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Flows and Spines

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Introduction

A compact two dimensional polyhedron P is called a *closed fake surface* (See [3].), if each point x of P has a regular neighborhood homeomorphic to one of the following three types described in Figure 1.

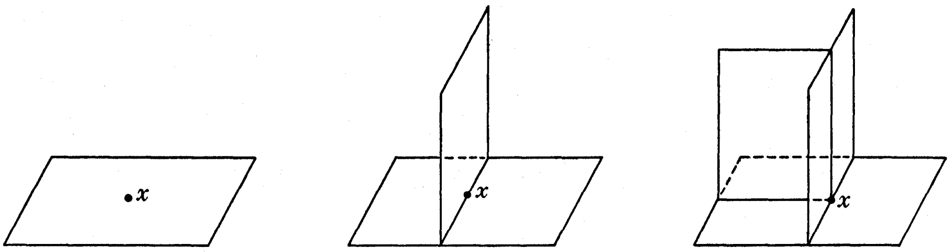

FIGURE 1

For a closed fake surface P, define

$$\mathfrak{S}'_i(P)=\{x\in P \mid \text{the regular neighborhood of } x \text{ is of type-}i\} \quad (i=1, 2, 3)\ .$$

The i-th singularity $\mathfrak{S}_i(P)$ is defined to be the closure of $\mathfrak{S}'_i(P)$ in P. A closed fake surface P is called a *standard spine* of a closed 3-manifold M, if it is embedded in M and $M-N(P)$ is homeomorphic to a 3-ball ($N(P)$ denotes a regular neighborhood of P in M). It is known ([2]) that any closed 3-manifold has a standard spine.

In this paper, we introduce a restricted class of standard spines, which we call flow-spines. In §1 we first define a "normal pair" which is a pair of a non-singular flow ψ_t on a closed 3-manifold M and its local section Σ. And we will show that a normal pair (ψ_t, Σ) determines flow-spines $P_-(\psi_t, \Sigma)$ and $P_+(\psi_t, \Sigma)$. Moreover it will be shown that on any closed 3-manifold there exists a normal pair. In §§2–4, we will exhibit methods for deciding the orientability and the fundamental group of the

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phase manifold by a flow-spine. And in §§ 5–6, we will show that, using the data about the third singularities of a flow-spine, we can reconstruct the phase manifold. As a consequence, we will see that a closed 3-manifold is completely determined by 1-dimensional data. In § 7 we present an example of methods for constructing a flow-spine with less third singularities than given one.

§ 1. Spines induced by a non-singular flow.

Throughout this paper, M will denote a closed smooth 3-manifold. Let ψ_t be a non-singular flow on M generated by a smooth vector field. A compact 2-dimensional submanifold of M with boundary is called a *compact local section* of ψ_t, if it is included in some open 2-dimensional submanifold which is nowhere tangential to ψ_t. For a compact local section Σ, we can take a positive number δ such that a mapping h defined by $h(x, t)=\psi_t(x)$ is a homeomorphism from $\Sigma\times(-\delta, \delta)$ onto $\{\psi_t(x)\mid x\in\Sigma,\ -\delta<t<\delta\}\subset M$. We call such a δ a *collar-size* for Σ and ψ_t, or simply for Σ.

Let Σ be a compact local section of ψ_t. We define two functions $T_+=T_+(\psi_t, \Sigma)$ and $T_-=T_-(\psi_t, \Sigma)$ on M as follows:

$$T_+(x)=\inf\{t>0\mid \psi_t(x)\in\Sigma\}$$
$$T_-(x)=\sup\{t<0\mid \psi_t(x)\in\Sigma\}$$
$$(T_+(x)=+\infty \text{ if } \psi_t(x)\notin\Sigma \text{ for any } t>0, \text{ and}$$
$$T_-(x)=-\infty \text{ if } \psi_t(x)\notin\Sigma \text{ for any } t<0).$$

For an $x\in M$ with $|T_\pm(x)|<\infty$, we define $\hat{T}_\pm(x)$ by

$$\hat{T}_\pm(x)=\psi_\sigma(x) \quad (\sigma=T_\pm(x)) .$$

Let Σ be a compact local section, and Σ' be another local section such that $\mathrm{Int}\,\Sigma'\supset\Sigma$. Then, for each point (x, t) on $\partial\Sigma\times\boldsymbol{R}$ with $\psi_t(x)\in\partial\Sigma$, we can take a small piece $\gamma=\gamma(x, t)$ of $\partial\Sigma$ and a smooth function $\omega\colon\gamma\to\boldsymbol{R}$ so that $x\in\gamma$, $\omega(x)=t$ and $\psi_{\omega(y)}(y)\in\Sigma'$ for any $y\in\gamma$. We say that $\partial\Sigma$ is *ψ_t-transversal* at $(x, t)\in\partial\Sigma\times\boldsymbol{R}$, if either $\psi_t(t)\notin\partial\Sigma$ or $\{\psi_{\omega(y)}(y)\mid y\in\gamma(x, t)\}$ intersects transversally with $\partial\Sigma$ at $\psi_{\omega(x)}(x)$ within Σ'.

Now we shall introduce the concept of the normality of a pair of a non-singular flow and its compact local section.

DEFINITION 1.1. A pair (ψ_t, Σ) of a non-singular flow ψ_t on M and its compact local section Σ is said to be a *normal pair* on M, if it satisfies the following four conditions:

(i) Σ is homeomorphic to a 2-disk,

(ii) $|T_{\pm}(\psi_t, \Sigma)(x)| < \infty$ for any $x \in M$,

(iii) $\partial\Sigma$ is ψ_t-transversal at $(x, T_+(\psi_t, \Sigma)(x))$ for any $x \in \partial\Sigma$,

(iv) if $x \in \partial\Sigma$ and $x_1 = \hat{T}_+(\psi_t, \Sigma)(x) \in \partial\Sigma$, then $\hat{T}_+(\psi_t, \Sigma)(x_1) \in \operatorname{Int}\Sigma$.

For a normal pair (ψ_t, Σ) on M, we define two subsets $P_-(\psi_t, \Sigma)$ and $P_+(\psi_t, \Sigma)$ of M by

$$P_-(\psi_t, \Sigma) = \Sigma \cup \{\psi_t(x) \mid x \in \partial\Sigma,\ T_-(\psi_t, \Sigma)(x) \leqq t \leqq 0\}$$
$$P_+(\psi_t, \Sigma) = \Sigma \cup \{\psi_t(x) \mid x \in \partial\Sigma,\ 0 \leqq t \leqq T_+(\psi_t, \Sigma)(x)\} .$$

In the remainder of this section, we shall show the following two theorems.

THEOREM 1.1. *On any closed 3-manifold, there exists a normal pair.*

THEOREM 1.2. *If (ψ_t, Σ) is a normal pair on M, then each of $P_-(\psi_t, \Sigma)$ and $P_+(\psi_t, \Sigma)$ is a standard spine of M.*

We call $P_{\pm}(\psi_t, \Sigma)$ *flow-spines* of M generated by a normal pair (ψ_t, Σ). In order to specify $P_-(\psi_t, \Sigma)$ (or $P_+(\psi_t, \Sigma)$), we call it a *negative flow-spine* (or *positive flow-spine* respectively).

PROOF OF THEOREM 1.1. Because the Euler number of M is zero, there exists a smooth non-singular flow ψ_t on M whose only limit sets are a finite collection of periodic orbits (see [8]). For such a flow ψ_t, if every periodic orbits intersect with a local section Σ, then $|T_{\pm}(\psi_t, \Sigma)(x)| < \infty$ for any $x \in M$.

Now take a flow ψ_t with the above properties, and choose compact local sections $\Sigma_1, \Sigma_2, \cdots, \Sigma_n$ so that each of Σ_j's is homeomorphic to a 2-disk and each periodic orbit of ψ_t intersects with one of $\operatorname{Int}\Sigma_j$ ($j = 1, \cdots, n$). And connect Σ_j's by local sections D_k ($k=1, \cdots, n-1$) as in Figure 2, so that $\Sigma_* = (\cup_j \Sigma_j) \cup (\cup_k D_k)$ is a compact local section homeo-

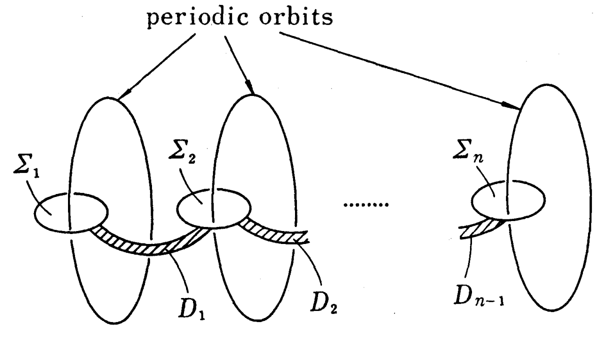


FIGURE 2

morphic to a 2-disk. We may assume that $\partial\Sigma_*$ contains no point on periodic orbits. Hence, using the same technique as in the proof of Lemma 9 of [6], we can deform Σ_* into Σ so that the pair (ψ_t, Σ) is a normal pair. This completes the proof.

PROOF OF THEOREM 1.2. First we shall show that $P_-(\psi_t, \Sigma)$ forms a closed fake surface whose third and second singularities are given by

$$\mathfrak{S}_3(P_-(\psi_t, \Sigma))=\{x\in \operatorname{Int}\Sigma \mid \hat{T}_+(x) \text{ and } \hat{T}_+^2(x) \text{ are both on } \partial\Sigma\}$$
$$\mathfrak{S}_2(P_-(\psi_t, \Sigma))=\hat{T}_-(\partial\Sigma)\cup\{\psi_t(x)\mid x\in\mathfrak{S}_3(P_-(\psi_t, \Sigma)),\ 0\leqq t\leqq T_+(x)\}\ .$$
$$(\hat{T}_+=\hat{T}_+(\psi_t, \Sigma) \text{ and } \hat{T}_+^2=\hat{T}_+\circ\hat{T}_+)$$

In fact, by the definition of normal pair, $P_-(\psi_t, \Sigma)$ is like as Figure 3 in a neighborhood of the orbit segment from a to $\hat{T}_+^2(a)$ $(a\in\mathfrak{S}_3(P_-(\psi_t, \Sigma)))$. This shows that the above defined sets are included in the third and the second singularities respectively. Moreover it is easy to see that $P_-(\psi_t, \Sigma)$ has no other singularities. Hence $P_-(\psi_t, \Sigma)$ forms a closed fake surface.

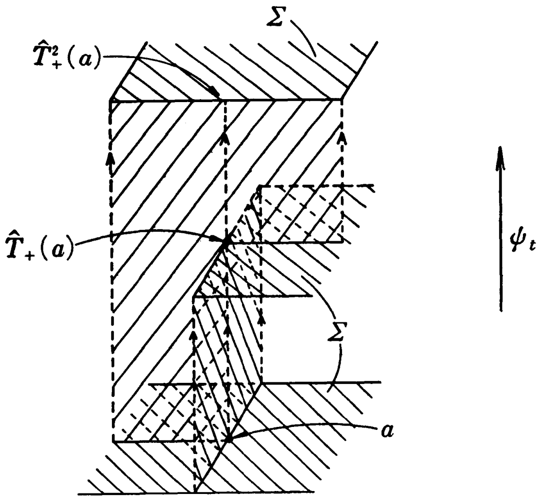


FIGURE 3

Next we shall show that the complement of a regular neighborhood of $P_-(\psi_t, \Sigma)$ is a 3-ball. Let Σ_1 and Σ_2 be compact local sections homeomorphic to a 2-disk such that $\operatorname{Int}\Sigma_1\supset\Sigma$ and $\operatorname{Int}\Sigma\supset\Sigma_2$. And define $V\subset M$ by

$$V=\{\psi_t(x)\mid x\in\Sigma_2,\ T_-(\psi_t, \Sigma_1)(x)+\delta\leqq t\leqq -\delta\}\ ,$$

where δ is a collar-size for Σ_1. If we choose Σ_1 and Σ_2 sufficiently close to Σ, then $M-V$ forms a regular neighborhood of $P_-(\psi_t, \Sigma)$. Furthermore V is homeomorphic to a subset $\tilde{V}$ of $\Sigma_2\times \boldsymbol{R}$ which is defined by

$$\tilde{V}=\{(x, t)\mid x\in\Sigma_2,\ T_-(\psi_t, \Sigma_1)(x)+\delta\leqq t\leqq -\delta\}\ .$$

Obviously $\tilde{V}$ is homeomorphic to a 3-ball, and hence also V is. This proves that $P_-(\psi_t, \Sigma)$ is a standard spine of M.

Quite analogously we can verify that also $P_+(\psi_t, \Sigma)$ is a standard spine of M. This completes the proof.

REMARK. (1) In the proof of Theorem 1.1, we used the Wilson's flow only for simplicity of the proof. Indeed, from any non-singular flow, we can construct a normal pair by an adequate choice of a compact local section and by a slight deformation of the flow.

(2) The third and the second singularities of $P_+(\psi_t, \Sigma)$ are given by

$$\mathfrak{S}_3(P_+(\psi_t, \Sigma)) = \{x \in \operatorname{Int} \Sigma \mid \hat{T}_-(x) \text{ and } \hat{T}_-^2(x) \text{ are both on } \partial\Sigma\}$$
$$\mathfrak{S}_2(P_+(\psi_t, \Sigma)) = \hat{T}_+(\partial\Sigma) \cup \{\psi_t(x) \mid x \in \mathfrak{S}_3(P_+(\psi_t, \Sigma)),\ T_-(x) \leqq t \leqq 0\}\ .$$

§ 2. Notation and definitions.

In the following three sections, we will fix a 3-manifold M and a normal pair (ψ_t, Σ) on it, and write $T_\pm$, $P_\pm$, etc. for $T_\pm(\psi_t, \Sigma)$, $P_\pm(\psi_t, \Sigma)$, etc.. In this section, we prepare some notation.

BASIC NOTATION.

(1) $\nu = \#\mathfrak{S}_3(P_-)$,

(2) $a_1, a_2, \cdots, a_\nu$ denote the elements of $\mathfrak{S}_3(P_-)$, namely $\{a_1, \cdots, a_\nu\} = \{x \in \operatorname{Int} \Sigma \mid \hat{T}_+(x)$ and $\hat{T}_+^2(x)$ are both on $\partial\Sigma\}$,

(3) $b_k = \hat{T}_+(a_k)$, $c_k = \hat{T}_+^2(a_k)$ and $d_k = \hat{T}_+^3(a_k)$ $(k = 1, \cdots, \nu)$, it is to be noticed that b_k and c_k are on $\partial\Sigma$ and d_k is in $\operatorname{Int} \Sigma$,

(4) $\Gamma_1, \Gamma_2, \cdots, \Gamma_\nu$ denote the connected components of $\partial\Sigma - \{b_1, \cdots, b_\nu\}$,

(5) $C_1, C_2, \cdots, C_{2\nu}$ denote the connected components of $\partial\Sigma - \{b_1, \cdots, b_\nu, c_1, \cdots, c_\nu\}$,

(6) μ denotes the number of connected components of $\Sigma - \hat{T}_-(\partial\Sigma)$,

(7) $D_1, D_2, \cdots, D_\mu$ denote the connected components of $\Sigma - \hat{T}_-(\partial\Sigma)$.

The assignments of numbers to a_k's, Γ_l's, C_m's and D_n's are assumed to be fixed once for all. It follows from the Euler-Poincaré formula that $\mu = \nu + 1$ if $\partial\Sigma \cup \hat{T}_-(\partial\Sigma)$ is connected.

DEFINITION 2.1. For each $k = 1, \cdots, \nu$, we define four integers $k(j)$ $(j = 1, \cdots, 4,\ 1 \leqq k(j) \leqq 2\nu)$ as follows: $m_j = k(j)$ iff the components C_{m_j} of $\partial\Sigma - \{b_1, \cdots, b_\nu, c_1, \cdots, c_\nu\}$ satisfy the following conditions (i)–(iv) (see Figure 4).

(i) C_{m_1} and C_{m_2} are components which have b_k as one of their end points.

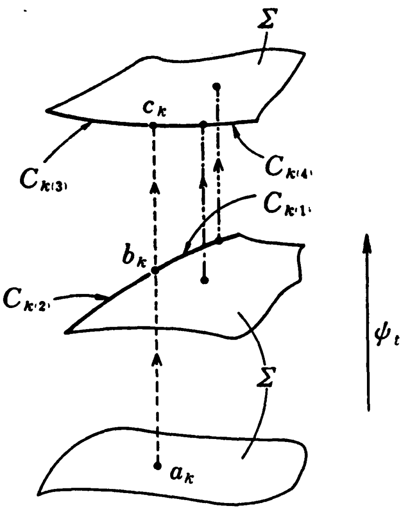


FIGURE 4

(ii) C_{m_3} and C_{m_4} are components which have c_k as one of their end points.

(iii) $T_+(x) \to T_+(b_k)$ if $x \to b_k$ within C_{m_1}, and $T_+(x) \not\to T_+(b_k)$ if $x \to b_k$ within C_{m_2}.

(iv) $T_-(x) \to T_-(c_k)$ if $x \to c_k$ within C_{m_4}, and $T_-(x) \not\to T_-(c_k)$ if $x \to c_k$ within C_{m_3}.

It is to be noticed that $k(j)$ may be equal to $k(j')$ for some $j' \neq j$.

DEFINITION 2.2. For each $k=1, \cdots, \nu$, we define three integers $k\{j\}$ ($j=1, 2, 3$, $1 \leqq k\{j\} \leqq \nu$) as follows: $l_j = k\{j\}$ iff the components Γ_{l_j} of $\partial\Sigma - \{b_1, \cdots, b_\nu\}$ satisfy that

(i) $\Gamma_{l_1} \supset C_{m_1}$,

(ii) $\Gamma_{l_2} \supset C_{m_2}$ and

(iii) $c_k \in \Gamma_{l_3}$.

DEFINITION 2.3. For each $m=1, \cdots, 2$, we define three integers $m\langle j\rangle$ ($j=1, 2, 3$, $1 \leqq m\langle j\rangle \leqq \mu$) as follows: $n_j = m\langle j\rangle$ iff the components D_{n_j} of

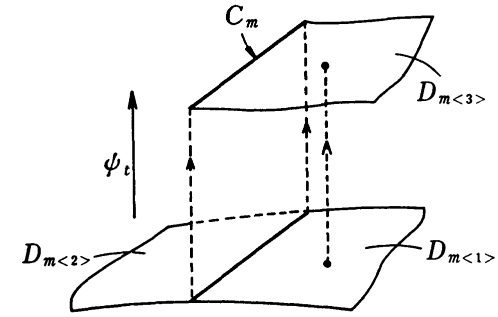


FIGURE 5

$\Sigma - \hat{T}_-(\partial\Sigma)$ satisfy the following conditions (i)-(iii) (see Figure 5).

(i) D_{n_1} and D_{n_2} are components which include $\hat{T}_-(C_m)$ in their boundary.

(ii) $T_+(x) \to T_+(x_0)$ if $x \to x_0 \in \hat{T}_-(C_m)$ within D_{n_1}.

(iii) D_{n_3} includes C_m in its boundary.

§ 3. Orientability.

In this section, we shall exhibit a method for reading off the orientability of M from a flow-spine.

Fix an orientation on Σ, and denote by $\widehat{xy}$ ($x, y \in \partial\Sigma$) the subarc of $\partial\Sigma$ going from x to y in the positive direction. For each $a_k \in \mathfrak{S}_3(P_-)$ ($k=1, \cdots, \nu$), we take four points w_k^j ($j=1, 2, 3, 4$) on $\partial\Sigma$ so that w_k^j is on $C_{k(j)}$, where $C_{k(j)}$ is the component defined in Definition 2.1. Then we have that

THEOREM 3.1. *M is orientable if and only if each $a_k \in \mathfrak{S}_3(P_-)$ satisfies either of the following conditions* (+) *or* (−).

$$(+) \qquad b_k \in \widehat{w_k^1 w_k^2} \quad \textit{and} \quad c_k \in \widehat{w_k^3 w_k^4}\,,$$

$$(-) \qquad b_k \in \widehat{w_k^2 w_k^1} \quad \textit{and} \quad c_k \in \widehat{w_k^4 w_k^3}\,.$$

The condition that a_k satisfies (+) or (−) is equivalent to the condition that $C_{k(1)}$ and $C_{k(3)}$ are on the same side of b_k and c_k respectively.

PROOF. Let $V_k \subset \Sigma$ be a neighborhood of b_k, and give to V_k the orientation derived by one of Σ. Then we can define the orientation of $\hat{T}_+(V_k)$ in two different ways. One of these is the orientation induced by $\hat{T}_+$, and the other is one obtained by restricting the orientation of Σ.

First we shall show that M is orientable if and only if the above two orientations of $\hat{T}_+(V_k)$ are coincide for any $k=1, \cdots, \nu$. Let $x \in M$ be an arbitrary point. Since (ψ_t, Σ) is a normal pair, we can find a $\tau \in \boldsymbol{R}$ and a local section $S_{x,\tau}$ such that $x \in S_{x,\tau}$, $\psi_\tau(x) \in \operatorname{Int}\Sigma$, and there is a continuous function $F\colon S_{x,\tau} \to \boldsymbol{R}$ such that $F(x)=\tau$ and $\hat{F}(y) \equiv \psi_{F(y)}(y) \in \Sigma$ for any $y \in S_{x,\tau}$. Then M is orientable if and only if the orientation on $S_{x,\tau}$ induced by $\hat{F}$ is independent of the choice of τ such that $\psi_\tau(x) \in \operatorname{Int}\Sigma$. And it can be easily seen that the orientation on $S_{x,\tau}$ is independent of τ if and only if the above mentioned two orientations on $\hat{T}_+(V_k)$ are coincide for any k. Therefore M is orientable if and only if the two orientations on $\hat{T}_+(V_k)$ are coincide for any k.

On the other hand, it follows immediately from the definition of the integers $k(j)$ that a_k satisfies (+) or (−) if and only if the above two orientations on $\hat{T}_+(V_k)$ are coincide (cf. Figure 4). This completes the proof.

This theorem shows that if M is orientable, then the points of $\mathfrak{S}_3(P_-)$ can be classified into two classes, those satisfying (+) and those satisfying (−). In the case where M is non-orientable, $\mathfrak{S}_3(P_-)$ can be classified into the following four cases:

$$\begin{array}{ll} (+) & a_k \text{ satisfies } (+)\,, \\ (-) & a_k \text{ satisfies } (-)\,, \\ (+^*) & b_k \in \widehat{w_k^1 w_k^2} \text{ and } c_k \in \widehat{w_k^4 w_k^3}\,, \\ (-^*) & b_k \in \widehat{w_k^2 w_k^1} \text{ and } c_k \in \widehat{w_k^3 w_k^4}\,. \end{array}$$

§ 4. Fundamental group.

In this section, we shall give methods for calculating the fundamental group of M by using a flow-spine.

We begin with some notation.

NOTATION.

(1) F_ν denotes a free group with the set $U=\{u_1, u_2, \cdots, u_\nu\}$ of free generators.

(2) F_μ denotes a free group with the set $V=\{v_1, v_2, \cdots, v_\mu\}$ of free generators.

(3) h_l is a U-word defined by

$$h_l = u_{l\{1\}}\, u_{l\{3\}}\, u_{l\{2\}}^{-1} \quad (l=1, \cdots, \nu)\,.$$

(4) η_m is a V-word defined by

$$\eta_m = v_{m\langle 1\rangle}\, v_{m\langle 3\rangle}\, v_{m\langle 2\rangle}^{-1} \quad (m=1, \cdots, 2\nu)\,.$$

(In (3) and (4), $l\{j\}$ and $m\langle j\rangle$ are those defined in Definitions 2.2 and 2.3.).

Then we get the following two presentations of $\pi_1(M)$, the fundamental group of M.

THEOREM 4.1. $\pi_1(M) = \langle u_1, \cdots, u_\nu; h_1, \cdots, h_\nu\rangle$.

THEOREM 4.2. $\pi_1(M) = \langle v_1, \cdots, v_\mu; \eta_1, \cdots, \eta_{2\nu}\rangle$.

PROOF OF THEOREM 4.1. Define $\tilde{\Gamma}_l$ $(l=1, \cdots, \nu)$ by

$$\tilde{\Gamma}_l = \{\psi_t(x) \mid x \in \Gamma_l,\ T_-(x) < t < 0\}\,,$$

and M_* by

$$M_* = (M - P_-) \cup \left(\bigcup_{l=1}^{\nu} \tilde{\Gamma}_l \right) = M - (\Sigma \cup \mathfrak{S}_2(P_-)) .$$

And denote by $L(M_*, x_0)$ the space of piecewise smooth loops in M_* with a base point x_0 which intersect transversally with each $\tilde{\Gamma}_l$. By p we denote the natural map from $L(M_*, x_0)$ onto the fundamental group $\pi_1(M, x_0)$.

Now let us define a map p_* from $L(M_*, x_0)$ to the free group F_ν on the free generators $U = \{u_1, \cdots, u_\nu\}$. Let $\gamma: [0, 1] \to M_*$ ($\gamma(0) = \gamma(1) = x_0$) be an element of $L(M_*, x_0)$, and let $\gamma(t_1), \gamma(t_2), \cdots, \gamma(t_r)$ $(t_1 < t_2 < \cdots < t_r)$ be the points on $\gamma \cap P_-$. Then we define $p_*(\gamma)$ by

$$p_*(\gamma) = u_{l_1}^{\varepsilon_1} u_{l_2}^{\varepsilon_2} \cdots u_{l_r}^{\varepsilon_r} \quad (\varepsilon_j = 1 \text{ or } -1) ,$$

where l_j is the number such that $\gamma(t_j) \in \tilde{\Gamma}_{l_j}$ and ε_j is defined as

$$\varepsilon_j = \begin{cases} 1 & \text{if} \quad \lim_{t \to t_j - 0} T_+(\gamma(t)) = T_+(\gamma(t_j)) \\ -1 & \text{if} \quad \lim_{t \to t_j + 0} T_+(\gamma(t)) = T_+(\gamma(t_j)) . \end{cases}$$

If γ has no intersection with P_-, then we put $p_*(\gamma) = 1$.

In order to verify the theorem, it is sufficient to show the following five conditions (a)-(e).

(a) $p_*(\gamma \circ \gamma') = P_*(\gamma) p_*(\gamma')$ ($\gamma \circ \gamma'$ denotes the composed loop),

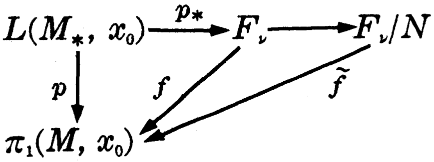
(b) p_* is surjective,

(c) if $p_*(\gamma') = p_*(\gamma)$, then γ' is homotopic to γ within M,

(d) $p(\gamma) = 1$ if $p_*(\gamma) \in N = N(h_1, \cdots, h_\nu)$ (the normal closure of $\{h_1, \cdots, h_\nu\}$),

(e) $p_*(\gamma) \in N$ if $p(\gamma) = 1$.

In fact, because of the conditions (a), (b) and (c), we can define a surjective homomorphism f from F_ν onto $\pi_1(M, x_0)$ by $f = p \circ p_*^{-1}$. By the condition (d) this f induces a homomorphism $\tilde{f}$ from F_ν/N onto $\pi_1(M, x_0)$. And the condition (e) implies the injectivity of $\tilde{f}$. Hence the above five conditions show the required presentation of $\pi_1(M)$.



Now take a compact local section Σ' such that $\operatorname{Int} \Sigma' \supset \Sigma$. Let δ be a positive number such that $T_-(\psi_t, \Sigma')(x) < -2\delta$ for any $x \in \Sigma'$. We assume that the base point x_0 is taken in $\psi_{-\delta}(\Sigma)$. Here we shall define a special

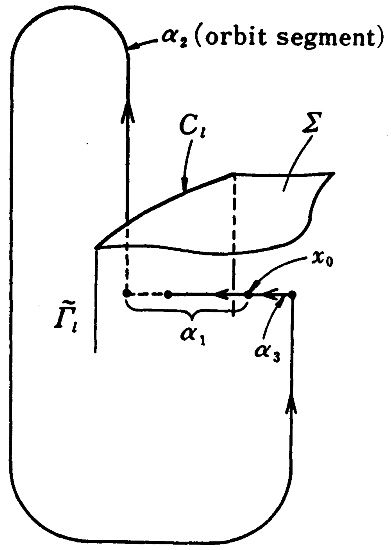


FIGURE 6

loop $\gamma_l \in L(M_*, x_0)$ for each $l=1, \cdots, \nu$. Let $\alpha_1: [0, 1]\to\Sigma'$ be an arc such that $\alpha_1(0)=\psi_\delta(x_0)\in\Sigma$, $\alpha_1\cap\partial\Sigma=\{\alpha_1(t_0)\}$ $(0<t_0<1)$ and $\alpha_1(t_0)\in\Gamma_l$. Let α_2 be the orbit segment $\alpha_2=\{\psi_t(\alpha_1(1))\,|\,0\leqq t\leqq T_+(\alpha_1(1))\}$. And let $\alpha_3: [0, 1]\to\Sigma$ be an arc such that $\alpha_3(0)=\hat{T}_+(\alpha_1(1))$ and $\alpha_3(1)=\psi_\delta(x_0)$. Then, putting $\gamma_l=\psi_{-\delta}(\alpha_1\circ\alpha_2\circ\alpha_3)$, we get a loop γ_l such that $p_*(\gamma_l)=u_l$ (see Figure 6).

Proof of (a) *and* (b). (a) is obvious by the definition of the map p_*. Now we shall show (b). Let $w=u_{l_1}^{\varepsilon_1}u_{l_2}^{\varepsilon_2}\cdots u_{l_r}^{\varepsilon_r}$ be any U-word ($\varepsilon_j=1$ or -1). Using the above defined loops γ_l, define a $\gamma\in L(M_*, x_0)$ by $\gamma=\gamma_{l_1}^{\varepsilon_1}\circ\gamma_{l_2}^{\varepsilon_2}\circ\cdots\circ\gamma_{l_r}^{\varepsilon_r}$. Then we have $p_*(\gamma)=w$. This shows that p_* is surjective.

Proof of (c). The condition (c) follows immediately from the facts that $M-P_-$ is simply connected, and that each $\tilde{\Gamma}_l$ is contractible in M.

Proof of (d). By (a), (b) and (c) we can see that $\{p_*(\gamma)\,|\,\gamma\in L(M_*, x_0), p(\gamma)=1\}$ is a normal subgroup of F_ν. Hence it is sufficient to show that for any k there is a $\gamma\in L(M_*, x_0)$ such that $p(\gamma)=1$ and $p_*(\gamma)=h_k$. Take a γ as in Figure 7. Then evidently $p(\gamma)=1$ and $p_*(\gamma)=h_k$. Therefore we get the condition (d).

Proof of (e). Let $\gamma\in L(M_*, x_0)$ be a loop with $p(\gamma)=1$. Then we can take an immersion $\iota: D^2\to M-\Sigma$ such that $\iota(\partial D^2)=\gamma$ and ι is transversal to $\mathfrak{S}_2(P_-)$. Let $\{z_1, \cdots, z_s\}$ be the inverse image $\iota^{-1}(\mathfrak{S}_2(P_-))$. For each z_j, take a loop β_j in D^2 which encircles z_j and does not encircle the other z_i's (see Figure 8). Then it is easy to see that $p_*(\iota(\beta_j))$ is conjugate to h_k if $z_j=\psi_t(a_k)$ for some t with $0<t<T_+(a_k)$ (cf. Figure 7). Moreover we can choose such β_j's that the loop $\partial D^2\circ\beta_s^{-1}\circ\cdots\circ\beta_2^{-1}\circ\beta_1^{-1}$ does not encircle any of z_j's, and hence $p_*(\iota(\partial D^2\circ\beta_s^{-1}\circ\cdots\circ\beta_2^{-1}\circ\beta_1^{-1}))=1$. This

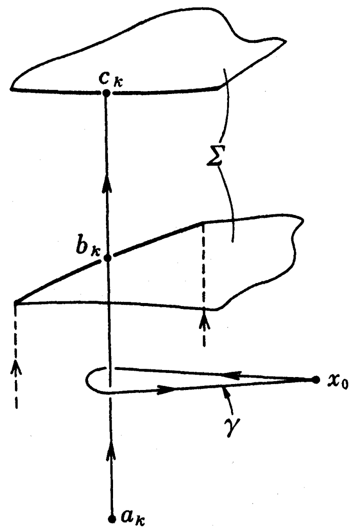


FIGURE 7

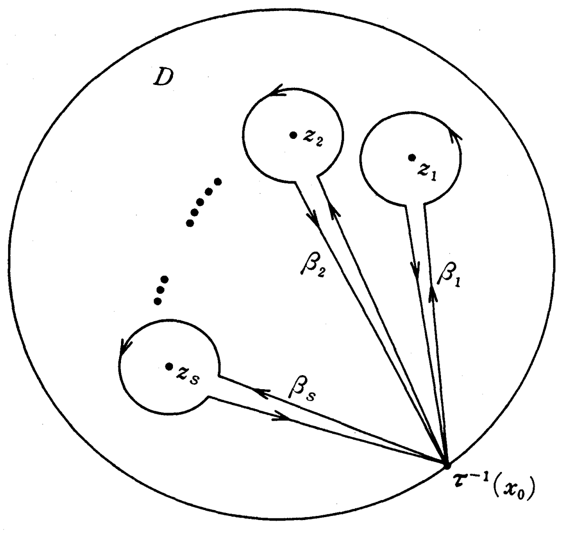


FIGURE 8

implies that $p_*(\gamma)=p_*(\iota(\beta_1\circ\beta_2\circ\cdots\circ\beta_s))$ is contained in the normal closure of $\{h_1, \cdots, h_\nu\}$. This completes the proof of (e), and so of Theorem 4.1.

PROOF OF THEOREM 4.2. Define M^* to be

$$M^*=M-\mathfrak{S}_2(P_-)\ ,$$

and $\tilde{D}_n$ $(n=1, \cdots, \mu)$ to be

$$\tilde{D}_n=D_n\cup(\cup\{\tilde{\Gamma}_l\,|\,\Gamma_l\subset\partial D_n\})\ ,$$

where D_n is the n-th component of $\Sigma-\hat{T}_-(\partial\Sigma)$. We denote by $L(M^*, x_0)$

the space of piecewise smooth loops in M^* with a base point x_0 which intersect transversally with each $\tilde{D}_n$.

We define a map p^* from $L(M^*, x_0)$ to the free group F_μ on the free generators $V=\{v_1, \cdots, v_\mu\}$ as follows. Let $\gamma: [0, 1]\to M^*$ $(\gamma(0)=\gamma(1)=x_0)$ be an element of $L(M^*, x_0)$, and let $\gamma(t_1), \gamma(t_2), \cdots, \gamma(t_r)$ $(t_1<t_2<\cdots<t_r)$ be the points on $\gamma\cap P_-$. Then we define $p^*(\gamma)$ by

$$p^*(\gamma)=v_{n_1}^{\varepsilon_1} v_{n_2}^{\varepsilon_2} \cdots v_{n_r}^{\varepsilon_r} \quad (\varepsilon_j=1 \text{ or } -1)\ ,$$

where n_j is the number such that $\gamma(t_j)\in\tilde{D}_{n_j}$ and ε_j is defined as

$$\varepsilon_j=\begin{cases} 1 & \text{if} \quad T_+(\gamma(t-\delta))<T_+(\gamma(t+\delta)) \text{ for any sufficiently small } \delta>0\ , \\ -1 & \text{if} \quad T_+(\gamma(t-\delta))>T_+(\gamma(t+\delta)) \text{ for any sufficiently small } \delta>0\ . \end{cases}$$

If γ has no intersection with P_-, then we put $p^*(\gamma)=1$.

For each $n=1, \cdots, \mu$, we can take a loop γ_n as in Figure 9 (a) which is in $p^{*-1}(v_n)$. And for each $m=1, \cdots, 2\nu$, we can take a loop β_m as in Figure 9 (b) which is in $p^{*-1}(\eta_m)$ and is contractible. Hence, in a quite similar way to the proof of Theorem 4.1, we can verify the presentation $\pi_1(M)=\langle v_1, \cdots, v_\mu; \eta_1, \cdots, \eta_{2\nu}\rangle$. This completes the proof.

Using Theorem 4.2, we can prove the following criterion of the non-triviality of the first homology.

THEOREM 4.3. *If M admits a normal pair (ψ_t, Σ) such that $\partial\Sigma\cup \hat{T}_-(\partial\Sigma)$ $(\hat{T}_-=\hat{T}_-(\psi_t, \Sigma))$ is not connected, then the first Betti-number*

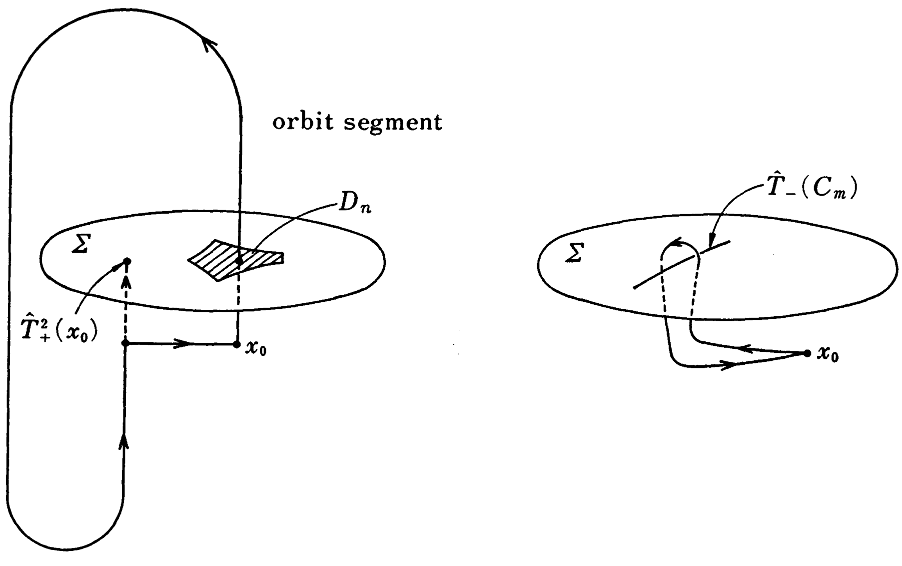


FIGURE 9

dim $H_1(M; \boldsymbol{R})$ *does not vanish.*

PROOF. If M is non-orientable, then its first Betti-number is obviously not zero. Hence we assume the orientability of M.

Define linear forms $f_l=f_l(x)$ $(l=1, \cdots, 2\nu)$ of μ-variables $x=(x_1, \cdots, x_\mu)$ by

$$f_l(x)=x_{l\langle 1\rangle}-x_{l\langle 2\rangle}+x_{l\langle 3\rangle}\ ,$$

where $l\langle j\rangle$ is the number defined in Definition 2.3. Then by Theorem 4.2 $H_1(M; \boldsymbol{R})$ is isomorphic to $\{x\in \boldsymbol{R}^\mu \mid f_l(x)=0 \text{ for any } l\}$. Moreover define $g_k(y)$ $(k=1, \cdots, \nu,\ y=(y_1, \cdots, y_{2\nu}))$ by

$$g_k(y)=y_{k(1)}-y_{k(2)}+y_{k(3)}-y_{k(4)}\ ,$$

where $k(j)$ is one defined in Definition 2.1. Then from the definitions of $l\langle j\rangle$ and $k(j)$ it follows that

$$g_k(f_1(x), f_2(x), \cdots, f_{2\nu}(x))\equiv 0$$

for any $k=1, \cdots, \nu$. And it is easy to see that the only linear relation between $g_1, \cdots, g_{2\nu}$ is given by

$$\sum_{k=1}^{\nu} i_k g_k(y)\equiv 0\ ,$$

where i_k is defined by

$$i_k=\begin{cases} 1 & \text{if } a_k \text{ satisfies the condition } (+) \text{ in Theorem 3.1}\ , \\ -1 & \text{if } a_k \text{ satisfies the condition } (-) \text{ in Theorem 3.1}\ . \end{cases}$$

Hence, among $f_1, \cdots, f_{2\nu}$, there are at most $\nu+1$ independent forms. This shows that the first Betti-number of M is not smaller than $\mu-(\nu+1)$.

On the other hand, applying the Euler-Poincaré formula to the planer graph $\partial\Sigma\cup\hat{T}_-(\partial\Sigma)$, we have that $\mu>\nu+1$ if $\partial\Sigma\cup\hat{T}_-(\partial\Sigma)$ is not connected. Therefore, if $\partial\Sigma\cup\hat{T}_-(\partial\Sigma)$ is not connected,then we get

$$\dim H_1(M; \boldsymbol{R})\geqq\mu-(\nu+1)>0\ .$$

This completes the proof.

§ 5. Examples.

In the next section, we will show that the phase manifold M can be reconstructed by the graphs $\hat{T}_-(\partial\Sigma)$ and $\hat{T}_+(\partial\Sigma)$. In this section, we shall explain by examples how we can draw the graphs $\hat{T}_\pm(\partial\Sigma)$ from the informations about the third singuralities.

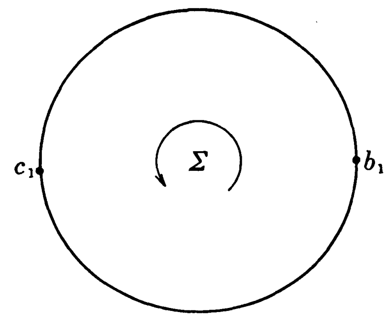


FIGURE 10

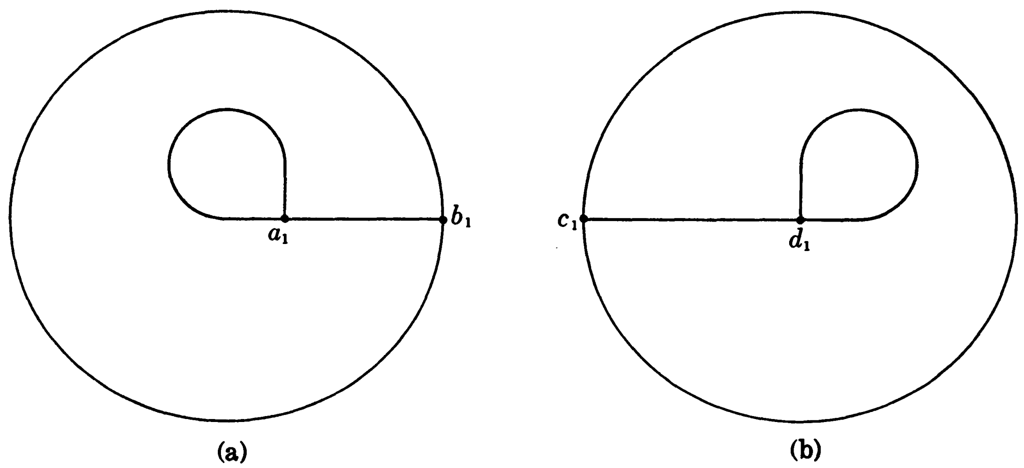
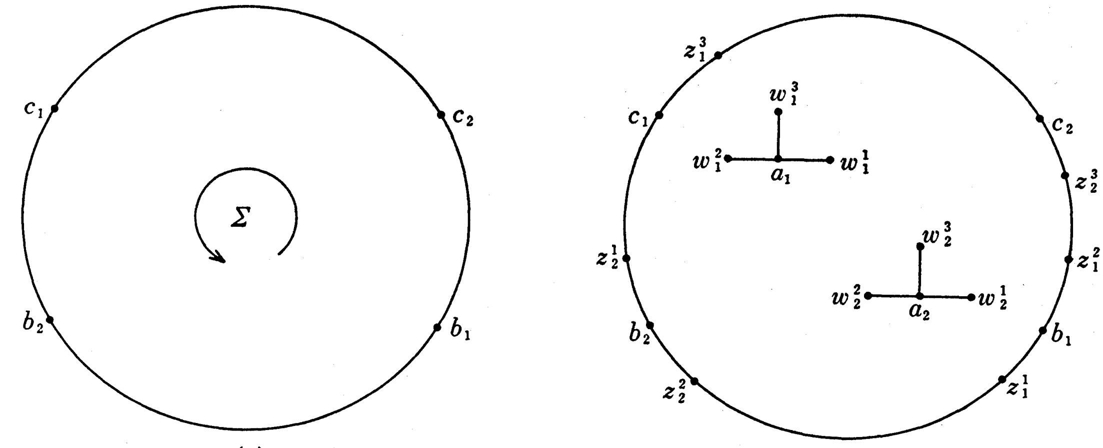


FIGURE 11

EXAMPLE I (The case of $\sharp\mathfrak{S}_3(P_-(\psi_t, \Sigma))=1$). First we shall consider the case where $\mathfrak{S}_3(P_-)=\{a_1\}$ consists of only one point. Let Σ be oriented, and $b_1=\hat{T}_+(a_1)$ and $c_1=\hat{T}_+^2(a_1)$ be arranged on $\partial\Sigma$ as in Figure 10. Suppose that a_1 satisfies the condition (+) in Theorem 3.1. Then $\widehat{b_1c_1}=C_{1(2)}=C_{1(3)}$ and $\widehat{c_1b_1}=C_{1(1)}=C_{1(4)}$ (see Definition 2.1 for the definition of $C_{1(j)}$). It follows from the definition of $C_{1(j)}$ that $\widehat{c_1b_1}$ is mapped by $\hat{T}_-$ to an arc joining b_1 to a_1, and $\widehat{b_1c_1}$ is mapped to one joining a_1 to a_1. Hence $\hat{T}_-(\partial\Sigma)$ is like as in Figure 11(a). Similarly $\hat{T}_+(\partial\Sigma)$ is like as in Figure 11(b).

Theorem 4.1 shows that if M admits a normal pair of this example, then $\pi_1(M)$ is trivial. Indeed, it can be shown that on the 3-sphere there really exists a normal pair of this example, and in this case the flow-spine P_- (or P_+) is so called the "abalone" (see the next section, and also [4]).

EXAMPLE II (Non-realizable case). Consider the case where $\mathfrak{S}_3(P_-)=\{a_1, a_2\}$ consists of two points, b_k and c_k ($k=1, 2$) are arranged as in Figure 12(a) and both a_1 and a_2 satisfy the condition (+). We shall show that there is no normal pair admitting such a case.



(a) (b)

FIGURE 12

Take points $z_k^j \in \partial\Sigma$ ($k=1, 2$, $j=1, 2, 3$) as in Figure 12(b). Then $\widehat{z_k^1 z_k^2}$ and $\widehat{z_k^3 c_k}$ must be mapped by $\hat{T}_-$ to the set drawn in Figure 12(b) ($w_k^j=\hat{T}_-(z_k^j)$). Now put $l_1=\widehat{z_1^2 z_2^3}$, $l_2=\widehat{c_2 z_1^3}$, $l_3=\widehat{c_1 z_2^1}$ and $l_4=\widehat{z_2^2 z_1^1}$. Then $\hat{T}_-(l_i)$ and $\hat{T}_-(l_j)$ ($i \neq j$) cannot intersect with each other. And $\hat{T}_-(l_1)$ joins w_1^2 to w_2^3, $\hat{T}_-(l_2)$ does b_2 to w_1^3, $\hat{T}_-(l_3)$ does b_1 to w_2^1, and $\hat{T}_-(l_4)$ dose w_2^2 to w_1^1. However we cannot draw such a graph. Hence this situation of the third singularities is not realized by any normal pair.

EXAMPLE III (Disconnected case). Next we shall give an example for which $\partial\Sigma \cup \hat{T}_-(\partial\Sigma)$ is not connected. Consider the case where the third singularities satisfies that

(i) $\mathfrak{S}_3(P_-)=\{a_1, a_2\}$ consists of two points,
(ii) b_k and c_k are arranged as in Figure 13,
(iii) a_1 satisfies $(+)$, and a_2 does $(-)$.

In this case, we can see that $\hat{T}_-(\widehat{c_2 c_1})$ is disjoint from $\partial\Sigma$.

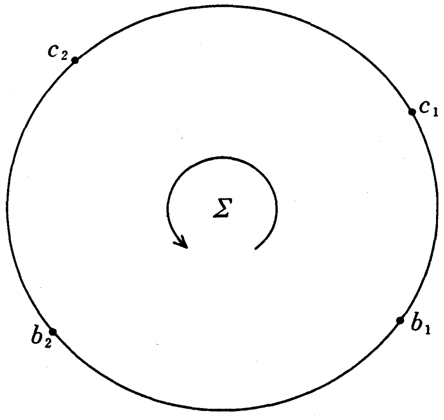


FIGURE 13

EXAMPLE IV (Non-orientable case). Consider the following case:

(i) $\mathfrak{S}_3(P_-)=\{a_1, a_2, a_3\}$ consists of three points,

(ii) b_k and c_k are arranged as in Figure 14,

(iii) a_1 satisfies $(+)$, a_2 dose $(-)$ and a_3 does $(+^*)$.

In this case, $\hat{T}_-(\partial\Sigma)$ and $\hat{T}_+(\partial\Sigma)$ are like as in Figure 15. In this case, M is non-orientable by Theorem 3.1.

The above examples show that we can draw the graphs corresponding to $\hat{T}_+(\partial\Sigma)$ and $\hat{T}_-(\partial\Sigma)$ by the following data about the third singularities:

(i) How b_k and c_k are arranged on $\partial\Sigma$?

(ii) Which of the condition $(+)$ or $(-)$ or $(+^*)$ or $(-^*)$ a_k satisfies?

Of course, some of these are impossible as Example II. We can easily check that if $\partial\Sigma\cup\hat{T}_-(\partial\Sigma)$ is connected, then the graph $\hat{T}_-(\partial\Sigma)$ is unique up to isotopy. We call the above data (i) and (ii) a *singularity-data* of

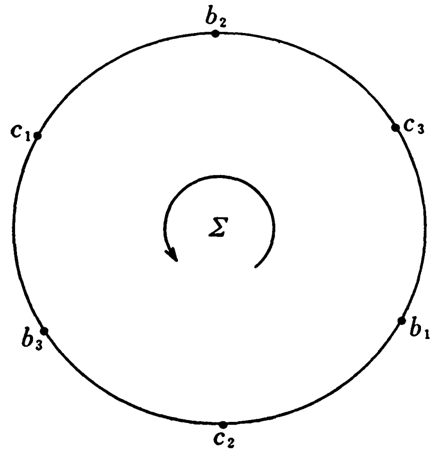


FIGURE 14

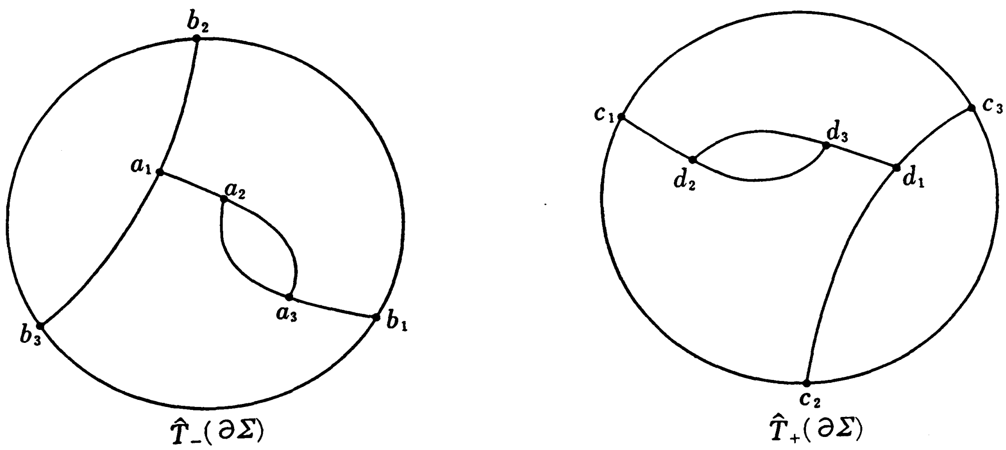


FIGURE 15

a flow-spine. And a singularity-data is said to be *realizable* if it admits graphs corresponding to $\hat{T}_{\pm}(\partial\Sigma)$.

However we have not yet proved whether there exists a normal pair for any realizable singularity-data. In the next section, we will see that a realizable singularity-data determines a 3-manifold. This strongly implies that any realizable singurality-data corresponds to a normal pair.

§ 6. Reconstruction of M.

Let (ψ_t, Σ) be a normal pair on M. Also in this section, we use the same notation as in § 2.

Let $B=B^3$ be the unit ball in $\boldsymbol{R}^3$, that is,

$$B=\{(x, y, z) \in \boldsymbol{R}^3 \mid x^2+y^2+z^2 \leqq 1\} .$$

And let ι be an embedding of Σ into ∂B such that $\iota(\partial\Sigma)=\partial B \cap \{z=0\}$, and ρ be a homeomorphism of ∂B defined by $\rho(x, y, z)=(x, y, -z)$. Then we define a spherical graph $G(\psi_t, \Sigma)$ by

$$G(\psi_t, \Sigma)=\iota(\partial\Sigma) \cup \iota(\hat{T}_-(\partial\Sigma)) \cup \rho(\iota(\hat{T}_+(\partial\Sigma))) .$$

This is a three-regular graph, namely each vertex is of order three. The vertexes, edges, and faces of $G(\psi_t, \Sigma)$ are given as follows:

(i) vertexes consist of $\iota(a_k)$, $\iota(b_k)$, $\iota(c_k)$ and $\rho(\iota(d_k))$ $(k=1, \cdots, \nu)$,

(ii) edges consist of $\iota(\hat{T}_-(C_l))$, $\iota(C_l)$ and $\rho(\iota(\hat{T}_+(C_l)))$ $(l=1, \cdots, 2\nu)$,

(iii) faces consist of $\iota(D_m)$ and $\rho(\iota(\hat{T}_+(D_m)))$ $(m=1, \cdots, \mu)$,

where the definitions of a_k, b_k, c_k, d_k, C_l and D_m are the same as in § 2.

Using this graph $G(\psi_t, \Sigma)$, we define an equivalence relation "$\sim$" on ∂B as follows:

(i) for vertexes of $G(\psi_t, \Sigma)$, $\iota(a_k) \sim \iota(b_k) \sim \iota(c_k) \sim \rho(\iota(d_k))$ for each $k=1, \cdots, \nu$,

(ii) if $x \in C_l$ for some $l=1, \cdots, 2\nu$, then we define $\iota(x) \sim \iota(\hat{T}_-(x)) \sim \rho(\iota(\hat{T}_+(x)))$,

(iii) if $x \in D_m$ for some $m=1, \cdots, \mu$, then we define $\iota(x) \sim \rho(\iota(\hat{T}_+(x)))$.

Then $B/\sim$ is a 3-manifold (cf. § 60 of [7]). Moreover $\partial B/\sim$ forms a standard spine of $B/\sim$. In what follows, we will show that $B/\sim$ is homeomorphic to M, and $\partial B/\sim$ is to $P_-(\psi_t, \Sigma)$.

Let $B_\delta=\{p=(x, y, z) \in \boldsymbol{R}^3 \mid \|p\|<\delta\}$ $(\|p\|^2=x^2+y^2+z^2,\ \delta<1)$. A collapsing map $c: (B/\sim)-B_\delta \to \partial B/\sim$ is given by $c(p)=p/\|p\|$ (for a "collapsing map", see [5]). We define an equivalence relation "$\underset{c}{\sim}$" on ∂B_δ by the following way:

$$p_1 \underset{c}{\sim} p_2 \quad \text{if and only if} \quad c(p_1)=c(p_2) \quad (p_1, p_2 \in \partial B_\delta) .$$

Then obviously $B/\sim$ is homeomorphic to $B_\delta/\underset{c}{\sim}$, and $\partial B/\sim$ is to $\partial B_\delta/\underset{c}{\sim}$.

The converse of the above fact is shown in [5]. Here we shall summarize this. Let P be a standard spine of M, and N be a regular neighborhood of P in M. Then we can define a collapsing map $c: N \to P$. And, using this c, we can define an equivalence relation "$\underset{c}{\sim}$" on ∂N in the same way as above. It is shown in [5] that M is homeomorpic to $(M-N)/\underset{c}{\sim}$, and P is to $\partial N/\underset{c}{\sim}$.

Using this theory of [5], we can show that

Theorem 6.1. *M is homeomorphic to $B/\sim$, and each of $P_-(\psi_t, \Sigma)$ and $P_+(\psi_t, \Sigma)$ is homeomorphic to $\partial B/\sim$.*

Proof. As in the proof of Theorem 1.2, we take compact local sections Σ_1 and Σ_2 such that each of them is homeomorphic to a 2-disk, $\operatorname{Int}\Sigma_1 \supset \Sigma$, and $\operatorname{Int}\Sigma \supset \Sigma_2$. And define V to be

$$V = \{\psi_t(x) \mid x \in \Sigma_2,\ T_-(\psi_t, \Sigma_1)(x) + \delta < t < -\delta\}\ ,$$

where δ is a collar-size for Σ_1. If we take Σ_j sufficiently close to Σ, then $N = M - V$ is a regular neighborhood of $P_-(\psi_t, \Sigma)$. Moreover, near the point $b_k \in \hat{T}_+(\mathfrak{S}_3(P_-))$, we can find three points $b_k^1 \in \partial\Sigma_2$, $b_k^2 \in \partial\Sigma_1$ and $b_k^3 \in \partial\Sigma_1$ such that $b_k^1 \in \partial\Sigma_2 \cap \hat{T}_-(\psi_t, \Sigma)(\partial\Sigma)$, $b_k^2 \in \partial\Sigma_1 \cap \hat{T}_-(\psi_t, \Sigma_1)(\partial\Sigma_2)$ and $b_k^3 \in \partial\Sigma_1 \cap \hat{T}_-(\psi_t, \Sigma_1)(\partial\Sigma_1)$ (see Figure 16). It is easy to see that we can take a collapsing map $c: N \to P_-(\psi_t, \Sigma)$ such that

(i) $(c|_{\partial N})^{-1}(\mathfrak{S}_3(P_-(\psi_t, \Sigma)))$ consists of $\psi_{-\delta}(a_k)$ and $\psi_\sigma(b_k^j)$ $(\sigma = T_-(\psi_t, \Sigma_1)(b_k^j) + \delta,\ k = 1, \cdots, \nu,\ j = 1, 2, 3)$, and

(ii) $(c|_{\partial N})^{-1}(\mathfrak{S}_2(P_-(\psi_t, \Sigma)))$ consists of the following four sets:

$$\psi_{-\delta}(\Sigma_2 \cap \hat{T}_-(\psi_t, \Sigma)(\partial\Sigma))$$
$$\partial N \cap \{\psi_t(x) \mid x \in \partial\Sigma_1 \cup \partial\Sigma_2,\ t = T_-(\psi_t, \Sigma_1)(x) + \delta\}$$
$$\{\psi_t(b_k^1) \mid T_-(\psi_t, \Sigma_1)(b_k^1) + \delta \leqq t \leqq -\delta,\ k = 1, \cdots, \nu\}$$
$$\{\psi_t(b_k^j) \mid T_-(\psi_t, \Sigma_1)(b_k^j) + \delta \leqq t \leqq \delta,\ k = 1, \cdots, \nu,\ j = 2, 3\}\ .$$

Let "$\underset{c}{\sim}$" be the equivalence relation on ∂N defined by such a collapsing

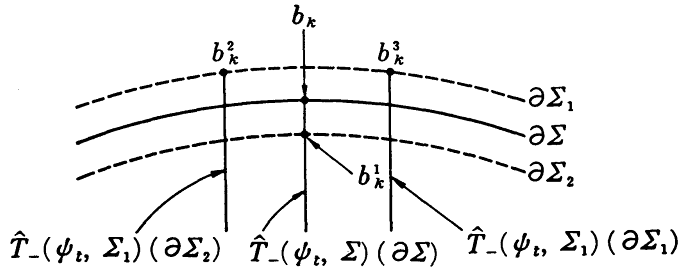


Figure 16

map c. Then, identifying ∂N with ∂B, and $\psi_{-\delta}(a_k)$, $\psi_\sigma(b_k^1)$, $\psi_\sigma(b_k^2)$ and $\psi_\sigma(b_k^3)$ with $\iota(a_k)$, $\iota(b_k)$, $\iota(c_k)$ and $\rho(\iota(d_k))$ respectively, we can see that the two equivalence relations "$\sim$" and "$\underset{c}{\sim}$" determine the same manifold. Hence, by the results of [5], $B/\sim$ is homeomorphic to M, and $\partial B/\sim$ is to $P_-(\psi_t, \Sigma)$.

Considering the time-reversed flow $\bar{\psi}_t=\psi_{-t}$, we can see that also $P_+(\psi_t, \Sigma)=P_-(\bar{\psi}_t, \Sigma)$ is homeomorphic to $\partial B/\sim$. This completes the proof.

By this theorem, the flow-spine of Example I in § 5 is obtained by the identification of a spherical graph indicated in Figure 17 (vertexes, edges and faces with the same names are identified in the indicated orientation). As is stated in § 5, this is the "abalone". On the other hand, for example, the "Bing's house with two rooms" (See p. 171 of [1].) cannot be a flow-spine, because its spherical graph is not constructed by any singularity-data.

It is to be noticed that if a realizable singularity-data is given, then we can define an equivalence relation on ∂B in the same way as above. Therefore we can say that a realizable singularity-data determines a 3-manifold.

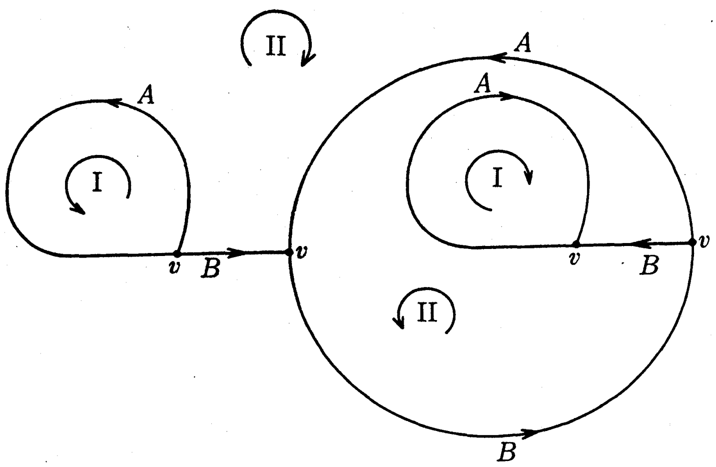


FIGURE 17

§ 7. Reducing methods.

Generally speaking, the fewer the third singularities of a standard spine are, the more we can know about the manifold (see [3], [4]). Hence it is important to find methods for decreasing the number of the third singularities. In the case of a flow-spine, we can find some new methods. In this section, we exhibit only one example of those methods.

Consider a singularity-data shown in Figure 18 (a), where a_1 satisfies

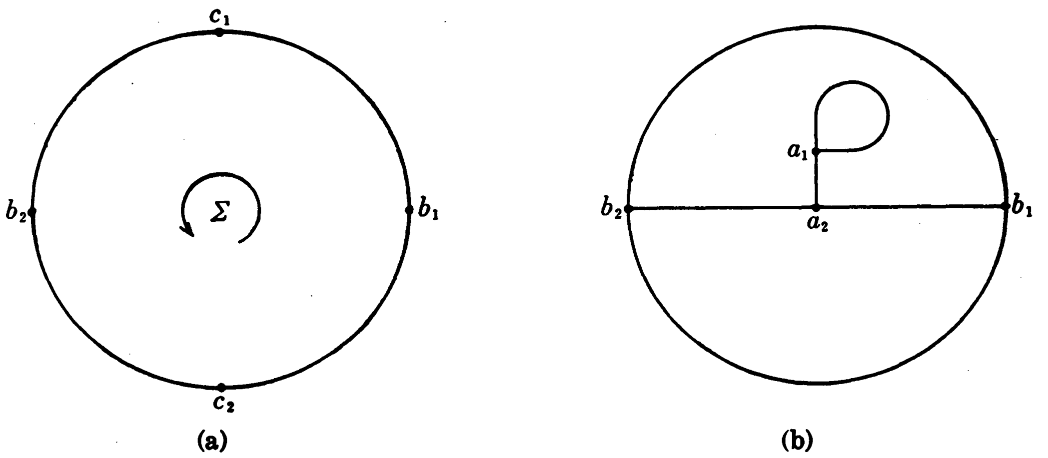


FIGURE 18

the condition (+) and a_2 does (−). Suppose that there is a normal pair (ψ_t, Σ) on M which has this singularity-data. Then we can see that $\hat{T}_-(\partial\Sigma)$ is like as in Figure 18 (b). In this case, $\pi_1(M)$ is trivial by Theorem 3.1. We shall show that, taking a new compact local section Σ', we can obtain a normal pair (ψ_t, Σ') such that $\#\mathfrak{S}_3(P_-(\psi_t, \Sigma'))=1$.

First take a compact 2-disk $Y\subset \operatorname{Int}\Sigma$ as in Figure 19 (a). And, setting $\gamma=Y\cap(\hat{T}_-(\psi_t, \Sigma)(\partial\Sigma))$, we choose a continuous function τ: $Y\rightarrow \boldsymbol{R}$ such that $\tau(x)=T_+(\psi_t, \Sigma)(x)$ for $x\in\gamma$ and $0<\tau(x)<T_+(\psi_t, \Sigma)(x)$ for $x\in Y-\gamma$. Then, for a compact local section $\Sigma'=\Sigma\cup\hat{\tau}(Y)$ $(\hat{\tau}(x)=\psi_{\tau(x)}(x))$, we have that (ψ_t, Σ') is a normal pair and $P_-(\psi_t, \Sigma')$ is like as in Figure 19 (b). Therefore $\#\mathfrak{S}_3(P_-(\psi_t, \Sigma'))=1$. Thus we can conclude that the phase manifold M is the 3-sphere, because it has the "abalone" as its spine.

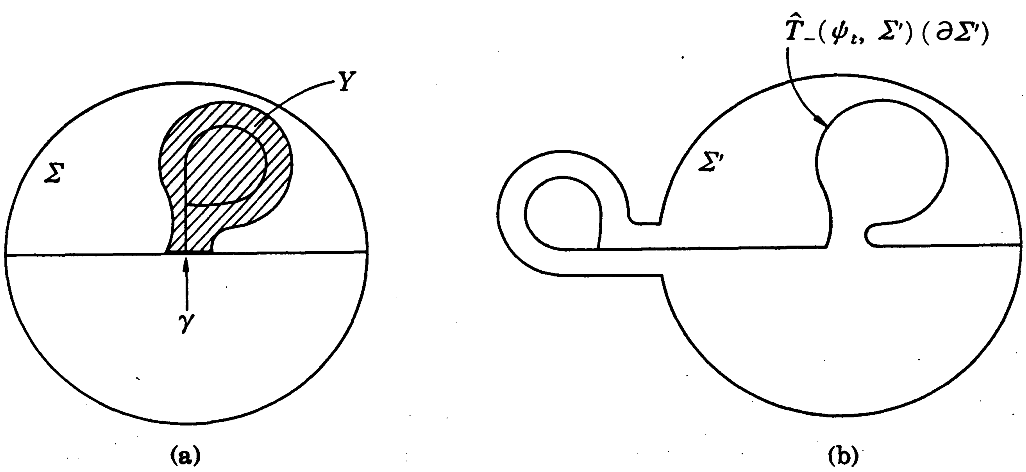


FIGURE 19

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