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On a Formula of Morita's Partition function $q(n)$

To the memory of Dr. Takehiko Miyata

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Introduction

It is well known that the number of conjugacy classes of $\mathfrak{sl}(2, \boldsymbol{C})$ in the Lie algebra of type $A_{n-1}=\mathfrak{sl}(n, \boldsymbol{C})$ is $p(n)-1$, where $p(n)$ is the number of partitions of n. Recently J. Morita [1] found that the number of conjugacy classes of $\mathfrak{sl}(2, \boldsymbol{C})$ in the Kac-Moody Lie algebra of type $A_{n-1}^{(1)}$ is finite and that this number is given by $q(n)-1$ where $q(n)$ is the function defined by (1), which we call Morita's partition function. But it is not easy to calculate $q(n)$ directly following the definition. In this note, using the convolution product, we give a formula of $q(n)$ (Theorem) which seems to have some significance in itself. We also give a combinatorial proof of this formula.

We would like to express great thanks to Professor Jun Morita for communicating this problem.

§1. Notations.

Let $\boldsymbol{Z}_+=\{1, 2, 3, \cdots\}$ be the set of positive integers. For $n\in\boldsymbol{Z}_+$, a partition of n is a sequence $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_r)$, where $\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_r$, $\lambda_i\in\boldsymbol{Z}_+$ and $\sum_i\lambda_i=n$. We write $\lambda\vdash n$ if λ is a partition of n. For a partition $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_r)$, we define a number $a(\lambda)$ by

$$a(\lambda)=G.C.D.(\lambda_1, \lambda_2, \cdots, \lambda_r)$$

the greatest common divisor.

We denote by $\mathfrak{Z}$ the set of functions from $\boldsymbol{Z}_+$ to the complex numbers $\boldsymbol{C}$. Let us denote by $f*g$ the convolution product of $f, g\in\mathfrak{Z}$, i.e.

$$f*g(n)=\sum_{d|n}f(d)g\left(\frac{n}{d}\right).$$

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This product is commutative and associative. The unit element of this product is $e \in \mathfrak{Z}$ defined by

$$e(n)=\begin{cases} 1 & \text{if } n=1 \\ 0 & \text{otherwise} . \end{cases}$$

(For the convolution product, see e.g. [2].)

§ 2. Formula.

DEFINITION. Morita's partition function $q(n)$ is defined by

(1) $$q(n)=\sum_{\lambda \vdash n} a(\lambda) .$$

THEOREM. *The following formula holds.*

(2) $$q(n)=\varphi * p(n) .$$

COROLLARY. *If n is a prime number, then*

(3) $$q(n)=p(n)+n-1 .$$

PROOF.

For $i \in \boldsymbol{Z}_{+}$, we define a function $f_i \in \mathfrak{Z}$ by

$$f_i(n)=\#\{\lambda \vdash n \mid a(\lambda)=i\} .$$

It is clear by definition that $f_i(n)=0$ if $i \nmid n$. By an operation multiplying $1/i$ to each component of λ, we get

(4) $$f_i(n)=f_1\left(\frac{n}{i}\right) .$$

On the other hand, by definition,

(5) $$q(n)=\sum_i f_i(n) ,$$

and

(6) $$p(n)=\sum_j f_i(n) .$$

Let us define two functions $1, \mathbf{1} \in \mathfrak{Z}$ by

$$1(n)=1$$
$$\mathbf{1}(n)=n \quad \text{for all} \quad n \in \boldsymbol{Z}_{+} .$$

Then (5) and (6) are reformulated as

(5′) $$q=\mathbf{1}*f_1$$

(6′) $$p=\mathbf{1}*f_1 .$$

Let μ, φ be the Möbius function and the Euler function respectively. Then we have $\mu*\mathbf{1}=e$, $\mathbf{1}*\mu=\varphi$ (the inversion formula) [2]. Multiplying both side of (6′) by μ, we get

$$f_1=\mu*p .$$

Therefore by (5′), we get

$$q=\mathbf{1}*\mu*p .$$ Q.E.D.

Now we will prove the formula (2) by a combinatorial argument. For a partition $\lambda=(\lambda_1, \lambda_2, \cdots \lambda_r)$ and $k\in \boldsymbol{Z}_+$, we set

$$k\lambda=(k\lambda_1, k\lambda_2, \cdots, k\lambda_r) .$$

It is clear that if $\lambda\vdash d$ then $k\lambda\vdash kd$ and $a(k\lambda)=ka(\lambda)$. Now we fix $n\in \boldsymbol{Z}_+$. Let $\mathfrak{p}(n)$ denote the set of all partitions whose sizes are divisors of n, that is

(A) $$\mathfrak{p}(n)=\bigcup_{d|n}\{\lambda' \mid \lambda'\vdash d\} \qquad \text{(disjoint union)} .$$

For a partition $\lambda'\vdash d$, we get a partition $\lambda\vdash n$, by $\lambda=(n/d)\lambda'$. Therefore $\mathfrak{p}(n)$ can also be expressed as

(B) $$\mathfrak{p}(n)=\bigcup_{\lambda\vdash n}\{\lambda' \mid k\lambda'=\lambda,\ k\in \boldsymbol{Z}_+\} \qquad \text{(disjoint union)} .$$

For a partition $\lambda'\in\mathfrak{p}(n)$, we define $\omega(\lambda')$ the weight of λ' by

$$\omega(\lambda')=\varphi\left(\frac{n}{d}\right) \qquad \text{if } \lambda'\vdash d .$$

Using the expression (A), we get

(A′) $$\sum_{\lambda'\in\mathfrak{p}(n)}\omega(\lambda')=\sum_{d|n}\varphi\left(\frac{n}{d}\right)p(d) .$$

On the other hand, if $k\lambda'=\lambda\vdash n$, then $\omega(\lambda')=\varphi(k)$, therefore

$$\sum_{k\lambda'=\lambda}\omega(\lambda')=\sum_{k|a(\lambda)}\varphi(k)=a(\lambda) .$$

Using the expression (B), we get

(B′) $$\sum_{\lambda'\in\mathfrak{p}(n)}\omega(\lambda')=\sum_{\lambda\vdash n}a(\lambda) .$$

From (A′) and (B′), $\sum_{\lambda\vdash n} a(\lambda)=\sum_{d\mid n} \varphi\frac{n}{d}p(d)$ therefore $q(n)=\varphi * p(n)$.

References

[1] J. Morita, Conjugate Classes of Three Dimensional Simple Lie Subalgebras of the Affine Lie Algebra $A_l^{(1)}$, Algebraic and Topological Theories, Kinokuniya. Tokyo, 1986.

[2] H. N. Shapiro, Introduction to the Theory of Numbers, John Wiley & Sons, Inc., New York, 1983.

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