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A New Characterization of Dragon and Dynamical System

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Introduction

The fractal sets called a twindragon and a dragon are encountered in a complex binary representation [7] and a paper folding curve [5], respectively. We have constructed in a previous paper [1] dynamical systems on the twindragon (Figure 1) and the tetradragon (Figure 2) tiled by four dragons which are obtained as realized domains for a two state Bernoulli shift and a some subshift with a finite coding from a Markov subshift [8], respectively.

We propose in this paper a new construction of a dragon different from the paper folding process and consider a dynamical system on a domain, tiled by four dragons, which are not the tetradragon. We call this domain a cross dragon. Moreover surprisingly we can show in Section 4 that this cross dragon system is actually a dual system [1] of a very simple group endomorphism.

Indeed the cross dragon system is obtained as a realization of a following Markov subshift. Let $M=(M_{j,k})$, $1\leqq j, k\leqq 4$, be a matrix such that

$$M=\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

We consider M as a structure matrix for a state space $\Gamma=\{0, i, -1+i, -1\}$ by a correspondence $\tau: \{1, 2, 3, 4\}\to\Gamma$ such that $\tau[1]=0$, $\tau[2]=i$, $\tau[3]=-1+i$ and $\tau[4]=-1$, that is, let V be a set of infinite sequences generated by the structure matrix M,

$$V=\{(\gamma_1, \gamma_2, \cdots); M_{\gamma_j,\gamma_{j+1}}=1,\ \gamma_j\in\Gamma \text{ for all } j\in N\},$$

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and σ a shift on V. Then the system (V, σ) is a Markov subshift. Define a realization map $\Phi: V \to Y \subset C$ such that

$$\Phi: (\gamma_1, \gamma_2, \cdots, \gamma_n, \cdots) \longrightarrow \sum_{k=1}^{\infty} \gamma_k (1+i)^{-k}$$

for each $(\gamma_1, \gamma_2, \cdots) \in V$, and let Y_γ be the set $\{z \in Y = \{\Phi(\gamma_1, \gamma_2, \cdots)\}; \gamma_1 = \gamma\}$ for $\gamma \in \Gamma$. Then we can see in Section 2 that each set Y_γ is the dragon whose construction is different from a paper folding process and the set Y is tiled by four dragons $\{Y_\gamma\}$, in spite of that Y is not the tetradragon. This is why we call Y a cross dragon (Figure 3). Also we can see in Section 3 that the cross dragon system (Y, T) can be defined as a realization of (V, σ) such that

$$Tz = (1+i)z - [(1+i)z]_C \quad \text{for} \quad z \in Y ,$$

where $[w]_C = \gamma$ if $w \in \gamma + (Y_{\gamma[1]} \cup Y_{\gamma[2]})$ for $M_{\gamma,\gamma[1]} = M_{\gamma,\gamma[2]} = 1$.

In Section 4 we will see in Theorem (4.1) that this cross dragon system (Y, T) is actually a dual system [1] of a group endomorphism T_L on the torus T^2 such that

$$T_L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \pmod 1 .$$

We remark that by Theorem (3.3) the cross dragon system (Y, T) is isomorphic to a simple system on the torus such that

$$T^{\dagger} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \pmod 1 .$$

§ 1. Properties of twindragon and dragon.

We summarize the properties of a twindragon and a drangon obtained in the previous paper [1]. Recall notations by Dekking [3] [4]. Let S be a finite set of symbols, S^* be the free semigroup generated from S by the equivalence relation $\sim$, which is defined as $W \sim V$ iff W and V determine the same word after cancellation, that is so-called reduced word. And let $\theta: S^* \to S^*$ be a semigroup endmorphism. Let $f: S^* \to C$ be a homeomorphism which satisfies

$$f(VW) = f(V) + f(W) , \qquad f(V^{-1}) = -f(V)$$

for all words $V, W \in S^*$. Define a map $K: S^* \to \mathscr{K}(C)$, the nonempty compact subsets of C, which satisfies

$$K[VW]=K[V]\cup(K[W]+f(V))$$

for all reduced words $V, W \in S^*$, by

$$K[s]=\{tf(s);\ 0\leqq t\leqq 1\} \quad \text{for} \quad s\in S\,.$$

This makes $K[s_1\cdots s_m]$ the polygonal line with vertices at 0, $f(s_1)$, $f(s_1)+f(s_2)$, $\cdots$, $f(s_1)+\cdots+f(s_m)$.

Let $S=\{a, b, c, d\}$ and the endomorphism θ_t be

$$\theta_t\colon a \longrightarrow ab,\ b\longrightarrow cb,\ c\longrightarrow cd,\ d\longrightarrow ad\ ,$$

and the homomorphism f be

$$f(a)=1=-f(c)\ , \qquad f(b)=-i=-f(d)\ .$$

Define the n-step twindragon D_n and n-step dragon H_n (or paperfolding dragon [5]) [1] [2] [3] [4] by

$$D_n=(1-i)^{-n}K[\theta_t^n(abcd)] \tag{1.1}$$

and

$$H_n=(1-i)^{-n}K[\theta_t^n(ab)]\ . \tag{1.2}$$

Notice that the n-step twindragon is tiled with two n-step dragon (Figure 1(b)), that is,

$$D_n=H_n\cup(-H_n+1-i)\ . \tag{1.3}$$

It is proved in [3] [4] that D_n and H_n converge to limit sets D_t and H_d respectively as $n\to\infty$ in the Hausdorff metric $d(\cdot,\cdot)$ where

$$d(A, B)=\sup\{\sup_{x\in A}\inf_{y\in B}|x-y|,\ \sup_{y\in B}\inf_{x\in A}|x-y|\}\ .$$

The sets D_t and H_d are called the twindragon and the dragon, respectively.

Now let sets X_B, $X_{B,0}$ and $X_{B,-i}$ be

$$X_B=\left\{\sum_{k=1}^{\infty} a_k(1-i)^{-k}\colon a_k\in\{0,\ -i\} \text{ for all } k\in \boldsymbol{N}\right\}\ ,$$

$$X_{B,0}=\left\{\sum_{k=1}^{\infty} a_k(1-i)^{-k}\colon a_1=0,\ a_k\in\{0,\ -i\} \text{ for all } k\geqq 2\right\}\ ,$$

$$X_{B,-i}=\left\{\sum_{k=1}^{\infty} a_k(1-i)^{-k}\colon a_1=-i,\ a_k\in\{0,\ -i\} \text{ for all } k\geqq 2\right\}\ .$$

Then followings were proved in [1]; X_B is similar to the twindragon D_t, that is,

$$X_B=(1-i)^{-1}D_t\ . \tag{1.4}$$

X_B is tiled by $X_{B,0}$ and $X_{B,-i}$ which are congruent each other and similar to X_B (Figure 1(a)), that is,

$$X_B=X_{B,0}\cup X_{B,-i} \quad\text{and}\quad \lambda(X_{B,0}\cap X_{B,-i})=0\ , \tag{1.5}$$

where λ is the Lebesgue measure on the plane. This fact indicates that

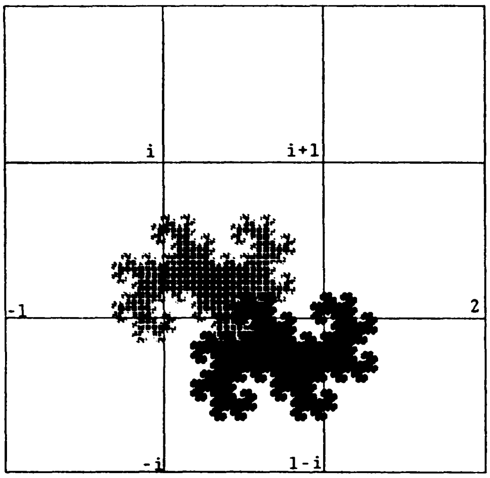

FIGURE 1(a). Twindragon X_B. X_B is similar to D_t, the limit set of twindragon curve (1.1), $X_B=(1-i)^{-1}D_t$. X_B is tiled by twindragons which are a meshed twin dragon $X_{B,0}$ and a dark twindragon $X_{B,-i}$, congruent to each other and similar to X_B, namely $X_B=X_{B,0}\cup X_{B,-i}$.

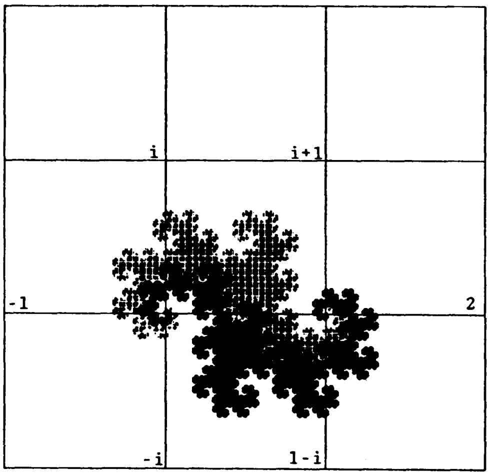

FIGURE 1(b). Twindragon X_B. X_B is also tiled by two dragons which are a meshed dragon $(1-i)^{-1}H_d$ and a dark dragon $-(1-i)^{-1}H_d+1$, where H_d is the limit set of dragon curve (1.2), namely $X_B=(1-i)^{-1}H_d\cup(-(1-i)^{-1}H_d+1)$.

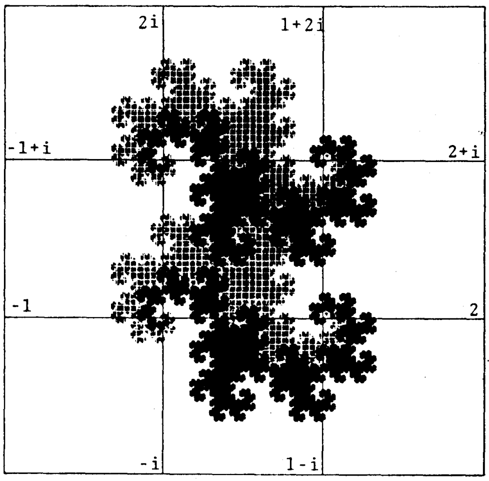

FIGURE 1(c). The plane is tiled by twindragons $\{X_B+m+in;\ m+in\in \boldsymbol{Z}(i)\}$. This figure indicates $X_B\cup(X_B+i)$, where each twindragon is tiled by two dragons. Notice that the cross dragon Y in Section 2 is included, namely $Y_{-1}\cup Y_0=(1-i)^{-1}H_d$ (meshed dragon with end points 0 and 1) and $Y_i\cup Y_{-1+i}=-(1-i)^{-1}H_d+1+i$ (dark dragon with end points $1+i$ and i) (cf. Figure 3).

twindragon is a selfsimilar fractal set of order 2. Finally the whole plane is tiled with twindragons (cf. Figure 1(a)(c)), that is,

$$\bigcup_{m+in\in Z(i)} X_{B(m+in)}=\boldsymbol{C}\ ,$$

(1.6) and

$$\lambda\Big(\bigcup_{m+in} \partial X_{B(m+in)}\Big)=0\ ,$$

where $X_{B(m+in)}=X_B+m+in$ and ∂A is a boundary of a set A.

Next recall $W^{(n)}$, which is a set of the revolving sequences $(\delta_1,\cdots,\delta_n)$ [1] [5]. We call a sequence $(\delta_1,\cdots,\delta_n)$, $\delta_j\in\{0,1,i,-1,-i\}$ for $1\leqq j\leqq n$, a revolving if nonzero digits repeat periodically following pattern from left to right,

$$\cdots\longrightarrow 1\longrightarrow -i\longrightarrow -1\longrightarrow i\longrightarrow 1\longrightarrow -i\longrightarrow\cdots.$$

Then $W^{(n)}$ is decomposed as following;

$$W^{(n)}=\bigcup_{\varepsilon\in\{0,1,2,3\}} W_\varepsilon^{(n)}\ ,$$

and

$$W_\varepsilon^{(n)}=W_{(\varepsilon,0)}^{(n}\cup W_{(\varepsilon,(-i)^\varepsilon)}^{(n)}\ ,$$

where $W_\varepsilon^{(n)}$ means a set of the revolving sequences whose first nonzero digit is $(-i)^{\varepsilon}$ and $W_{(\varepsilon,\delta)}^{(n)}$ a subset of $W_{\varepsilon}^{(n)}$ whose first digit is δ (refer to [1] for more precise definitions). Put

$$W_{\varepsilon}^{*(n)}=\overline{W_{\varepsilon}^{(n)}} \quad \text{and} \quad W_{(\varepsilon,\bar{\delta})}^{*(n)}=\overline{W_{(\varepsilon,\delta)}^{(n)}} ,$$

where $\overline{}$ means to take a complex conjugate for each digit of $(\delta_1, \cdots, \delta_n)$.

Let sets $X_{(\varepsilon,\delta)}^{(n)}$ and $X_{(\varepsilon,\delta)}^{*(n)}$ be

$$X_{(\varepsilon,\delta)}^{(n)}=\left\{\sum_{k=1}^{n} \delta_k(1+i)^{-k} : (\delta_1, \cdots, \delta_n) \in W_{(\varepsilon,\delta)}^{(n)}\right\} ,$$

and

$$X_{(\varepsilon,\delta)}^{*(n)}=\left\{\sum_{k=1}^{n} \delta_k^*(1-i)^{-k} : (\delta_1^*, \cdots, \delta_n^*) \in W_{(\varepsilon,\delta)}^{*(n)}\right\} .$$

$X_{\varepsilon}^{(n)}$, $X^{(n)}$, $X_{\varepsilon}^{*(n)}$, and $X^{*(n)}$ are defined in a similar way. Then followings were proved in [1]; the sets of points $\{X_{(\varepsilon,\delta)}^{*(n)}\}$ are congruent to each other and similar to a set of folding points of $(n-3)$-step dragon H_{n-3}, to express more precisely, for $n \geqq 3$ and $\varepsilon \in \{0, 1, 2, 3\}$

$$e^{-i\pi\varepsilon/2}(1-i)^3 X_{(\varepsilon,0)}^{*(n)}=\{\text{folding points of } H_{n-3}\} . \tag{1.7}$$

Furthermore $\{X_{\varepsilon}^{*(n)}\}$ are similar to a set of folding points of $(n-2)$-step dragon H_{n-2} and

$$e^{-i\pi\varepsilon/2}(1-i)^2 X_{\varepsilon}^{*(n)}=\{\text{folding points of } H_{n-2}\} . \tag{1.8}$$

Taking $n \to \infty$, the set $X_{\varepsilon}^{*(n)}$ and $X_{(\varepsilon,\delta)}^{*(n)}$ converge to limit sets X_{ε}^* and $X_{(\varepsilon,\delta)}^*$ in the Hausdorff metric, respectively, and so X^* is tiled by sets of

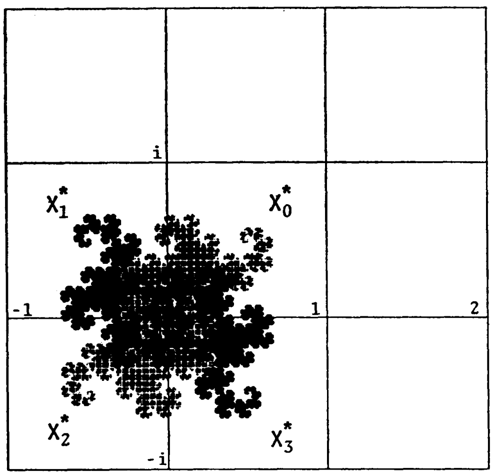

FIGURE 2(a). Tetradragon X^*. X^* is tiled by four dragons $\{X_{\varepsilon}^*;$ $\varepsilon=\{0,1,2,3\}\}$, namely $X_{\varepsilon}^*=e^{i\varepsilon\pi/2}(1-i)^{-2}H_d$ and $X^*=\cup X_{\varepsilon}^*$.

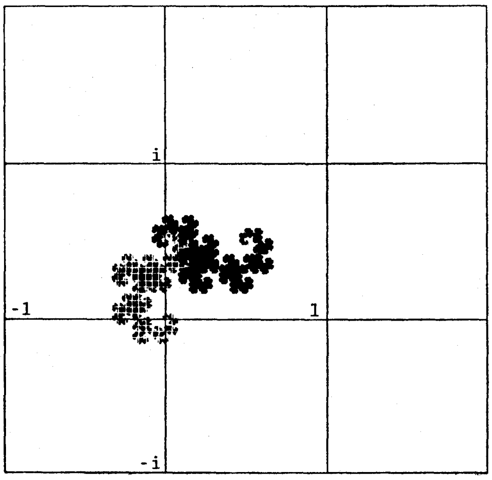

FIGURE 2(b). Dragon X_0^*. X_0^* is tiled by two dragons which are meshed dragon $X^*_{(0,0)}$ and dark dragon $X^*_{(0,1)}$. Notice that the dragon X_0^* coincides with Y_{-1}, a part of the cross dragon Y in Section 2 (Figure 3).

dragons $\{X_\varepsilon^*\}$ (Figure 2(a)) and each X_ε^* is also tiled by dragons $X^*_{(\varepsilon,0)}$ and $X^*_{(\varepsilon,i\varepsilon)}$ (Figure 2(b)), that is,

(1.9) $$X^* = \bigcup_{\varepsilon\in\{0,1,2,3\}} X_\varepsilon^* \quad \text{and} \quad \lambda(X_\varepsilon^* \cap X_{\varepsilon'}^*) \quad \text{for} \quad \varepsilon \neq \varepsilon' ,$$

and

(1.10) $$X_\varepsilon^* = X^*_{(\varepsilon,0)} \cup X^*_{(\varepsilon,i\varepsilon)} \quad \text{and}$$
$$\lambda(X^*_{(\varepsilon,0)} \cup X^*_{(\varepsilon,i\varepsilon)}) = 0 .$$

This fact indicates that the dragons X_ε^* are also selfsimilar fractal sets of order 2. We call the set X^* a tetradragon. Finally the Lebesgue measure of each $X^*_{(\varepsilon,\delta)}$ is

(1.11) $$\lambda(X^*_{(\varepsilon,\delta)}) = 1/8 .$$

The statements for $\{X_{(\varepsilon,\delta)}\}$ are obtained by taking the complex conjugate.

By the way another approach for the selfsimilar fractal set K is proposed by Hutchinson [6] using a set of contraction maps. A method of constructing such set K is shown in the following theorem,

THEOREM 1.1 (Hutchinson [6]). (i) *Let $\mathcal{L} = \{S_0, \cdots, S_{N-1}\}$ be a finite set of contraction maps on a complete metric space. Then there exists a unique closed bounded set K such that $K = \bigcup_{j=1}^{N-1} S_j(K)$.*

(ii) *For arbitrary set A let $\mathcal{L}(A) = \bigcup_{j=1}^{N-1} S_j(A)$ and $\mathcal{L}^p(A) = \mathcal{L}(\mathcal{L}^{p-1}(A))$, then $\mathcal{L}^p(A) \to K$ in the Hausdorff metric for closed bounded A.*

We call the above set K a $\mathscr{L}$-invariant set.

For $\mathscr{L}=\{S_0, \cdots, S_{N-1}\}$ let $\mathscr{L}^n(z_0)$ be

$$\mathscr{L}^n(z_0)=\bigcup_{(j_1,\cdots,j_n)} S_{j_n}\circ S_{j_{n-1}}\circ\cdots\circ S_{j_1}(z_0) . \tag{1.12}$$

where $(j_1, \cdots, j_n)\in\prod_{k=1}^n\{0, \cdots, N-1\}$ and $z_0\in C$. Then a desired set K can be obtained by taking $n\to\infty$ for (1.12).

Now we put contraction maps as following; for $\varepsilon\in\{0, 1, 2, 3\}$

$$T_0(z)=(1-i)^{-1}z \quad\text{and}\quad T_1(z)=(1-i)^{-1}(z-i) , \tag{1.13}$$

$$G^*_{0,\varepsilon}(z)=(1-i)^{-1}z \quad\text{and}\quad G^*_{1,\varepsilon}(z)=(1-i)^{-1}(iz+i^\varepsilon) , \tag{1.14}$$

$$G_{0,\varepsilon}(z)=(1+i)^{-1}z \quad\text{and}\quad G_{1,\varepsilon}(z)=(1+i)^{-1}(-iz+(-i)^\varepsilon) . \tag{1.15}$$

PROPOSITION 1.2. *For* $(j_1, \cdots, j_n)\in\prod_{k=1}^n\{0, 1\}$

(i) $\mathscr{L}^n(0)=X_B^{(n)}$, $T_0(\mathscr{L}^n(0))=X_{B,0}^{(n+1)}$, *and* $T_1(\mathscr{L}^n(0))=X_{B,-i}^{(n+1)}$, *where* $\mathscr{L}^n(z)=\bigcup_{(j_1,\cdots,j_n)} T_{j_n}\circ\cdots\circ T_{j_1}(z)$, *and* $\{T_0, T_1\}$*-invariant set coincides with* X_B, *that is,*

$$X_B=T_0(X_B)\cup T_1(X_B) , \qquad \lambda(T_0(X_B)\cap T_1(X_B))=0 .$$

(ii) $\mathscr{L}^n(0)=X_\varepsilon^{*(n)}$, $G^*_{0,\varepsilon}(\mathscr{L}^n(0))=X_{(\varepsilon,0)}^{*(n+1)}$, *and* $G^*_{1,\varepsilon}(\mathscr{L}^n(0))=X_{(\varepsilon,i\varepsilon)}^{*(n+1)}$ *where* $\mathscr{L}^n(z)=\bigcup_{(j_1,\cdots,j_n)} G^*_{j_n,\varepsilon}\circ\cdots\circ G^*_{j_1,\varepsilon}(z)$, *and* $\{G^*_{0,\varepsilon}, G^*_{1,\varepsilon}\}$*-invariant set coincides with* X_ε^*, *that is,*

$$X_\varepsilon^*=G^*_{0,\varepsilon}(X_\varepsilon^*)\cup G^*_{1,\varepsilon}(X_\varepsilon^*) , \qquad \lambda(G^*_{0,\varepsilon}(X_\varepsilon^*)\cap G^*_{1,\varepsilon}(X_\varepsilon^*))=0 .$$

The similar statements for $G_{0,\varepsilon}$ and $G_{1,\varepsilon}$ also hold.

PROOF. It is verified from the definitions of the contraction maps. □

To summarize results obtained in this section: The twindragon is regarded as the limit set of n-step twindragon curve D_n and also as the complex binary expansion X_B and as well as $\{T_0, T_1\}$-invariant set. The twindragon is also obtained as an interior of a limit of a closed curve $K_n=(1-i)^{-n}K[\theta^n(aba^{-1}b^{-1})]$, where $\theta(a)=ab$ and $\theta(b)=ba^{-1}$ for $S=\{a, b\}$, $f(a)=1$ and $f(b)=-i$ [1] [3]. Also a dragon is constructed as the limit set of n-step paper folding dragon curve H_n and as the revolving expansion X_ε^* and as $\{G^*_{0,\varepsilon}, G^*_{1,\varepsilon}\}$-invariant set.

We give another construction of the dragon in next section.

§ 2. Biased revolving sequences and cross dragon.

In this section we construct the dragon by a new procedure. Let

M be the structure matrix and V the set of one sided infinite seuences generated by M and σ a shift operator on V. We call V a set of biased revolving sequences. Then (V, σ) is a subshift of finite type, namely V is a closed subset of $\prod_{k=1}^{\infty} \Gamma$ and shift invariant $\sigma V = V$. Notice that nonzero entries of the structure matrix can be written as $M_{\tau[k],\tau[(k+1) \bmod 4]} = M_{\tau[k],\tau[(k+2) \bmod 4]} = 1$ for $1 \leqq k \leqq 4$. We denote these two admissible states which follow $\gamma = \tau[k]$ with $\gamma[1] = \tau[(k+1) \bmod 4]$ and $\gamma[2] = \tau[(k+2) \bmod 4]$. Denote a set of all finite biased revolving sequences with length n by $V^{(n)}$. Let $V_\gamma^{(n)}$ be

$$V_\gamma^{(n)} = \{(\gamma_1, \cdots, \gamma_n) \in V^{(n)};\ \gamma_1 = \gamma\} . \tag{2.1}$$

PROPERTY 2.1.
(i)

$$V^{(n)} = \bigcup_{\gamma \in \{0, i, -1+i, -1\}} V_\gamma^{(n)} ,$$

(ii)

$$\sigma V_\gamma^{(n)} = V_{\gamma[1]}^{(n-1)} \cup V_{\gamma[2]}^{(n-1)} ,$$

where σ is defined by $\sigma(\gamma_1, \cdots, \gamma_n) = (\gamma_2, \cdots, \gamma_n)$ for $(\gamma_1, \cdots, \gamma_n) \in V_\gamma^{(n)}$ and $M_{\gamma,\gamma[1]} = M_{\gamma,\gamma[2]} = 1$.
(iii)

$$iV_\gamma^{(n)} + i = V_{\gamma[1]}^{(n)} \quad \text{and} \quad -V_\gamma^{(n)} + (-1+i) = V_{\gamma[2]}^{(n)} ,$$

where $aV^{(n)} + b = \{(a\gamma_1 + b, \cdots, a\gamma_n + b)\}$ for $V^{(n)} = \{(\gamma_1, \cdots, \gamma_n)\}$.

PROOF. (i) and (ii) are obvious. In order to prove (iii), it is enough to notice that symbols 0, i, $-1+i$ and -1, which can be considered as points on the plane, are obtained from a symbol by rotating by angle $\pi j/2$, $j = 1, 2, 3$, around $(-1+i)/2$. Indeed, for example,

$$e^{i\pi/2}\{V_0^{(n)} - (-1+i)/2\} + (-1+i)/2 = V_i^{(n)} ,$$

and

$$e^{i\pi}\{V_0^{(n)} - (-1+i)/2\} + (-1+i)/2 = V_{-1+i}^{(n)} . \qquad \square$$

We realize a biased revolving sequence $(\gamma_1, \cdots, \gamma_n)$ to a point $p(\gamma_1, \cdots, \gamma_n)$ of $\boldsymbol{C}$ by the realization map Φ defined in the Introduction

$$p(\gamma_1, \cdots, \gamma_n) = \sum_{k=1}^{n} \gamma_k (1+i)^{-k} . \tag{2.2}$$

Corresponding to the sets of sequence $V^{(n)}$ and $V_\gamma^{(n)}$, let sets of points $Y^{(n)}$ and $Y_\gamma^{(n)}$ be

$$(2.3)\qquad \begin{aligned} Y^{(n)} &= \{p(\gamma_1, \cdots, \gamma_n);\ (\gamma_1, \cdots, \gamma_n) \in V^{(n)}\}\ , \quad \text{and} \\ Y_\gamma^{(n)} &= \{p(\gamma_1, \cdots, \gamma_n);\ (\gamma_1, \cdots, \gamma_n) \in V_\gamma^{(n)}\}\ . \end{aligned}$$

By Property 2.1 we obtain:

PROPOSITION 2.2.
(i)

$$Y^{(n)} = \bigcup_{\gamma \in \{0, i, -1+i, -1\}} Y_\gamma^{(n)}\ ,$$

(ii)

$$(1+i)\, Y_\gamma^{(n)} - \gamma = Y_{\gamma[1]}^{(n-1)} \cup Y_{\gamma[2]}^{(n-1)} \quad \textit{for} \quad n \geqq 2\ ,$$

where $aA+b=\{ax+b;\ x \in A\}$ *for a set* A,
(iii)

$$i\, Y_\gamma^{(n)} + \sum_{k=1}^{n} i(1+i)^{-k} = Y_{\gamma[1]}^{(n)} \quad \textit{and} \quad -Y_\gamma^{(n)} + \sum_{k=1}^{n} (-1+i)(1+i)^{-k} = Y_{\gamma[2]}^{(n)}\ ,$$

that is, $Y_{\gamma[1]}^{(n)}$ *and* $Y_{\gamma[2]}^{(n)}$ *are obtained by rotating* $Y_\gamma^{(n)}$ *by angle* $\pi/2$ *and* π, *respectively, around* $\sum_{k=1}^{n} (-1+i)/2(1+i)^{-k}$.

LEMMA 2.3. $Y_\gamma^{(n)} = (1+i)^{-1}\{i\, Y_\gamma^{(n-1)} + \gamma + \sum_{k=1}^{n-1} i(1+i)^{-k}\} \cup (1+i)^{-1} \times \{-Y_\gamma^{(n-1)} + \gamma + \sum_{k=1}^{n-1} (-1+i)(1+i)^{-k}\}$.

PROOF. From Property 3.1

$$\begin{aligned} V_\gamma^{(n)} &= (\gamma, \underbrace{0, \cdots, 0}_{n-1}) + \{(0,\ V_{\gamma[1]}^{(n-1)}) \cup (0,\ V_{\gamma[2]}^{(n-1)})\} \\ &= (\gamma, \underbrace{0, \cdots, 0}_{n-1}) + \{(0,\ iV_\gamma^{(n-1)} + i) \cup (0,\ -V_\gamma^{(n-1)} + (-1+i))\}\ , \end{aligned}$$

where $(0, V^{(n-1)}) = \{(0, \gamma_1, \cdots, \gamma_{n-1})\} \in V^{(n)}$ for $V^{(n-1)} = \{(\gamma_1, \cdots, \gamma_{n-1})\}$. By the relation above we obtain the result. □

This lemma shows that each set $Y_\gamma^{(n)}$ is a recurrent set of order 2, namely the n-step set $Y_\gamma^{(n)}$ is obtained from two $(n-1)$-step sets $Y_\gamma^{(n-1)}$ for each γ.

It is verified by the definition of $Y_\gamma^{(n)}$ that

$$(2.4)\qquad d(Y_\gamma^{(n)},\ Y_\gamma^{(n+1)}) \leqq \left(\frac{1}{\sqrt{2}}\right)^n$$

in the Hausdorff metric. Then there exist limit sets Y and Y_γ such that $Y^{(n)}$ and $Y_\gamma^{(n)}$ converge to Y and Y_γ, respectively, in the Hausdorff metric. Taking $n \to \infty$ in Proposition 2.2 and Lemma 2.3, we obtain, PROPOSITION 2.4. *Let* $Y=\{\sum_{k=1}^{\infty}\gamma_k(1+i)^{-k}: (\gamma_1, \gamma_2, \cdots)\in V\}$ *and* $Y_\gamma=\{\sum_{k=1}^{\infty}\gamma_k(1+i)^{-k}; (\gamma_1, \gamma_2, \cdots)\in V_\gamma\}$. *Then sets* Y *and* Y_γ, $\gamma\in\Gamma$, *satisfy following properties*:

(i)
$$Y=\bigcup_{\gamma\in\{0,i,-1+i,-1\}} Y_\gamma ,$$

(ii)
$$(1+i)Y_\gamma-\gamma=Y_{\gamma[1]}\cup Y_{\gamma[2]} ,$$

(iii)
$$iY_\gamma+1=Y_{\gamma[1]} \quad and \quad -Y_\gamma+1+i=Y_{\gamma[2]} ,$$

that is, sets $\{Y_\gamma\}$ *are congruent to each other and obtained by rotating some* Y_γ, *by angles* $\pi k/2$, $k=1, 2, 3$, *around* $(1+i)/2$.

(iv)
$$Y_\gamma=(1+i)^{-1}(iY_\gamma+\gamma+1)\cup(1+i)^{-1}(-Y_\gamma+\gamma+1+i) .$$

Let contraction maps $F_{0,\gamma}$ and $F_{1,\gamma}$ on the plane be

$$F_{0,\gamma}(z)=(1+i)^{-1}(iz+\gamma+1) \quad \text{and}$$
$$F_{1,\gamma}(z)=(1+i)^{-1}(-z+\gamma+1+i) . \tag{2.5}$$

Then from Proposition 2.4 (iv) we can say that the limit sets $\{Y_\gamma\}$ are $\{F_{0,\gamma}, F_{1,\gamma}\}$-invariant sets satisfying relations

$$Y_\gamma=F_{0,\gamma}(Y_\gamma)\cup F_{1,\gamma}(Y_\gamma) \quad \text{for each} \quad \gamma\in\Gamma. \tag{2.6}$$

THEOREM 2.5. *Let sets* Y_γ, $\gamma\in\{0, i, -1+i, -1\}$ *satisfy the relation* (2.6) *and* $Y=\bigcup_{\gamma\in\{0,i,-1+i,-1\}}Y_\gamma$. *Then*

(i) *each set* Y_γ *is a dragon with* $\lambda(Y_\gamma)=1/4$ *and end point besides the common* $(1+i)/2$ *is* 0 *for* Y_{-1}, 1 *for* Y_0, $1+i$ *for* Y_i, i *for* Y_{-1+i}.

(ii) *the set* Y *is tiled by* $\{Y_\gamma\}$, *that is,*

$$Y=\bigcup_{\gamma\in\{0,i,-1+i,-1\}} Y_\gamma \quad and \quad \lambda(Y_\gamma\cap Y_{\gamma'})=0 \quad for \quad \gamma\neq\gamma'$$

(*see Figure* 3).

PROOF. (i) Notice that the contraction maps $F_{0,\gamma}$ and $F_{1,\gamma}$ for $\gamma=-1$ coincide with $G^*_{0,\varepsilon}$ and $G^*_{1,\varepsilon}$ for $\varepsilon=0$ in Section 1, namely

$$F_{0,-1}(z)=G^*_{0,0}(z) \quad \text{and} \quad F_{1,-1}(z)=G^*_{1,0}(z) .$$

As discussed in Section 1, the set Y_{-1} satisfying

$$Y_{-1}=F_{0,-1}(Y_{-1})\cup F_{1,-1}(Y_{-1}) ,$$

is a dragon $(1-i)^{-2}H_d$ with $\lambda(Y_{-1})=1/4$ and end points are 0 and $(1+i)/2$ (Figure 2 (b)), and

(*) $$\lambda(F_{0,-1}(Y_{-1}) \cap F_{1,-1}(Y_{-1}))=0 .$$

Then from Proposition 2.4 (iii) we obtain (i).

(ii) A set $Y_0 \cup Y_i$ is tiled by Y_0 and Y_i owing to (*) and Proposition 2.4 (iii). Using Proposition 2.4 (iii), it is shown that each set $Y_\gamma \cup Y_{\gamma[1]}$ is tiled by Y_γ and $Y_{\gamma[1]}$. Proposition 2.4 (iii) also indicates that the set $Y_{-1} \cup Y_0$ also forms a dragon $(1-i)^{-1}H_d$ with end points 0 and 1 since similar condition holds for $X_i^* = X_{(i,0)}^* \cup X_{(i,i)}^*$. Moreover by (1.3), (1.4) and (1.6) we can see that the twindragon X_B has anoter tiling form (Figure 1 (b)), that is,

$$X_B = (1-i)^{-1}H_d \cup (-(1-i)^{-1}H_d + 1) ,$$

and

$$\lambda(X_B \cap (X_B + i)) = 0 .$$

Thus we obtain the following relation,

$$\lambda((1-i)^{-1}H_d \cap \{-(1-i)^{-1}H_d + 1 + i\}) = 0 .$$

Since $(1-i)^{-1}H_d = Y_{-1} \cup Y_0$,

$$\lambda((Y_{-1} \cup Y_0) \cap (Y_i \cup Y_{-1+i})) = 0 ,$$

that is evident from Proposition 2.4 (iii), which was to be demonstrated (cf. Figure 1 (c) and Figure 3). □

It is verified that $Y_{-1} = X_0^*$, $Y_0 = X_1^* + 1$, $Y_i = X_2^* + 1 + i$ and $Y_{-1+i} = X_3^* + i$. We call the set Y a cross dragon (Figure 3).

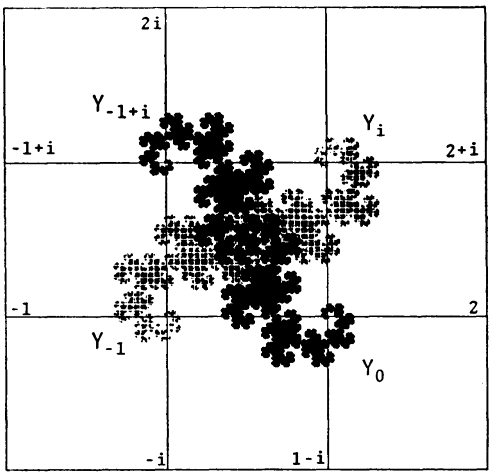

FIGURE 3. Cross dragon Y. Y is tiled by four dragons $\{Y_\gamma;\ \gamma = \{0, i, -1+i, -1\}\}$ in a different manner from tetradragon X^* (Figure 2). Notice that $Y \subset (X_B \cup X_B + i))$ (Figure 1(c)).

§ 3. Dynamical system on cross dragon.

We can define a dynamical system on the cross dragon. Since the dynamical system is constructed in the same manner as the previous one in Section 6 of [1], we state propositions without proof.

We consider the map $\hat{T}$ for each point $z \in Y$:

$$\hat{T}: z \longrightarrow (1+i)z \quad \text{for} \quad z \in Y . \tag{3.1}$$

Then we obtain by Proposition 2.4 (ii),

$$\hat{T} Y_\gamma = \gamma + (Y_{\gamma[1]} \cup Y_{\gamma[2]}) .$$

We prepare following sets $\hat{U}_\gamma$ and U_γ for each $\gamma \in \Gamma$;

$$\begin{aligned} &\hat{U}_0 = Y_i \cup Y_{-1+i} , \qquad \hat{U}_i = i + (Y_{-1+i} \cup Y_{-1}) , \\ &\hat{U}_{-1+i} = -1 + i + (Y_{-1} \cup Y_0) , \qquad \hat{U}_{-1} = -1 + (Y_0 \cup Y_i) , \quad \text{and} \\ &U_\gamma = \hat{U}_\gamma - \gamma . \end{aligned} \tag{3.2}$$

We call $\hat{U}_\gamma$ a neighbourhood of integer γ.

Define a map T for $z \in Y \backslash \cap_{\gamma \in \Gamma} \partial Y_\gamma$ by

$$Tz = (1+i)z - [(1+i)z]_G , \tag{3.3}$$

where $[w]_G = \gamma$ if $w \in \hat{U}_\gamma$. Then the map T satisfies

$$T Y_\gamma = Y_{\gamma[1]} \cup Y_{\gamma[2]} \quad \text{for each} \quad \gamma \in \Gamma , \tag{3.4}$$

that is, the partition $\{Y_\gamma; \gamma \in \Gamma\}$ of Y is a Markov partition for the map T. Let $\gamma_k(z)$ be

$$\gamma_k(z) = [(1+i)T^{k-1}z]_G \quad \text{for} \quad k \geqq 1 . \tag{3.5}$$

Then we have

THEOREM 3.1. *Let Y be the cross dragon and T be the cross dragon map* (3.3). *Then*

(i) *the transformation (Y, T) induces an expansion*

$$z = \sum_{k=1}^{\infty} \gamma_k(z)(1+i)^{-k} \quad \text{for} \quad z \in Y \backslash \bigcup_{k=0}^{\infty} T^{-k}(\bigcap_{\gamma \in \Gamma} \partial Y_\gamma) ,$$

(ii) *the Lebesgue measure λ is invariant with respect to (Y, T),*

(iii) *let μ be a Markov invariant measure for the system (V, σ) with the transition probability P and stationary probability Π such that*

$$P=\begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}, \qquad \Pi=(1/4, 1/4, 1/4, 1/4),$$

then, the dynamical system (Y, T, λ) *is isomorphic to* (V, σ, μ) *and consequently* (Y, T, λ) *is ergodic.*

Identifying the complex plane with $\boldsymbol{R}^2$, we can show that the set Y can be regarded as a covering space of the torus $\boldsymbol{T}^2$ because of the tiling properties of twindragon (1.6) and the set $\{Y_\gamma\}$.

COROLLARY 3.2. *Let*

$$L=\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

which induces an expanding endomorphism T_L *on the torus* $\boldsymbol{T}^2$. *Then there exists a Markov partition* $\{Y_\gamma; \gamma \in \Gamma\}$ *on the torus for* $T_L: \boldsymbol{T}^2 \to \boldsymbol{T}^2$, *so that the dynamical system* $(\boldsymbol{T}^2, T_L, \lambda)$ *with this partition is isomorphic to the one sided subshift of finite type* (V, σ, μ).

This corollary says that there exists a "fractal" Markov partition with respect to the expanding endomorphism T_L $\left(\text{For general expanding endomorphisms } T_L,\ L=\begin{pmatrix} a & b \\ c & d \end{pmatrix},\ \text{see Bedford [9]}\right)$.

Moreover we introduce a simple system $(Y^\dagger, T^\dagger, \lambda^\dagger)$ as follows: let $Y^\dagger=\{x+iy;\ 0 \leqq x, y<1\}$ and a map $T^\dagger$ be

$$T^\dagger z=(1-i)z+(-1+i)-[(1-i)z+(-1+i)] \quad \text{for} \quad z \in Y^\dagger, \tag{3.6}$$

where $[w]=[\mathrm{Re}(w)]+i[\mathrm{Im}(w)]$ for $z \in \boldsymbol{C}$, and the sequence of integer $\{\xi_k(z); k \in \boldsymbol{N}\}$ be

$$\xi_k(z)=[(1-i)T^{\dagger k-1}z+(-1+i)] \quad \text{for each} \quad z \in Y^\dagger. \tag{3.7}$$

Then we can verify that the transformation $(Y^\dagger, T^\dagger)$ induces a expansion

$$z=\sum_{k=1}^{\infty}(\xi_k(z)-(-1+i))(1-i)^{-k} \quad \text{for} \quad \text{a.e. } z \in Y^\dagger, \tag{3.8}$$

and has the Lebesgue measure as an invariant measure $\lambda^\dagger$ and also the partition $\{Y_\gamma^\dagger; \gamma \in \Gamma\}$, where $Y_\gamma^\dagger=\{z \in Y^\dagger; \xi_1(z)=\gamma\}$, is a Markov partition, that is,

$$T^\dagger Y_\gamma^\dagger = Y_{\gamma[1]}^\dagger \cup Y_{\gamma[2]}^\dagger. \tag{3.9}$$

Therefore $T^{\dagger}$-admissible sequences $\{(\xi_1(z), \xi_2(z), \cdots)\}$ which have the same structure of the sequences generated by the cross dragon system (Y, T). Thus we obtain:

THEOREM 3.3. *The dynamical systems (Y, T, λ) and $(Y^{\dagger}, T^{\dagger}, \lambda^{\dagger})$ are isomorphic to each other as an endomorphism, that is there exists measure preserving invertible map Ψ defined on Y such that*

$$T^{\dagger} \circ \Psi = \Psi \circ T .$$

§ 4. Dual map and natural extension of cross dragon system.

We show that the cross dragon system (Y, T, λ) is nothing but the dual map [1] of a very simple system.

Let $Y^* = \{x+iy;\ 0 \leqq x, y < 1\}$ and a map T^* be

$$T^* z = (1+i)z - [(1+i)z] \quad \text{for} \quad z \in Y^* . \tag{4.1}$$

Hence a set $\{[(1+i)z];\ z \in Y^*\}$ coincides with $\Gamma = \{0, i, -1+i, -1\}$. We can easily verify that the transformation (Y^*, T^*) is well defined on Y^* and has the Lebesgue measure λ^* on Y^* as an invariant measure and also induces a expansion for a.e. $z \in Y^*$ such that

$$z = \sum_{k=1}^{\infty} \eta_k(z)(1+i)^{-k} , \tag{4.2}$$

where

$$\eta_k(z) = [(1+i) T^{*k-1} z] .$$

Let a set Y_η^* be

$$Y_\eta^* = \left\{ \sum_{k=1}^{\infty} \eta_k(z)(1+i)^{-k};\ z \in Y^* \ \text{and} \ \eta_1(z) = \eta \right\} . \tag{4.3}$$

Then we can see that the sets $\{Y_\eta^*;\ \eta \in \Gamma\}$ are four triangles with vertices $0, 1$ for Y_0^*, $1, 1+i$ for Y_i^*, $1+i, i$ for Y_{-1+i}^*, $i, 0$ for Y_{-1}^* and $(1+i)/2$ in common, and the domain Y^* is tiled by these triangles, that is,

$$Y^* = \bigcup_{\eta \in \{0, i, -1+i, -1\}} Y_\eta^* \quad \text{and} \quad \lambda(Y_\eta^* \cap Y_{\eta'}^*) = 0 \quad \text{for} \quad \eta \neq \eta' . \tag{4.4}$$

Let M^* be a structure matrix such that

$$M_{j,k}^* = \begin{cases} 1 & \text{if} \quad T^* Y_{\tau[j]}^* \cap Y_{\tau[k]}^* \neq \emptyset \\ 0 & \text{if} \quad T^* Y_{\tau[j]}^* \cap Y_{\tau[k]}^* = \emptyset . \end{cases}$$

Let V^* and V^*_η be

$$V^*=\{(\eta_1, \eta_2, \cdots); \eta_j \in \Gamma \text{ and } M^*_{\eta_j,\eta_{j+1}}=1 \text{ for all } j\geqq 1\} \tag{4.5}$$

$$V^*_\eta=\{(\eta_1, \eta_2, \cdots) \in V^*; \eta_1=\eta\} . \tag{4.6}$$

It is easily verified that every element of V^* has the same admissibility as the sequence $(\eta_1(z), \eta_2(z), \cdots)$ induced by (Y^*, T^*), Notice that

$${}^tM=M^* ,$$

and so for any $(\eta_1, \cdots, \eta_n) \in V^{*(n)}$ a sequence $(\eta_n, \cdots, \eta_1)$, which is a backward sequence of it, is an element of $V^{(n)}$. In this sense we call (V^*, σ^*) is a dual symbolic system [1] for (V, σ). Thus we obtain,

THEOREM 4.1. *The cross dragon system (Y, T, λ) is a dual system for the system (Y^*, T^*, λ^*).*

The natural extension [1] of the symbolic system (V, σ) is $(\tilde{V}, \tilde{\sigma})$ such that

$$\tilde{V}=\{(\cdots, \gamma_{-2}, \gamma_{-1}, \gamma_0, \gamma_1, \gamma_2, \cdots); \gamma_k \in \Gamma \text{ and } M_{\gamma_k,\gamma_{k+1}}=1 \text{ for all } k \in \boldsymbol{Z}\} , \tag{4.7}$$

and $\tilde{\sigma}$ is a shift operator on $\tilde{V}$.

LEMMA 4.2. *The set $\tilde{V}$ is decomposed as follows;*

$$\begin{aligned}\tilde{V}&=\bigcup_{\gamma \in \{0,i,-1+i,-1\}} (V^*_\eta \cup V^*_{\eta'})\cdot V_\gamma\\ &=\bigcup_{\gamma \in \{0,i,-1+i,-1\}} V^*_\gamma\cdot(V_{\gamma[1]} \cup V_{\gamma[2]}) ,\end{aligned}$$

where for $(\eta_1, \eta_2, \cdots) \in V^*$ and $(\gamma_1, \gamma_2, \cdots) \in V$, $(\eta_1, \eta_2, \cdots)\cdot(\gamma_1, \gamma_2, \cdots)=(\cdots, \eta_2, \eta_1, \gamma_1, \gamma_2, \cdots)$ and $M_{\eta,\gamma}=M_{\eta',\gamma}=M_{\gamma,\gamma[1]}=M_{\gamma,\gamma[2]}=1$.

The proof is easily derived from the admissibilities of V and V^*.

THEOREM 4.3. *Let a set $\tilde{Y}$ be a subset of $\boldsymbol{C}^2$ such that*

$$\begin{aligned}\tilde{Y}&=\bigcup_{\gamma\in\Gamma}\bigcup_{\eta} Y^*_\eta \times Y_\gamma\\ &=\bigcup_{\gamma\in\Gamma}\bigcup_{\delta} Y^*_\gamma \times Y_\delta\end{aligned}$$

where $\eta \in \{\eta'; M_{\eta',\gamma}=1\}$ and $\delta \in \{\delta'; M_{\gamma,\delta'}=1\}$ for $\gamma \in \Gamma$, and a map $\tilde{T}$ be for $(w, z) \in Y^*_\eta \times Y_\gamma$

$$\tilde{T}(w, z)=((1+i)^{-1}(w+\gamma), Tz) .$$

Then the system $(\tilde{Y}, \tilde{T}, \tilde{\lambda})$ *is a natural extension of the cross dragon system* (Y, T, λ), *where* $\tilde{\lambda}$ *is the Lebesgue measure on* $\tilde{Y}$.

PROOF. The decompositions of $\tilde{V}$ in Lemma 4.2 reduce to the decompositions of their realization $\tilde{Y}$ with a realization map $\tilde{\Phi}$ for $(\eta_1, \eta_2, \cdots)\times(\gamma_1, \gamma_2, \cdots) \in \tilde{V}$ such that

$$\tilde{\Phi}: (\eta_1, \eta_2, \cdots)\cdot(\gamma_1, \gamma_2, \cdots) \longrightarrow \left(\sum_{k=1}^{\infty} \eta_k(1+i)^{-k}, \sum_{j=1}^{\infty} \gamma_j(1+i)^{-j}\right) .$$

We can see by Property 2.1 and Lemma 4.2 that if $\tilde{\omega} \in V_\eta^* \cdot V_\gamma$ then $\tilde{\omega}$ is translated by $\tilde{\sigma}$ bijectively to

$$\tilde{\sigma}\tilde{\omega} \in V_\gamma^* \cdot (V_{\gamma[1]} \cup V_{\gamma[2]}) .$$

The realization $(\tilde{V}, \tilde{\sigma})$ is nothing but

$$\tilde{T}(w, z)=((1+i)^{-1}(w+\gamma), Tz) \quad \text{for} \quad (w, z) \in Y_\eta^* \times Y_\gamma .$$

Therefore the map $\tilde{T}$ is well defined and bijection. It is easily verified that the Lebesgue measure $\tilde{\lambda}$ is invariant with respect to $(\tilde{Y}, \tilde{T})$. □

CORORALLY 4.4. *The dynamical system* $(\tilde{Y}, \tilde{T}^{-1}, \tilde{\lambda})$ *is a natural extension of* (Y^*, T^*, λ^*).

We can say by Corollary 4.4 that the cross dragon system (Y, T, λ) is the dual system of the simple system (Y^*, T^*, λ^*).

We point out here that the dynamical system $(Y^\dagger, T^\dagger, \lambda^\dagger)$ in Section 3 is also the dual system for (Y^*, T^*, λ^*) which has a simple domain in contrast with (Y, T, λ).

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