(m)-|w|} & \text{if } r(m)>0\,, \\ 0 & \text{if } r(m)=0\,, \end{cases} \tag{1.76}$$

and

$$\begin{aligned}\chi(w, m, j, x)=&-s(a^c_{|w^*_m|+1})\operatorname{sgn} w^*_m\lambda^{-w}\lambda^{a(1,q(m)+r(m))} \\ &\chi(j+q(m)+r(m)-|w|, x)+\chi^*(w, m, j, x)\,,\end{aligned} \tag{1.77}$$

where

$$\chi^*(w, m, j, x)=\begin{cases} -s(a^c_{|w^*_m|+1})\operatorname{sgn} w^*_m\lambda^{-w}\lambda^{a(1,r(m))}\chi(j+r(m)-|w|, x) & \text{if } r(m)>0\,, \\ 0 & \text{if } r(m)=0\,. \end{cases} \tag{1.78}$$

Let

$$\chi^+(n, x)=\begin{cases}\lambda^{-a(1,n)} & \text{if } a(1, n)x \text{ is admissible}\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{1.79}$$

$$(i, j](k)=\begin{cases}1 & \text{if } i<k\leqq j\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{1.80}$$

§2. Renewal equation and Fredholm determinant.

In this section, we will introduce renewal equations on $s(n, x)$ and $w(n, x)$, and prepare several lemmas and theorems which we use in the next section. We will show in Lemma 2-1 and Lemma 2-2 that the type of words determines symbolic structure of the mapping F.

LEMMA 2-1. i) *A word $w=a_1\cdots a_n$ is admissible if and only if, for each i $(1\leqq i\leqq n)$, one of the following holds:*

a) $a_i\neq 0$,

b) $a_i=0$ *and* $a_i\cdots a_n\leqq a(1, n-i+1)$.

ii) *For admissible words $w, w'\in W(F)$, $ww'\in W(F)$ if and only if one of the following holds:*

a) *w is of the type $(0, 0)$,*

b) *w is of the type $(p, 0)$ $(p>0)$ and $a(1, p)w'\leqq a(1, p+|w'|)$,*

c) *w is of the type (p, q) $(p, q>0)$ and $a(1, p)w'\leqq a(1, p+|w'|)$ and $a(1, q)w'\leqq a(1, q+|w'|)$.*

iii) *If w is complete, then $F^{(|w|)}((w))\supset(0, 1)$.*

Proof of i). If $n=1$, the assertion is trivial. Suppose that the assertion holds up to $n-1$. For a word $a_1\cdots a_n$ for which $a_2\cdots a_n$ is admissible, if $a_1\neq 0$, $(a_1\cdots a_n)$ is admissible since $F((a_1))\supset(0, 1)$. On the other hand, if $a_1=0$ and $a_1\cdots a_n\leqq a(1, n)$, then $F(1)<x$ for some $x\in(a_2\cdots a_n)$. This completes the proof.

Proof of ii). By i), it is sufficient to show that if

$$a(1, p+q)w'\leqq a(1, p+q+|w'|) \tag{2.1}$$

and

$$a(1, q)w'\leqq a(1, q+|w'|)\ , \tag{2.2}$$

then for any r $(r\neq p+q)$ which satisfies

$$w=w''a(1, r)\quad (w''\in W(F))\ , \tag{2.3}$$

$$a(1, r)w'\leqq a(1, r+|w'|) \tag{2.4}$$

holds.

case 1) Suppose that $\operatorname{sgn} a(1, p+q-r)=+1$. Then

$$\begin{aligned} a(1, p+q-r)a(1, r)w'&=a(1, p+q)w'\leqq a(1, p+q+|w'|)\\ &=a(1, p+q-r)a(p+q-r+1, p+q+|w'|)\ , \end{aligned} \tag{2.5}$$

and from the admissibility, we have the last term of (2.5) equaling

$$a(1, p+q-r)a(1, r+|w'|)\ . \tag{2.6}$$

Thus

$$a(1, r)w'\leqq a(1, r+|w'|)\ . \tag{2.7}$$

case 2) Suppose that $\operatorname{sgn} a(1, p+q-r)=-1$ and that $r\neq q$. Since $a(1, q-r)=+1$, as in the case 1) we get

$$a(1, r)w'\leqq a(1, r+|w'|)\ . \tag{2.8}$$

This completes the proof.

Proof of iii). By ii), if w is of the type $(0, 0)$, $ww'\in W(F)$ for every $w'\in W(F)$. This shows that $F^{(|w|)}((w))\supset(0, 1)$.

Lemma 2-2. *Suppose that θ_p is of the type $(|\theta_q|, |\theta_r|)$. Then θ_{p+1} is either of the type $(|\theta_q|, |\theta_{p+1}|)$ or $(|\theta_{p+1}|, 0)$. Moreover, if θ_{p+1} is of the type $(|\theta_{p+1}|, 0)$, then θ_{r+1} is of the type $(|\theta_{r+1}|, 0)$.*

PROOF. Suppose that θ_{p+1} is of the type $(|\theta_s|, |\theta_t|)$ $(0<t\leqq r)$. Then by the assumption of the lemma,

$$a(1, |\theta_{r+1}|)>a(1, |\theta_{s+q}|)a(1, |\theta_t|) \tag{2.9}$$

and

$$a^c_{|\theta_{s-q}|+1}=0 . \tag{2.10}$$

Since sgn $a(1, |\theta_{s-q}|)=+1$, we get by (2.9)

$$a(|\theta_{s-q}|+1, |\theta_{r+1}|)>a(1, |\theta_t|) . \tag{2.11}$$

In view of (2.10), this contradicts the admissibility of θ_{r+1}. Now assume that θ_{p+1} is of the type $(\theta_{p+1}, 0)$. Suppose that θ_{r+1} is of the type $(|\theta_s|, |\theta_t|)$ $(t>0)$. Then

$$a(1, |\theta_{r+1}|)>a(|\theta_q|+1, |\theta_{q+r-1}|) \tag{2.12}$$

and

$$a^c_{|\theta_{q+s}|+1}=0 . \tag{2.13}$$

By (2.12), we get

$$\begin{aligned} a(1, |\theta_s|)a(1, |\theta_t|)&=a(1, |\theta_{r+1}|)>a(|\theta_q|+1, |\theta_{q+r+1}|) \\ &=a(1, |\theta_s|)a(|\theta_{q+s}|+1, |\theta_{q+r+1}|) . \end{aligned} \tag{2.14}$$

Since sgn $a(1, |\theta_s|)=-1$, we get by (2.14)

$$a(1, |\theta_t|)<a(|\theta_{q+s}|+1, |\theta_{q+r+1}|) . \tag{2.15}$$

In view of (2.13), this contradicts the admissibility of θ_p.

In order to write down the relevent renewal equations, we need two more lemmas. Since

$$s(n, x)=\sum_p \bar{r}(p, n, x) , \tag{2.16}$$

we first calculate $\bar{r}(p, n, x)$.

LEMMA 2-3. i) *If θ_p is of the type $(|\theta_p|, 0)$ and θ_{p+1} is of the type $(|\theta_{p+1}|, 0)$, then*

$$\bar{r}(p, n, x)=r(p, n, x)+\chi(n, x)(|\theta_p|, |\theta_{p+1}|](n) . \tag{2.17}$$

ii) *If θ_p is of the type $(|\theta_s|, |\theta_r|)$ and θ_{p+1} is of the type $(|\theta_s|, |\theta_{r+1}|)$, then*

$$(2.18)\qquad \bar{r}(p, n, x)=\chi^{+}(n, x)(|\theta_p|, |\theta_{p+1}|](n)\ .$$

iii) *If* θ_p *is of the type* $(|\theta_s|, |\theta_r|)$ $(r>0)$ *and* θ_{p+1} *is of the type* $(|\theta_{p+1}|, 0)$, *then*

$$(2.19)\qquad \bar{r}(p, n, x)=\sum_{q=s}^{p-1}\bar{r}(q, n, x)+r(p, n, x)+\chi(n, x)(|\theta_s|, |\theta_{p+1}|](n) \\ -\chi^{+}(n, x)(|\theta_s|, |\theta_p|](n)\ .$$

PROOF. By Lemma 2-1, i) and ii) are trivial. We will prove iii) when $\operatorname{sgn}\theta_p=+1$. For the câse $\operatorname{sgn}\theta_p=-1$, the proof is almost the same. Let $t=t(p)$, which we defined in (1.32). Then

$$(2.20)\qquad \bar{r}(p, n, x)=\lambda^{-\theta_p}\sum_{j\geq 1}k(p, j)s(n-|\theta_p|-j, x) \\ -\lambda^{-\theta_p}\sum_{j\geq 1}k(r, j)s(n-|\theta_p|-j, x)-\lambda^{-\theta_s\theta_{r+1}}s(n-|\theta_s\theta_{r+1}|, x) \\ +\chi(n, x)(t, |\theta_{p+1}|](n)+\lambda^{-\theta_s}\chi(n-|\theta_s|, x)(t, |\theta_s\theta_{r+1}|](n) \\ +\chi^{+}(n, x)(|\theta_p|, t](n)\ .$$

On the other hand, since $\operatorname{sgn}\theta_r=-1$,

$$(2.21)\qquad \sum_{q\geq r+1}\bar{r}(q, n-|\theta_s|, x)=s(n-|\theta_s|, x)-\sum_{q=0}^{r}\bar{r}(q, n-|\theta_s|, x) \\ =s(n-|\theta_s|, x)-\sum_{q=0}^{r-1}\bar{r}(q, n-|\theta_s|, x)-\{\lambda^{-\theta_r}s(n-|\theta_p|, x) \\ -\lambda^{-\theta_r}\sum_{j\geq 1}k(p, j)s(n-|\theta_p|-j, x)-\lambda^{-\theta_{r+1}}s(n-|\theta_s\theta_{r+1}|, x) \\ +\chi(n-|\theta_s|, x)(|\theta_p|, |\theta_s\theta_{r+1}|](n)\}\ .$$

Moreover, since

$$(2.22)\qquad \lambda^{-\theta_s}s(n-|\theta_s|, x)-\bar{r}(0, n-|\theta_s|, x)=\lambda^{-\theta_s}\{s(n-|\theta_s|, x) \\ -\sum_{j\geq 1}k(0, j)s(n-|\theta_s|-j, x)-\chi(n-|\theta_s|, x)(0, |\theta_1|](n-|\theta_s|)\} \\ =r(s, n, x)+\lambda^{-\theta_{s+1}}s(n-|\theta_{s+1}|, x)-\chi(n, x)(|\theta_s|, |\theta_{s+1}|](n) \\ -\lambda^{-\theta_s}\chi(n-|\theta_s|, x)(0, |\theta_1|](n-|\theta_s|)$$

and

$$(2.23)\qquad \chi(n, x)(|\theta_s|, |\theta_{s+1}|](n)+\lambda^{-\theta_s}\chi(n-|\theta_s|, x)(0, |\theta_1|](n-|\theta_s|) \\ =\chi^{+}(n, x)(|\theta_s|, |\theta_{s+1}|](n)\ ,$$

repeating this procedure, we get

$$(2.24)\qquad \lambda^{-\theta_s}s(n-|\theta_s|,x)-\sum_{q=0}^{r-1}\bar{r}(q,n-|\theta_s|,x)=\sum_{q=s}^{s+r-1}r(q,n,x)$$
$$+\lambda^{-\theta_p}s(n-|\theta_p|,x)-\chi^{+}(n,x)(|\theta_s|,|\theta_p|](n)\,.$$

Combining (2.20) and (2.21), we get the proof.

LEMMA 2-4. *Let*

$$(2.25)\qquad b_{p,j}=k(p,j)-(\sum_{a\neq 0}\lambda^{-a})k(p,j-1)\,.$$

Then

$$(2.26)\quad \text{i)}\quad b_{p,j}=\lambda^{\theta_p}\lambda^{-a(1,|\theta_p|+1)}\quad \text{if}\quad j=1\quad\text{and}\quad \operatorname{sgn} a_{p+1}(1,1)s(a_{p+1,2})=+1\,.$$

$$(2.27)\quad \text{ii)}\quad b_{p,j}=\lambda^{\theta_p}b_{|\theta_p|+j}\quad \text{if}\quad 2\leqq j<|w^c_{p+1}|\,.$$

$$(2.28)\quad \text{iii)}\quad b_{p,|w^c_{p+1}|}=\lambda^{\theta_p}b_{|\theta_{p+1}|}\quad \text{if}\quad \operatorname{sgn} w^c_{p+1}=+1\,.$$

$$(2.29)\quad \text{iv)}\quad b_{p,|w^c_{p+1}|}=\lambda^{\theta_p}(b_{|\theta_{p+1}|}-\lambda^{-\theta_{p+1}})\quad \text{if}\quad \operatorname{sgn} w^c_{p+1}=-1\,.$$

$$(2.30)\quad \text{v)}\quad b_{p,|w^c_{p+1}|+1}=\lambda^{\theta_p}(\sum_{a<0}\lambda^{-a})\lambda^{-\theta_{p+1}}\quad \text{if}\quad \operatorname{sgn} w^c_{p+1}=+1\,.$$

$$(2.31)\quad \text{vi)}\quad b_{p,|w^c_{p+1}|+1}=\lambda^{\theta_p}(\sum_{a>0}\lambda^{-a})\lambda^{-\theta_{p+1}}\quad \text{if}\quad \operatorname{sgn} w^c_{p+1}=-1\,.$$

$$(2.32)\quad \text{vii)}\quad b_{p,j}=0\quad \text{otherwise}\,.$$

PROOF.

$$(2.33)\qquad k(p,j)=\sum_{2\leqq i\leqq j^*}(\sum\nolimits^{p,i}\lambda^{-a})\lambda^{-a_{p+1}(1,i-1)}(\sum_{a\neq 0}\lambda^{-a})^{j-i}+k^*(p,j)\,,$$

where

$$(2.34)\qquad j^*=\min\{j,|w^c_{p+1}|\}\,,$$

$\sum^{p,i}$ is the sum over all $a\in A$ which satisfies

$$a<a_{p+1,i}\quad\text{and}\quad a\neq 0\,,\quad\text{if}\quad \operatorname{sgn} a_{p+1}(1,i-1)=+1\,,$$
$$a>a_{p+1,i}\quad\text{and}\quad a\neq 0\,,\quad\text{if}\quad \operatorname{sgn} a_{p+1}(1,i-1)=-1\,,$$

and

$$(2.35)\quad \text{i)}\quad k^*(p,j)=\lambda^{\theta_p}(\sum_{a<0}\lambda^{-a})\lambda^{-\theta_{p+1}}(\sum_{a\neq 0}\lambda^{-a})^{j-|\theta_{p+1}|+|\theta_p|-1}$$
$$\text{if}\quad \operatorname{sgn} w^c_{p+1}=+1\quad\text{and}\quad j>|w^c_{p+1}|\,,$$

$$(2.36)\quad \text{ii)}\quad k^*(p,j)=\lambda^{\theta_p}(\sum_{a>0}\lambda^{-a})\lambda^{-\theta_{p+1}}(\sum_{a\neq 0}\lambda^{-a})^{j-|\theta_{p+1}|+|\theta_p|-1}$$
$$\text{if}\quad \operatorname{sgn} w^c_{p+1}=-1\quad\text{and}\quad j>|w^c_{p+1}|\,,$$

(2.37) iii) $k^*(p, j)=\lambda^{-a_{p+1}(1,j)}$

if $j<|w^c_{p+1}|$ and j satisfies one of the following:

a) $\operatorname{sgn} a_{p+1}(1, j)=+1$, $a_{p+1,j+1}>0$ and $a_{p+1,j}\neq 0$,

b) $\operatorname{sgn} a_{p+1}(1, j)=-1$, $a_{p+1,j+1}<0$ and $a_{p+1,j}\neq 0$,

(2.38) iv) $k^*(p, j)=0$ otherwise.

Hence, the proof easily follows.

The relevant renewal equation takes the following form.

THEOREM 2-5. i) *The radii of convergence for* $\chi(z; x)$ *and* $w(z; x)$ *are greater than* e^{ε}.

ii) *For* $|z|<e^{\varepsilon}$,

$$w(z; x)=\chi(z; x)/\Phi(z) . \tag{2.39}$$

PROOF. The radii of convergence for $\chi(z; x)$ and $w(z; x)$ both equal

$$\exp\Big(\varlimsup_{n\to\infty} \frac{1}{n} \log \lambda^{a(1,n)}\Big)=\exp\Big(\varlimsup_{n\to\infty} \frac{1}{n} \log (F^{(n)})'(1)\Big)\geqq e^{\varepsilon} . \tag{2.40}$$

Using Lemma 2-3 and Lemma 2-4 to $s(n, x)-(\sum_{a\neq 0} \lambda^{-a})s(n-1, x)$, we get

$$s(n, x)=\sum_{j=0}^{n} b_j s(n-j, x)+\chi(n, x)-(\sum_{a\neq 0} \lambda^{-a})\chi(n-1, x) . \tag{2.41}$$

This is the renewal equation for $s(n, x)$. Now we show the renewal equation for $w(n, x)$, that is,

$$\begin{aligned} w(n, x)&=\sum_{j=0}^{n}(\sum_{a\neq 0} \lambda^{-a})^j s(n-j, x) \\ &=\sum_{j=0}^{n} b_j w(n-j, x)+\chi(n, x) . \end{aligned} \tag{2.42}$$

This shows ii).

To calculate the decay rate of correlation, we need to express $w(w, m, n, x)$ in terms $w(n, x)$. For this purpose, we must consider $w(m, n, x)$. To beginning with, we have to calculate the values of $r(p, m, n, x)$ and $k(p, m, j)$.

LEMMA 2-6. *Let* θ_p *be of the type* $(|\theta_q|, |\theta_r|)$. *Then we have the following*:

(2.43) i) $r(p, m, n, x)=r(p, n, x)$ *if* $m=|\theta_p|$,

(2.44) ii) $r(p, m, n, x)=\lambda^{-\theta_p}\sum_j k(p, m-|\theta_p|, j)s(n-|\theta_p|-j, x)$
$+\chi(n, x)(m-1, |\theta_{p+1}|](n)$,
if $|\theta_p|\leqq m\leqq|\theta_{p+1}|$, $m>t(p)$ *and* $\operatorname{sgn}\theta_p=+1$,

(2.45) iii) $r(p, m, n, x)=\lambda^{-a(1,m)}w(n-m, x)-\lambda^{-\theta_p}\sum_j k(p, m-|\theta_p|, j)$
$\times s(n-|\theta_p|-j, x)-\lambda^{-\theta_{p+1}}s(n-|\theta_{p+1}|, x)$
$+\chi(n, x)(m-1, |\theta_{p+1}|](n)$,
if $|\theta_p|\leqq m\leqq|\theta_{p+1}|$, $m>t(p)$ *and* $\operatorname{sgn}\theta_p=-1$,

(2.46) iv) $r(p, m, n, x)=\chi^+(n, x)(m-1, |\theta_{p+1}|](n)$,
if θ_{p+1} *is of the type* $(|\theta_q|, |\theta_{r+1}|)$ (*hence* $m\leqq t(p)$) ,

(2.47) v) $r(p, m, n, x)=\sum_{s=q}^{p-1} r(s, n, x)+\lambda^{-\theta_p}\sum_j (k(p, m-|\theta_p|, j)$
$-k(r, m-|\theta_p|, j)+k(r, j))s(n-|\theta_p|-j, x)$
$+\chi(n, x)(m-1, |\theta_{p+1}|](n)-\chi^+(n, x)(|\theta_q|, |\theta_p|](n)$,
if θ_{p+1} *is of the type* $(|\theta_{p+1}|, 0)$, $m\leqq t(p)$ *and* $\operatorname{sgn}\theta_p=+1$,

(2.48) vi) $r(p, m, n, x)=\sum_{s=0}^{p-1} r(s, n, x)+\lambda^{-\theta_p}s(n-|\theta_p|, x)$
$-\lambda^{-\theta_p}\sum_j (k(p, m-|\theta_p|, j)-k(r, m-|\theta_p|, j)$
$+k(r, j))s(n-|\theta_p|-j, x)-\lambda^{-\theta_{p+1}}s(n-|\theta_{p+1}|, x)$
$+\chi(n, x)(m-1, |\theta_{p+1}|](n)-\chi^+(n, x)(|\theta_q|, |\theta_p|](n)$,
if θ_{p+1} *is of the type* $(|\theta_{p+1}|, 0)$, $m\leqq t(p)$ *and* $\operatorname{sgn}\theta_p=-1$.

Proof. As Lemma 2–3, we can prove i), ii), iii) and iv). Thus we will prove v), the proof of vi) is almost the same.

(2.49)
$$r(p, m, n, x)=\lambda^{-\theta_p}\sum_j k(p, m-|\theta_p|, j)s(n-|\theta_p|-j, x)$$
$$-\lambda^{-\theta_p}\sum_j k(r, m-|\theta_p|, j)s(n-|\theta_p|-j, x)-\lambda^{-\theta_q\theta_{r+1}}s(n-|\theta_q\theta_{r+1}|, x)$$
$$+\chi^+(n, x)(m-1, t(p)](n)+\chi(n, x)(t(p), |\theta_{p+1}|](n)$$
$$+\lambda^{-\theta_q}\chi(n-|\theta_q|, x)(t(p), |\theta_q\theta_{r+1}|](n)+\lambda^{-\theta_q}\sum_{s\geqq r+1} r(s, n-|\theta_q|, x) .$$

On the hand, as in Lemma 2–3,

(2.50)
$$\sum_{s\geqq r+1} r(s, n-|\theta_q|, x)=s(n-|\theta_q|, x)-\sum_{s=0}^{r-1} r(s, n-|\theta_q|, x)$$
$$-\lambda^{-\theta_r}s(n-|\theta_p|, x)+\lambda^{-\theta_r}\sum_j k(r, j)s(n-|\theta_p|-j, x)$$
$$+\lambda^{-\theta_{r+1}}s(n-|\theta_q\theta_{r+1}|, x)-\chi(n-|\theta_q|, x)(|\theta_p|, |\theta_q\theta_{r+1}|](n) .$$

Combining (2.49) and (2.50), we get the proof.

LEMMA 2-7. i) *For* $1\leqq m\leqq|w_{p+1}^{c}|$,

$$(2.51)\qquad k(p,m,j)-(\sum_{a\neq 0}\lambda^{-a})k(p,m,j-1)$$

$$=\begin{cases} k(p,j)-(\sum_{a\neq 0}\lambda^{-a})k(p,j-1) & \text{if } j>m\,, \\ \lambda^{-a_{p+1}(1,m)} & \text{if } j=m \text{ and } \operatorname{sgn} a_{p+1}(1,m)s(a_{p+1,j+1})=+1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

ii) *Let* θ_p *be of the type* $(|\theta_q|, |\theta_r|)$. *Then for* $|\theta_p|<m\leqq t(p)$,

$$(2.52)\qquad \{k(p,m-|\theta_p|,j)-k(r,m-|\theta_p|,j)+k(r,j)\}$$
$$-(\sum_{a\neq 0}\lambda^{-a})\{k(p,m-|\theta_p|,j-1)-k(r,m-|\theta_p|,j-1)+k(r,j-1)\}$$
$$=k(p,j)-(\sum_{a\neq 0}\lambda^{-a})k(p,j-1)\,.$$

The proofs are almost the same as in Lemma 2-4, thus we omit them.

LEMMA 2-8. *For* $n>m>0$,

$$(2.53)\qquad w(m,n,x)$$

$$=\begin{cases} \lambda^{-a(1,q)}\{w(n-q,x)-(\sum_{s(a)=+1}\lambda^{-a})w(n-q-1,x)\} \\ \quad +\sum_{j\geqq q+1}b_j w(n-j,x)+\chi(n,x) \\ \quad \text{if } a(1,m) \text{ is of the type } (q,r)\ (r>0)\,, \\ \lambda^{-a(1,m)}+\sum_{j\geqq m+1}b_j w(n-j,x)+\chi(n,x)\,, \\ \quad \text{if } a(1,m) \text{ is of the type } (m,0) \text{ and } \operatorname{sgn} a(1,m)s(a_{m+1}^{c})=+1\,. \\ \sum_{j\geqq m+1}b_j w(n-j,x)+\chi(n,x) \\ \quad \text{if } a(1,m) \text{ is of the type } (m,0) \text{ and } \operatorname{sgn} a(1,m)s(a_{m+1}^{c})=-1\,. \end{cases}$$

PROOF. Using Lemma 2-6 and Lemma 2-7 to

$$w(m,n,x)-\sum_{j\geqq 1}(\sum_{a\neq 0}\lambda^{-a})^{j}w(m,n-j,x)\,,$$

we can prove the lemma.

Now we can calculate $w(w,m,n,x)$.

LEMMA 2-9. *For a word* $w\in W(F)$,

i) *if one of the following holds*:

1) $w<a(m+1,m+|w|)$ *and* $s(a_{m+1}^{c})=+1$,

2) $w>a(m+1,m+|w|)$ *and* $s(a_{m+1}^{c})=-1$,

then

$$w(w,\ m,\ n,\ x)=0\ , \tag{2.54}$$

ii) *otherwise,*

$$\begin{aligned} w(w,\ m,\ n,\ x)&=\operatorname{sgn} w_m^*(-s(a^c_{w_m^*+1}))\lambda^{-w}\lambda^{a(1,q(m)+r(m))} \\ &\quad\times\{\sum_{j\geqq q(m)+1} b_j w(n+q(m)+r(m)-|w|-j,x)+\chi(n+q(m)+r(m)-|w|,x)\} \\ &\quad+w_1(w,\ m,\ n,\ x)+w_2(w,\ m,\ n,\ x)\ , \end{aligned} \tag{2.55}$$

where

$$\begin{aligned} &w_1(w,\ m,\ n,\ x) \\ &=\begin{cases} \lambda^{-w}\lambda^{a(1,q(m)+r(m))}\lambda^{-a(1,q(m))}w(n-|w|+r(m),\ x) \\ \qquad -w_3(w,\ m,\ n,\ x)\ , \\ \qquad \textit{if } \operatorname{sgn} a(1,\ p(m))s(a^c_{p(m)+1})\operatorname{sgn} w_m^* s(a^c_{|w_m^*|+1})=-1\ , \\ 0 \qquad \textit{otherwise}\ , \end{cases} \end{aligned} \tag{2.56}$$

$$w_3(w,\ m,\ n,\ x)=\begin{cases} (\sum_{a\neq 0}\lambda^{-a})w(n-|w|+r(m)-1,\ x) & \textit{if}\quad r(m)>0\ , \\ 0 \qquad \textit{if}\quad r(m)=0\ , \end{cases} \tag{2.57}$$

$$w_2(w,\ m,\ n,\ x)=\begin{cases} \lambda^{-w}w(n-|w|,\ x) & \textit{if}\quad r^*(m)>r(m)>0 \textit{ and } \\ & \qquad s(a^c_{r^*(m)+1})s(a^c_{r(m)+1})=-1\ , \\ 0 & \textit{otherwise}\ . \end{cases} \tag{2.58}$$

PROOF. From the assumption of $w(w,\ m,\ n,\ x)$, if $w>a(m+1,\ m+|w|)$,

$$w(w,\ m,\ n,\ x)=\begin{cases} 0 & \text{if}\quad a^c_{m+1}\leqq 0\ , \\ \lambda^{-w}\lambda^{a(1,q+r)}w(q+r,\ n+q+r-|w|,\ x) & \text{if}\quad a^c_{m+1}>0\ , \end{cases} \tag{2.59}$$

where w is of the type $(q,\ r)$. Thus by Lemma 2–8, the assertion is proved. For the case $w<a(m+1,\ m+|w|)$, the proof is almost the same. Now assume that $w=a(m+1,\ m+|w|)$. Then

$$w(w,\ m,\ n,\ x)=\sum\lambda^{-wv}\ , \tag{2.60}$$

where the sum is over all v which satisfies

$$|v|=n-|w|\ , \tag{2.61}$$

and

$$\begin{aligned} a(1,\ m+|w|)vx&\leqq a(1,\ \infty) &&\text{if}\quad \operatorname{sgn} a(1,\ m)s(a^c_{m+1})=-1\ , \\ a(1,\ m+|w|)vx&>a(1,\ \infty) &&\text{if}\quad \operatorname{sgn} a(1,\ m)s(a^c_{m+1})=+1\ . \end{aligned} \tag{2.62}$$

Thus if $a(1, m+|w|)$ is of the type $(m+|w|, 0)$,

$$(2.63)\qquad w(w, m, n, x)=\begin{cases}\lambda^{a(1,m)}w(m+|w|, n+m, x)\ , & \text{if}\quad \operatorname{sgn} a(1, m)s(a^c_{m+1})=-1\ ,\\ \lambda^{-w}w(n-|w|, x)-\lambda^{a(1,m)}w(m+|w|, n+m, x)\ , & \text{if}\quad \operatorname{sgn} a(1, m)s(a^c_{m+1})=+1\ .\end{cases}$$

Thus also by Lemma 2-8, we can prove the assertion. For the other cases, the proofs are more complicated but we can prove them in a similar way.

COROLLARY 2-10.

$$(2.64)\qquad w(w, m, n, x)=\sum_{j\ge|w|} b_j(w, m)w(n-j, x)+\chi(w, m, n, x)\ .$$

The proof is trivial.

Now we will show that 1 is one of the Fredholm eigenvalues.

LEMMA 2-11. *Suppose that* $\xi>0$. *Then we get*:

$$(2.65)\qquad \text{i)}\quad \Phi(1)=0\ .$$

ii) *For any word* $w\in W(F)$,

$$(2.66)\qquad \int\langle w\rangle(x)\chi(m, x)dx=-s(a^c_{m+1})\operatorname{sgn} a(1, m)\lambda^{-a(1,m)}\sum_j b_j(w, m)\ .$$

PROOF. Let

$$(2.67)\qquad W(n, 0)=\bigcap_{x\in(0)} W(n, x)\ ,$$

$$(2.68)\qquad W(w, m, n, 0)=\bigcap_{x\in(0)} W(w, m, n, x)$$

and

(2.69) $F(w, m, n)=$ the set of words $v=a_1\cdots a_n\in W(w, m, n, 0)$ such that $a_1\cdots a_i\notin W(w, m, n, 0)$ for any $1\leqq i\leqq n-1$.

Then we define

$$(2.70)\qquad w(n, 0)=\sum_{v\in W(n,0)}\lambda^{-v}\ ,$$

$$(2.71)\qquad w(w, m, n, 0)=\sum_{v\in W(w,m,n,0)}\lambda^{-v}\ ,$$

$$(2.72)\qquad f(w, m, n)=\sum_{v\in F(w,m,n)}\lambda^{-v}$$

and we denote the generating functions of $w(n, 0)$, $w(w, m, n, 0)$ and $f(w, m, n)$ by $w(z; 0)$, $w(z; w, m)$ and $f(z; w, m)$. Then as for Theorem 2–5 and Corollary 2–10, we get

$$w(z; 0)=(\Phi(z))^{-1}, \tag{2.73}$$

and

$$w(z; w, m)=(\sum_j b_j(w, m)z^j)w(z; 0)-w^*(w, m)z^{|w|}, \tag{2.74}$$

where

$$w^*(w, m)=\begin{cases}1 & \text{if } r(m)>0 \text{ and } s(a^c_{q(m)+r(m)+1})s(a^c_{r(m)+1})=-1, \\ 0 & \text{otherwise}.\end{cases} \tag{2.75}$$

Since

$$w(w, m, n, 0)=\sum_{i=1}^n f(w, m, i)w(n-i, 0), \tag{2.76}$$

we get

$$w(z; w, m)=f(z; w, m)w(z; 0). \tag{2.77}$$

Hence by (2.73) and (2.74)

$$f(z; w, m)=\sum_j b_j(w, m)z^j-w^*(w, m)z^{|w|}\Phi(z). \tag{2.78}$$

Thus, by virtue of (2.78), we get

$$\begin{aligned}&-s(a^c_{m+1})\operatorname{sgn} a(1, m)\lambda^{a(1,m)}\int \langle w\rangle(x)\chi(m, x)dx \\ &=\sum_i\sum_j f(w, m, i)(\sum_{a\neq 0}\lambda^{-a})^j(1-\sum_{a\neq 0}\lambda^{-a}) \\ &=f(1; w, m) \\ &=\sum_j b_j(w, m)-w^*(w, m)\Phi(1).\end{aligned} \tag{2.79}$$

Hence, for $w=\phi$ and $m=0$, since $w^*(\phi, 0)=0$ and $b_j(w, m)=b_j$, we get

$$\sum_j b_j=1, \tag{2.80}$$

that is, $\Phi(1)=0$. This completes i). Substituting this to (2.79), we get ii).

§3. The decay of correlation.

In this section, we will prove Theorem 0–1 and Theorem 0–2. We first show that $\rho(x)$ is the density of the invariant probability measure with respect to the mapping F.

LEMMA 3-1. *Assume that* $\xi>0$ *and that* $\eta<1$. *Then we get:*

(3.1) i) $\rho(x)\geqq 0$.

(3.2) ii) $\int \rho(x)dx=1$.

iii) *For any* $f\in L^1$,

$$\int f(x)\rho(x)dx=\int f(F(x))\rho(x)dx \text{ .} \tag{3.3}$$

iv) *For any word* $w\in W(F)$,

$$\int \langle w\rangle(x)\rho(x)dx=-(\Phi'(1))^{-1}\sum_m\sum_j(-s(a^c_{m+1}))\mathrm{sgn}\, a(1,m)\times b_j(w,m)\lambda^{-a(1,m)} \text{ .} \tag{3.4}$$

PROOF OF iv). By virtue of (2.66), this is trivial.

PROOF OF iii). For any word $w\in W(F)$, we get:
1) If $a^c_{m+1}\leqq 0$,

$$\sum_{a\in A}\chi(m,ax)\lambda^{-a}=\sum_{ax\leqq a(m+1,\infty)}\mathrm{sgn}\, a(1,m)\lambda^{-a(1,m)a}=b_{m+1}+\chi(m+1,x) \text{ .} \tag{3.5}$$

2) If $a^c_{m+1}>0$, then as above,

$$\sum_{a\in A}\chi(m,ax)\lambda^{-a}=b_{m+1}+\chi(m+1,x) \text{ .} \tag{3.6}$$

Hence, appealing to Lemma 2-11, we get

$$\begin{aligned}\int\langle w\rangle(F(x))\chi(1;x)dx&=\sum_{a\in A}\lambda^{-a}\int\langle w\rangle(x)\sum_{m=0}^{\infty}\chi(m,ax)dx\\&=\int\langle w\rangle(x)\sum_{m=0}^{\infty}b_{m+1}dx+\int\langle w\rangle(x)\sum_{m=0}^{\infty}\chi(m+1,x)dx\\&=\int\langle w\rangle(x)\chi(1;x)dx \text{ .}\end{aligned} \tag{3.7}$$

On the other hand, by the assumption, $\{(w): w\in W(F)\}$ is a generator. This proves iii).

PROOF OF ii). in iv), put $w=\phi$. Then, since $w^*_m=a(1,m)$, (3.8) $$b_j(\phi, m) = -s(a^c_{m+1})\mathrm{sgn}\, a(1, m) b_{j+m} \lambda^{a(1,m)} .$$

Thus

(3.9) $$\int \chi(1; x)dx = \sum_{m=0}^{\infty} \int \chi(m, x)dx = \sum_m \sum_j b_{j+m}$$
$$= \sum_{n=1}^{\infty} nb_n = -\Phi'(1) .$$

PROOF OF i).

(3.10) $$\rho(x) = -\Phi'(1)^{-1}\chi(1; x) = \lim_{z \uparrow 1}(1-z)\chi(z; x)/\Phi(z)$$
$$= \lim_{z \uparrow 1}(1-z)w(z; x) \geqq 0 .$$

This proves i).

LEMMA 3-2. *Suppose that* $\xi > 0$ *and* $\eta < 1$. *Then for* $\varepsilon > 0$ *which satisfies* $(\eta+\varepsilon)e^{\xi-\varepsilon} > 1$ *and for a function* $f \in BV$,

(3.11) i) $\nu^\varepsilon(f) < \infty$.

ii) *For sufficiently large* n,

(3.12) $$\int |f(x) - \{(f_1)_n + (f_2)_n\}(x)| dx < V(f)e^{-n(\xi-\varepsilon)} ,$$

if $f_1 + f_2 = f$ *and both* f_i $(i=1, 2)$ *are monotone.*

PROOF. Assume that a function f is monotone. Then for sufficiently large n,

(3.13) $$\nu^\varepsilon_n(f) < V(f)(\eta+\varepsilon)^{-1}((\eta+\varepsilon)e^{\xi-\varepsilon} - 1)^{-1}$$

and

(3.14) $$\int |f(x) - (f_1)_n(x)| dx < V(f)e^{-n(\xi-\varepsilon)} .$$

This proves the lemma.

LEMMA 3-3. *Assume that* $\varepsilon > 0$ *and* $\eta < 1$. *Then for any* $\varepsilon > 0$, *there exists a constant* K_1 *such that*

(3.15) $$|w(n, x) - \rho(x)| < K_1(\eta+\varepsilon)^n .$$

PROOF. By Theorem 2-5 and the definition of ρ,

(3.16) $$w(z; x) - \rho(x)(1-z)^{-1} = \chi(z; x)/\Phi(z) + (\Phi'(1))^{-1}\chi(1; x)(1-z)^{-1}$$

is analytic in $|z|<\eta^{-1}$. Thus for $|z|<(\eta+\varepsilon)^{-1}$, there exists a constant K_1 such that

$$|w(z;x)-\rho(x)(1-z)^{-1}|<K_1 . \tag{3.17}$$

This completes the proof.

LEMMA 3-4. *Assume that $\xi>0$ and $\eta<1$. Then for any $\varepsilon>0$ which satisfies $(\eta+\varepsilon)e^{\xi-\varepsilon}>1$, $w \in W(F)$ and $g \in L^1$, there exists a constant K_2 such that for $n>|w|$,*

$$\left|\int \langle w\rangle(x)g(F^{(n)}(x))\rho(x)dx-\int \langle w\rangle(x)\rho(x)dx \int g(x)\rho(x)dx\right| \\ <K_2\|g\|(\eta+\varepsilon)^n\nu^{\varepsilon}((w)) . \tag{3.18}$$

PROOF. By Lemma 3-1,

$$\begin{aligned} \int \langle w\rangle(x)g(F^{(n)}(x))\rho(x)dx &= \sum_{|v|=n-|w|} \lambda^{-wv} \int g(x)\rho(wvx)dx \\ &= -(\Phi'(1))^{-1} \sum_{|v|=n-|w|} \lambda^{-wv} \int g(x) \sum_m \chi(m, wvx)dx \\ &= -(\Phi'(1))^{-1} \sum_m \lambda^{-a(1,m)} \int g(x)(-s(a^c_{m+1}))\mathrm{sgn}\, a(1, m)w(w, m, n, x)dx \\ &= (\Phi'(1))^{-1} \sum_m \lambda^{-a(1,m)} s(a^c_{m+1})\mathrm{sgn}\, a(1, m) \int g(x) \\ &\quad \times \left\{\sum_{j=|w|}^{n} b_j(w, m)w(n-j, x)+\chi(w, m, n, x)\right\}dx \\ &= (\Phi'(1))^{-1} \sum_m \lambda^{-a(1,m)} s(a^c_{m+1})\,\mathrm{sgn}\, a(1, m)\left\{\int g(x) \sum_{j=|w|}^{n} b_j(w, m)\right. \\ &\quad \times (w(n-j, x)-\rho(x))dx-\int g(x) \sum_{j=n+1}^{\infty} b_j(w, m)\rho(x)dx \\ &\quad \left.+\int g(x)\chi(w, m, n, x)dx\right\}+\int \langle w\rangle(x)\rho(x)dx \int g(x)\rho(x)dx . \end{aligned} \tag{3.19}$$

On the other hand, from the definition, there exists a constant K_3 such that

$$|b_j(w, m)| , \quad |\chi(w, m, j, x)|<K_3\lambda^{-w}e^{-(j-|w|)(\xi-\varepsilon)} . \tag{3.20}$$

Hence, by Lemma 3-3 and using the fact that ρ is bounded, we get the proof.

PROOF OF THEOREM 0-1. By the assumption that $\xi>0$, Lemma 3-4 shows that the dynamical system $([0, 1], \mu, F)$ is mixing. For $f \in BV$,

$$(3.21)\qquad \left|\int Q^n f(x)g(x)d\mu\right| \leqq \left|\int Q^n\{(f_1)_n+(f_2)_n\}(x)g(x)d\mu\right| + \left|\int Q^n[f-\{(f_1)_n+(f_2)_n\}](x)g(x)d\mu\right| ,$$

where f_i $(i=1, 2)$ are monotone and $f_1+f_2=f$. Then by Lemma 3-4

$$(3.22)\qquad \left|\int Q^n f(x)g(x)d\mu\right| < K_2(\eta+\varepsilon)^n \|g\|(\nu_n^\varepsilon(f_1)+\nu_n^\varepsilon(f_2)) + \left|\int [f(x)-\{(f_1)_n(x)+(f_2)_n(x)\}]g(F^{(n)}(x))d\mu\right| + \int |f(x)-\{(f_1)_n(x)+(f_2)_n(x)\}|\, d\mu \left|\int g d\mu\right| .$$

Since

$$(3.23)\qquad \left|\int g(F^{(n)}(x))d\mu\right| = \left|\int g(x)d\mu\right| < \infty ,$$

and by Lemma 3.2,

(3.24) $\lim_{n\to\infty} e^{n(\xi-\varepsilon)}$ {the second and the third term of the right hand side of (3.22)}$=0$.

Hence,

$$(3.25)\qquad \lim_{n\to\infty} (\eta+\varepsilon)^{-n} \int Q^n f(x)g(x)d\mu = 0 .$$

This completes the proof.

Now we will prove Theorem 0–2. We devide the proof into six steps.

i) If a word w is complete, then it is trivial that $\lambda^w>1$.

ii) If $\lambda^{w_1^c}\geqq 1$, since any other 0-word is complete, $\lambda^w\geqq 1$ for any word w. This contradicts the assumption. Thus we get

$$(3.26)\qquad \lambda^{w_1^c}<1 .$$

iii) If $\operatorname{sgn} w_1^c=-1$, it is trivial that there exists a periodic point with its expansion $w_1^c w_1^c\cdots$ and $w_i^c=w_1^c$ for any i.

iv) Assume that $\operatorname{sgn} w_1^c=+1$. Since for $m<n$,

$$(3.27)\qquad \lim_{x\to c} F^{(m)}(x) \notin (0) \quad (x\in(0)) ,$$

a word which is of the form $w=w_1^c a_1^c\cdots a_m^c$ $(m<n)$ satisfies either

(3.28) a) $\lambda^w>1$,

or

(3.29) b) $(w0)=\phi$.

v) By iv), we get

$$\lambda^{w_1^c w_2^c}<1 . \tag{3.30}$$

vi) Repeating above arguments, we get that one of the following holds:

1) There exists n such that $\operatorname{sgn} w_1^c\cdots w_n^c=-1$, and there exists a periodic point with its expansion $w_1^c\cdots w_n^c w_1^c\cdots w_n^c w_1^c\cdots$ which coincides with the expansion of c.

2) For any n, $\operatorname{sgn} w_1^c\cdots w_n^c=+1$ and $\lambda^{w_1^c\cdots w_n^c}<1$. But, since

$$\lim_{n\to\infty}\overline{(w_1^c\cdots w_n^c)}=\{0\} , \tag{3.31}$$

the case 2) can occur only if

$$\inf\{x\in(0)\}=0 \tag{3.32}$$

and

$$\lambda^0<1 , \tag{3.33}$$

that is, the expansion of c equals $00\cdots$ and 0 is an attractive fixed point. This completes the proof.

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