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# Lower Bounds for the Unknotting Numbers of Certain Iterated Torus Knots

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## Introduction

In [5], Milnor asked if the unknotting number of an algebraic knot is equal to its genus, where an algebraic knot is the one which occurs as an isolated singularity of a complex plane curve (see Problem 4 of Gordon [2] also). It is known that an algebraic knot is an iterated torus knot and that all torus knots are algebraic. It is a consequence from a result of Murasugi [6] that the problem of Milnor is affirmative for the $(2, m)$-torus knot. It is not hard to see that the unknotting number of an algebraic knot is at most its genus. It is more difficult to give lower bounds for the unknotting number. Weintraub [7] showed that the unknotting number of $(m-1, m)$-torus knot is at least $(m^2-5)/4$ if $m$ is odd, and $(m^2-4)/4$ if $m$ is even. Yamamoto [9] showed that the unknotting number of $(l, 2kl\pm 1)$-torus knot is at least $(k(l^2-1)-2)/2$ if $l$ is odd, and $(kl^2-2)/2$ if $l$ is even. In this note we give lower bounds for the unknotting numbers of certain iterated torus knots as follows (see §1 for definitions and notations);

THEOREM. *The unknotting number of the* $([l_1, 2kl_1-1], [l_2, 2kl_1l_2+(l_1-1)l_2-1], \cdots, [l_q, 2kl_1\cdots l_q+(l_{q-1}-1)l_q-1])$*-iterated torus knot is at least* $(k((l_1\cdots l_q)^2-1)-2)/2$ *if* $l_1\cdots l_q$ *is a power of an odd prime, and* $(k(l_1\cdots l_q)^2-2)/2$ *if* $l_1\cdots l_q$ *is even, where* $k$ *is a positive integer and* $q, l_1, \cdots, l_q$ *are integers greater than one.*

With our notation, the $([l_1, m_1], \cdots, [l_q, m_q])$-iterated torus knot is algebraic if and only if $m_il_{i+1}<m_{i+1}$ for all $i=1, \cdots, q-1$ by Brauner [1] (cf. Lê [4]). Then we note that the $([l_1, 2kl_1-1], [l_2, 2kl_1l_2+(l_1-1)l_2-1])$-

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iterated torus knot in Theorem is algebraic.

## § 1. Preliminaries.

Let $L_{l,m}$ be the link of $l+m$ components such that $l$ components of $L_{l,m}$ are parallel to each other and $m$ components of $L_{l,m}$ are parallel to each other as in Fig. 1, where $L_{2,4}$ is pictured. Let $\alpha$ and $\beta$ be the

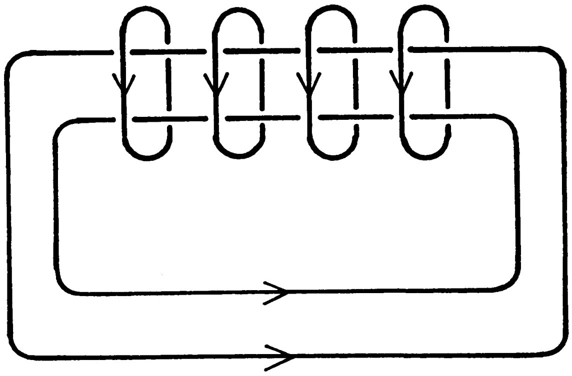

FIGURE 1. $L_{2,4}$

generators of $H_2(S^2\times S^2-\mathrm{Int}\,B^4,\ \partial(S^2\times S^2-\mathrm{Int}\,B^4);\ Z)$, and $l$ and $m$ be positive integers. Then the class $l\alpha+m\beta$ can be represented by $l+m$ properly embedded disks with boundaries $L_{l,m}$. The class can be also represented by the disk obtained from the $l+m$ properly embedded disks by connecting the boundaries with $l+m-1$ strips on $\partial(S^2\times S^2-\mathrm{Int}\,B^4)=S^3$ (see Kervaire-Milnor [3]). By $K_{l,m}$, we denote the boundary of the disk. In Yamamoto [9], the following is shown;

LEMMA. *Let $u$ be the unknotting number of $K_{l,m}$. Suppose that $l$ and $m$ are divisible by a positive integer $d$. Then*

$$u \geqq \begin{cases} \dfrac{lm(d^2-1)}{2d^2}-1 & \text{if } d \text{ is a power of an odd prime}\,, \\[2ex] \dfrac{lm}{2}-1 & \text{if } d \text{ is even}\,. \end{cases}$$

Next, we define iterated torus knots. Let $V$ be a solid torus in the 3-sphere $S^3$. A *longitude* is an essential simple closed curve on $\partial V$ which is homologous to 0 in $S^3-\mathrm{Int}\,V$ and a *meridian* is an essential simple closed curve on $\partial V$ which is homotopic to 0 in $V$. The $(l, \mu)$*-torus link* is the one which lies on an unknotted torus in $S^3$ and sweeps around it $l$ times in the longitude and $\mu$ times in the meridian. When $l$ and $\mu$ are relatively prime, it is a knot and called the $(l, \mu)$*-torus knot.* Let

$(l_1, \mu_1), \cdots, (l_q, \mu_q)$ be a sequence of pairs of relatively prime integers. The $((l_1, \mu_1), \cdots, (l_q, \mu_q))$-iterated torus knot is defined inductively as follows; we assume that we have already constructed the $((l_1, \mu_1), \cdots, (l_{q-1}, \mu_{q-1}))$-iterated torus knot. Then the $((l_1, \mu_1), \cdots, (l_q, \mu_q))$-*iterated torus knot* is defined to be the one which lies on the boundary of a tubular neighborhood of the $((l_1, \mu_1), \cdots, (l_{q-1}, \mu_{q-1}))$-iterated torus knot and sweeps around it $l_q$ times in the longitude and $\mu_q$ times in the meridian.

Let $m_i$ be

$$m_1 = \mu_1 ,$$
$$m_i = \mu_i + m_{i-1} l_i - \mu_{i-1} l_{i-1} l_i \quad \text{for} \quad i=2, \cdots, q .$$

Then, by Brauner [1], the $((l_1, \mu_1), \cdots, (l_q, \mu_q))$-iterated torus knot is an algebraic knot if and only if $m_{i-1} l_i < m_i$ for all $i=2, \cdots, q$, and the knot is associated with the polynomial $f\colon (C^2, 0) \to (C, 0)$ whose Puiseux characteristic pairs are $((l_1, m_1), , \cdots, (l_q, m_q))$ (cf. Lê [4]). For the convenience, we call the $((l_1, \mu_1), \cdots, (l_q, \mu_q))$-iterated torus knot the $([l_1, m_1], \cdots, [l_q, m_q])$-*iterated torus knot* in this note.

## §2. Proof of Theorem.

Firstly, we prove Theorem in the case of $q=2$. It is sufficient to prove that there is a disk representing $l_1 l_2 \alpha + k l_1 l_2 \beta \in H_2(S^2 \times S^2 - \mathrm{Int}\, B^4, \partial(S^2 \times S^2 - \mathrm{Int}\, B^4); Z)$ with boundary the $([l_1, 2kl_1 - 1], [l_2, 2kl_1 l_2 + (l_1 - 1) l_2 - 1])$-iterated torus knot by Lemma. In [9], it is shown that the $(l_1 l_2, 2kl_1 l_2)$-torus link is obtained from $L_{l_1 l_2, k l_1 l_2}$ by connecting the components with $kl_1 l_2$ strips. We denote each component of the $(l_1 l_2, 2kl_1 l_2)$-torus link by $K_i$ as in Fig. 2. The $(l_1 l_2, 2kl_1 l_2)$-torus link is considered as the link such

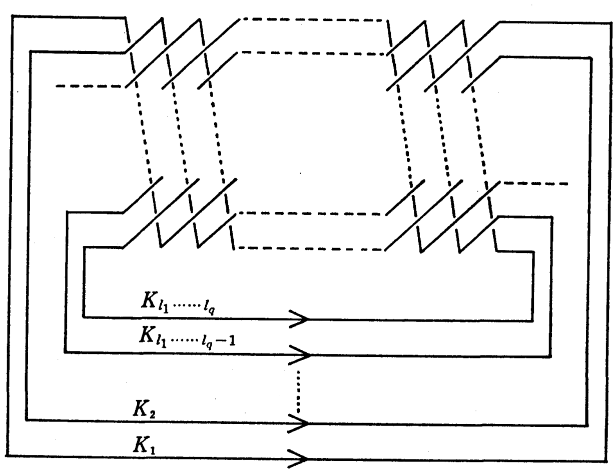


FIGURE 2. The $(l_1 \cdots l_q, 2kl_1 \cdots l_q)$-torus link

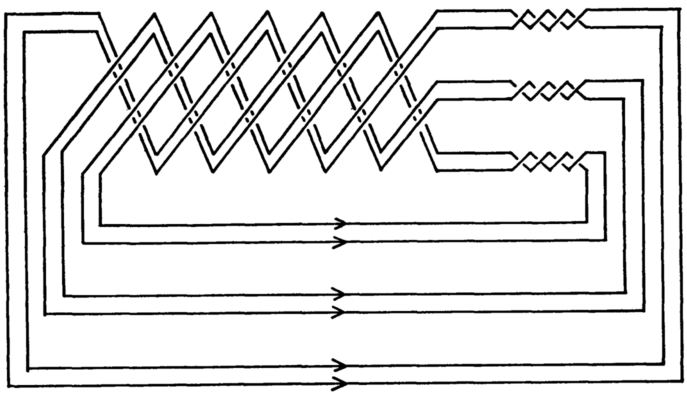

FIGURE 3. The (6, 12)-torus link

that the components $K_{il_2+1}, \cdots, K_{(i+1)l_2}$ form the $(l_2, 2kl_2)$-torus link on each component of the $(l_1, 2kl_1)$-torus link for $i=0, \cdots, l_1-1$ as in Fig. 3. In Fig. 3, near the rightmost crossing points where $K_1, \cdots, K_{l_2}$ and $K_{il_2+1}, \cdots, K_{(i+1)l_2}$ cross, we connect $K_j$ and $K_{il_2+j}$ with a strip as in Fig. 4 for all $j=1, \cdots, l_2$ and for all $i=1, \cdots, l_1-1$, and next we connect $K_1, \cdots, K_{l_2}$ with $l_2-1$ strips, as in Fig. 5, near the rightmost crossing points where $K_1, \cdots, K_{l_2}$ cross. Then, from its figure, the resulting knot is the $([l_1, 2kl_1-1], [l_2, 2kl_1l_2+(l_1-1)l_2-1])$-iterated torus knot (cf. Yamamoto [8]). This shows that $l_1l_2\alpha+kl_1l_2\beta$ is represented by a properly embedded disk with boundary the $([l_1, 2kl_1-1], (l_2, [2kl_1l_2+(l_1-1)l_2-1])$-iterated torus knot in $S^2\times S^2-\text{Int}\, B^4$.

Next we consider in the case of $q\geqq 3$. As above, the $(l_1\cdots l_q, 2kl_1\cdots l_q)$-torus link is obtained from $L_{l_1\cdots l_q, kl_1\cdots l_q}$ by connecting the components with

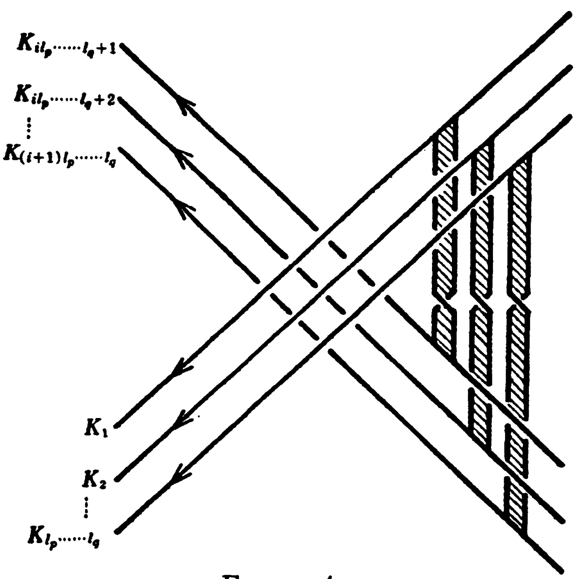


FIGURE 4

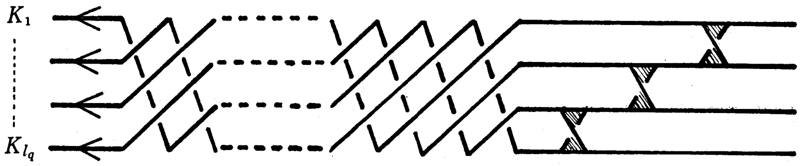


FIGURE 5

$kl_1 \cdots l_q$ strips. We denote each component of the link by $K_i$ as in Fig. 2. If we identity $K_{il_p \cdots l_q+1}, \cdots, K_{(i+1)l_p \cdots l_q}$ with one string, say $K_{p,i}$, for each $i=0, \cdots, l_1 \cdots l_{p-1}-1$, then $K_{p,0}, \cdots, K_{p,l_1 \cdots l_{p-1}-1}$ form the $(l_1 \cdots l_{p-1}, 2kl_1 \cdots l_{p-1})$-torus link. As in the case $q=2$, near the rightmost crossing points where $K_1, \cdots, K_{l_p \cdots l_q}$ and $K_{il_p \cdots l_q+1}, \cdots, K_{(i+1)l_p \cdots l_q}$ crooss, we connect $K_j$ and $K_{il_p \cdots l_q+j}$ with a strip as in Fig. 4 for all $j=1, \cdots, l_p \cdots l_q$ and for all $i=1, \cdots, l_1 \cdots l_{p-1}-1$, and we do this for all $p=1, \cdots, q-1$. Then we connect $K_1, \cdots, K_{l_q}$ with $l_q-1$ strips, as in Fig. 5, near the rightmost crossing points where $K_1, \cdots, K_{l_q}$ cross. Then, from its figure, the resulting knot is the $([l_1, 2kl_1-1], [l_2, 2kl_1l_2+(l_1-1)l_2-1], \cdots, [l_q, 2kl_1 \cdots l_q+(l_{q-1}-1)l_q-1])$-iterated torus knot. This shows that $l_1 \cdots l_q\alpha+kl_1 \cdots l_q\beta$ is represented by a properly embedded disk with boundary the $([l_1, 2kl_1-1], [l_2, 2kl_1l_2+(l_1-1)l_2-1], \cdots, [l_q, 2kl_1 \cdots l_q+(l_{q-1}-1)l_q-1])$-iterated torus knot in $S^2 \times S^2 - \operatorname{Int} B^4$. By Lemma, we have Theorem. This completes the proof of Theorem.

REMARK. The genus $g_q$ of the $([l_1, 2kl_1-1], [l_2, 2kl_1l_2+(l_1-1)l_2-1], \cdots, [l_q, 2kl_1 \cdots l_q+(l_{q-1}-1)l_q-1])$-iterated torus knot is given by the equation

$$\begin{aligned} g_q = k(l_1 \cdots l_q)^2 + \frac{1}{2}\Big( & -(2kl_1+l_1+1)(l_2 \cdots l_q)^2 \\ & + \sum_{i=2}^{q-1} (l_i-1)(l_{i+1} \cdots l_q)^2(2kl_1 \cdots l_i+(l_{i-1}-1)l_i-1) \\ & + (l_q-1)(2kl_1 \cdots l_q+(l_{q-1}-1)l_q-1)\Big) \end{aligned}$$

(cf. [8]). Our lower bounds is about a half of the first term of above.

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