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On a Bernoulli Property for Multi-dimensional Mappings with Finite Range Structure

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Introduction

In the previous paper [4], we considered ergodic properties of a mapping T defined on a bounded domain $X\subset R^d$ satisfying a "local Renyi's condition". The purpose of this paper is to prove that such a mapping T is weak Bernoulli if it admits a finite absolutely continuous invariant measure.

The mapping we consider is characterized by a certain type of partition $Q=\{X_a: a\in I\}$ of X and a finite number of subsets $U_0(=X)$, U_1, $\cdots$, U_N of X satisfying some special properties (see §1 for precise definitions). We shall call such a transformation T a multi-dimensional mapping with a finite range structure. If such a T satisfies the Renyi's condition, in addition, then it is known that T has a finite absolutely continuous invariant measure, and furthermore, under some additional conditions one can prove that Q is a weak Bernoulli partition ([9], [18]). On the other hand, when X is an interval of R^1, Ledrappier established in [6] the weak Bernoulli property for a transformation T having a similar characterization under some further hypothesis, such as the existence of a finite invariant measure with positive entropy, but without assuming that T satisfies the Renyi's condition (cf. [2]). The main ingredient of his proof, which is patterned after the work of Sinai [15] (cf. [16]) and Ratner [12], is the use of Rohlin's formula for proving the absolute continuity of some conditional measures.

In this paper we establish a sufficient condition for a multi-dimensional mapping with a finite range structure to have the weak Bernoulli property when they do not necessarily satisfy the Renyi's condition. We do need, however, to make several assumptions on the transformation; some of these assumptions seem to be essential, while the others are seen

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to be purely technical (see §1). Under the assumptions explained in detail in §1, we will show that T is exact (§2), and that "Rohlin's formula" holds for T (§3). In §4, we construct a natural extension for T and consider conditional measures with respect to the extension. By using Rohlin's formula, we prove the absolute continuity of the conditional measures, and this will lead us to the desired conclusion along the line of argument used in [6] and [12]. Since the reasoning follows more or less the same pattern we will only sketch the outline of the argument used for this part.

The remainder (§5) of the paper is devoted to the discussion of three examples of multi-dimensional mapping with a finite range structure. One of these examples is a one parameter family of maps of an interval, and the others arise from the number theory: an inhomogeneous diophantine approximation problem and complex continued fractions. None of these transformations satisfy the Renyi's condition, but all of them do satisfy a "local Renyi's condition". We will show that these transformations satisfy all of the assumptions made in §1, and therefore weak Bernoulli.

§1. Notations and results.

DEFINITION. A mapping T on a bounded domain $X \subset R^d$ is called *"a multi-dimensional mapping with a finite range structure"* if there exist a countable partition $Q=\{X_a: a \in I\}$ of X and a finite number of subsets $\{U_0, U_1, \cdots, U_N\}$ of X satisfying the following Conditions (1)~(4):

(1) Each X_a is a measurable connected subset with piecewise smooth boundary.

(2) Each U_k has a positive measure.

(3) For each X_a the mapping $T|_{X_a}$ restricted on X_a is injective, of class C^1, and $\det DT|_{X_a} \neq 0$.

(4) If $\mathrm{int}(X_{a_1}) \cap \mathrm{int}(T^{-1}X_{a_2}) \cap \cdots \cap \mathrm{int}(T^{-(n-1)}X_{a_n}) \neq \emptyset$, then $T^n(X_{a_1} \cap T^{-1}X_{a_2} \cap \cdots \cap T^{-(n-1)}X_{a_n}) = U_k$ for some $k \in \{0, 1, \cdots, N\}$.

To state our results, we introduce some notations. If $\mathrm{int}(X_{a_1}) \cap \mathrm{int}(T^{-1}X_{a_2}) \cap \cdots \cap \mathrm{int}(T^{-(n-1)}X_{a_n}) \neq \emptyset$, we denote $X_{a_1} \cap T^{-1}X_{a_2} \cap \cdots \cap T^{-(n-1)}X_{a_n}$, by $X_{a_1 \cdots a_n}$, and call it *a cylinder of rank n* with respect to T. $\mathscr{L}^{(n)}$ denotes the family of all cylinders $X_{a_1 \cdots a_n}$ of rank n, and $\mathscr{L} = \bigcup_{n=1}^{\infty} \mathscr{L}^{(n)}$. If $X_{a_1 \cdots a_n} \in \mathscr{L}^{(n)}$, we call the sequence $(a_1 \cdots a_n)$ *T-admissible*. Denote the set of all T-admissible sequences of length n by $A(n)$. We write Ψ_a for $(T|_{X_a})^{-1}$ and define inductively

$$\Psi_{a_1 \cdots a_n} = \Psi_{a_1 \cdots a_{n-1}} \circ \Psi_{a_n} .$$

For a constant $C \geqq 1$, we call a cylinder $X_{a_1 \cdots a_n}$ an "*R.C-cylinder*" if it satisfies "Renyi's condition", i.e.

$$\sup_{x \in T^n X_{a_1 \cdots a_n}} |\det D\Psi_{a_1 \cdots a_n}(x)| \leqq C \cdot \inf_{x \in T^n X_{a_1 \cdots a_n}} |\det D\Psi_{a_1 \cdots a_n}(x)| .$$

Let $R(C.T)$ denote the set of all $R.C$-cylinders. We define for $C \geqq 1$

$$\mathscr{D}_n = \{X_{a_1 \cdots a_n} \in \mathscr{L}^{(n)} : X_{a_1 \cdots a_j} \in \mathscr{L} \backslash R(C.T) \text{ for } 1 \leqq j \leqq n\} ,$$
$$D_n = \bigcup_{X_{a_1 \cdots a_n} \in \mathscr{D}_n} X_{a_1 \cdots a_n} ,$$
$$\beta_n = \{X_{a_1 \cdots a_n} \in \mathscr{L}^{(n)} : X_{a_1 \cdots a_{n-1}} \in \mathscr{D}_{n-1}, X_{a_1 \cdots a_n} \in R(C.T)\} ,$$
$$B_n = \bigcup_{X_{a_1 \cdots a_n} \in \beta_n} X_{a_1 \cdots a_n} .$$

Let $\lambda(\cdot)$ be the normalized Lebesgue measure on X.

We now state some conditions to be used in our results.

(C.1) (*generator condition*)
$\bigvee_{m=1}^{\infty} T^{-m} Q = \varepsilon$, *i.e. the partition into points.*

Assume that there exists a constant $C \geqq 1$ *such that*

(C.2) (*transitivity condition*)
for each j *with* $0 \leqq j \leqq N$, *there exists a cylinder* $X_{a_1 \cdots a_{s_j}}$ *contained in* U_j *such that* $X_{a_1 \cdots a_{s_j}} \in R(C.T)$ *and* $T^{s_j} X_{a_1 \cdots a_{s_j}} = X$,

(C.3) *if* $X_{a_1 \cdots a_n} \in R(C.T)$, *then* $X_{b_1 \cdots b_k a_1 \cdots a_n} \in R(C.T)$ *for any* $(b_1 \cdots b_k a_1 \cdots a_n) \in A(k+n)$,

(C.4) $\sum_{n=1}^{\infty} \lambda(D_n) < +\infty$.

Under the above conditions, we have

THEOREM 1. *T is exact.*

REMARK 1. In previous paper [4], we showed that T is ergodic and has a finite invariant measure μ equivalent to λ under the same conditions (C.1)∼(C.4).

Assume further

(C.5) *for all* $n > 0$,

$$W_n \equiv \sum_{m=0}^{\infty} \left(\sum_{X_{k_1 \cdots k_m} \in \mathscr{D}_m} \left(\sup_{y \in T^m X_{k_1 \cdots k_m} \cap \left(\bigcup_{j=1}^{n} B_j \right)} |\det D\Psi_{k_1 \cdots k_m}(y)| \right) \right)$$
$$< +\infty ,$$

(C.6) $^{\#}\mathscr{D}_1 < +\infty$,

(C.7) *there exists a positive integer* l *such that for all* $n > 0$ *and all* $X_{a_1 \cdots a_n} \in \mathscr{D}_n$

$$\frac{\sup_{x\in T^nX_{a_1\cdots a_n}} |\det D\Psi_{a_1\cdots a_n}(x)|}{\inf_{x\in T^nX_{a_1\cdots a_n}} |\det D\Psi_{a_1\cdots a_n}(x)|}=O(n^l) ,$$

(C.8) $\log|\det(DT(\cdot))|\in\mathscr{L}^1(X, \lambda)$.
Then we have

THEOREM 2. *Rohlin's formula* (R) *is true.*

$$\text{(R):} \quad h(T)=\int_X \log|\det DT(x)|d\mu(x) .$$

REMARK 2. In general, the density of μ is not bounded (for example, see §5). For this reason, we need some technical Conditions (C.5) and (C.6). These conditions allow us to have the following properties: The density of μ is bounded on $(D_n)^c$ for each $n>0$, and therefore for all $n>0$, there exists $M(n)$ such that

$$\frac{\mu(X_{a_1\cdots a_n})}{\lambda(X_{a_1\cdots a_n})}<M(n) \quad \text{for any } X_{a_1\cdots a_n}\in\mathscr{L}^{(n)} .$$

(This is proved in §3).

REMARK 3. (C.7) is a weaker Renyi's condition. It is easy to see that we can replace $\mathscr{D}_n$ by $\mathscr{L}^{(n)}$, and this condition allows us to have

$$\inf_{x\in X}|\det DT(x)|>0 .$$

For main theorem, we also suppose
(C.4)* $\sum_{n=1}^{\infty}\lambda(D_n)\log n<+\infty$,
(C.9) *there exists a positive integer* k_0 *which satisfies the following; if* $X_{a_1\cdots a_n}\in\mathscr{D}_n^c$ *and* $X_{a_2\cdots a_n}\in\mathscr{D}_{n-1}$, *then*

$$X_{a_1\cdots a_n}\subset\bigcup_{j=1}^{k_0} B_j .$$

THEOREM 3. *Let* T *be a multi-dimensional mapping with a finite range structure satisfying* (C.1)~(C.9). *Then* Q *is a weak Bernoulli partition with respect to* T.

§2. Proof of Theorem 1.

From a basic fact proved in Rohlin's paper [13], it is sufficient to show that for all measurable sets E of positive measure with measurable images TE, T^2E, $\cdots$,

$$\lim_{n\to\infty} \mu(T^n E) = \mu(X) .$$

Let ε be a positive number. By Lemma 2.1 in §2 of [4], the reexists $X_{a_1\cdots a_n} \in R(C.T)$ such that

$$\lambda(E \cap X_{a_1\cdots a_n}) > (1-\varepsilon)\lambda(X_{a_1\cdots a_n}) ,$$

and therefore

$$\varepsilon \cdot \lambda(X_{a_1\cdots a_n}) > \lambda(X_{a_1\cdots a_n} \cap E^c) . \tag{2.1}$$

It follows from (2.1) and relation

$$\lambda(T^n X_{a_1\cdots a_n} \cap (T^n E)^c) \leqq \int_{E^c \cap X_{a_1\cdots a_n}} |\det DT^n(x)|\, d\lambda(x)$$

that

$$\lambda(T^n X_{a_1\cdots a_n} \cap (T^n E)^c) < C\cdot\varepsilon\cdot\lambda(T^n X_{a_1\cdots a_n}) . \tag{2.2}$$

From (C.2), for $T^n X_{a_1\cdots a_n} = U_j$, there exists $X_{a_1\cdots a_{sj}} \subset U_j$ such that

$$X_{a_1\cdots a_{sj}} \in R(C.T) \quad \text{and} \quad T^{sj} X_{a_1\cdots a_{sj}} = X ,$$

hence

$$\lambda(X_{a_1\cdots a_{sj}} \cap (T^n E)^c) < C\cdot\varepsilon\cdot\lambda(T^n X_{a_1\cdots a_n}) . \tag{2.3}$$

Put $D = \min_{0\leqq j\leqq N}(\lambda(X_{a_1\cdots a_{sj}})/\lambda(U_j))$, then

$$\lambda(X_{a_1\cdots a_{sj}} \cap (T^n E)^c) < \frac{C\cdot\varepsilon}{D}\lambda(X_{a_1\cdots a_{sj}}) . \tag{2.4}$$

By virtue of the properties of $X_{a_1\cdots a_{sj}}$, it follows from (2.4) that

$$\begin{aligned}
&\lambda(T^{sj}(X_{a_1\cdots a_{sj}} \cap (T^n E)^c)) \\
&\quad \leqq C\{\inf_{x\in X_{a_1\cdots a_{sj}}} |\det DT^{sj}(x)|\}\lambda(X_{a_1\cdots a_{sj}} \cap (T^n E)^c) \\
&\quad < \frac{C^2\cdot\varepsilon}{D}\cdot\lambda(X_{a_1\cdots a_{sj}})\cdot \inf_{x\in X_{a_1\cdots a_{sj}}} |\det DT^{sj}(x)| \\
&\quad < \frac{C^2\cdot\varepsilon}{D} .
\end{aligned}$$

Using the equality

$$\lambda(T^{sj}(X_{a_1\cdots a_{sj}} \cap (T^n E)^c)) = 1 - \lambda(T^{sj}(X_{a_1\cdots a_{sj}} \cap T^n E)) ,$$

we obtain

$$\lambda(T^{n+s_j}E)>\lambda(T^{s_j}(X_{a_1\cdots a_{s_j}}\cap T^n E))>1-\frac{C^2\cdot\varepsilon}{D}.$$

From this and the fact that $\mu\sim\lambda$, the theorem follows (cf. [4]). □

§3. The proof of Theorem 2.

We now prepare some lemmas to be used in the proof of Theorem 2. We note that in [4] T-invariant ergodic measure μ such that $\mu\sim\lambda$ was given by

$$\mu(A)=\sum_{m=0}^{\infty}\nu(T^{-m}A\cap D_m)$$

for any measurable set A, where $\nu\sim\lambda$ and there is a constant $G>1$ satisfying

$$G^{-1}\leqq\frac{d\nu}{d\lambda}\leqq G.$$

As we have announced in introduction, we first prove

LEMMA 3.1. *There exists a monotone increasing sequence* $\{\hat{M}(n)\}_{n>0}$ *such that for any* $X_{a_1\cdots a_{n+k}}\subset B_n$

$$\frac{\mu(X_{a_1\cdots a_{n+k}})}{\lambda(X_{a_1\cdots a_{n+k}})}<\hat{M}(n).$$

Furthermore, we have for all $n>0$ *and* $X_{a_1\cdots a_n}\in\mathscr{L}^{(n)}$,

$$G^{-1}\leqq\frac{\mu(X_{a_1\cdots a_n})}{\lambda(X_{a_1\cdots a_n})}\leqq M(n), \tag{3.1}$$

where

$$M(n)=\max\left\{\hat{M}(n),\ \max_{X_{a_1\cdots a_n}\in\mathscr{D}_n}\left\{\frac{\mu(X_{a_1\cdots a_n})}{\lambda(X_{a_1\cdots a_n})}\right\}\right\}.$$

PROOF. We note that the following equality is true.

$$X_{k_1\cdots k_m a_1\cdots a_n}=\Psi_{k_1\cdots k_m}(T^m X_{k_1\cdots k_m}\cap X_{a_1\cdots a_n}).$$

From this, we have

$$\begin{aligned}\mu(X_{a_1\cdots a_{n+k}})&\leqq G\sum_{m=0}^{\infty}\Big(\sum_{X_{k_1\cdots k_m}\in\mathscr{D}_m}\lambda(X_{k_1\cdots k_m a_1\cdots a_{n+k}})\Big)\\&=G\sum_{m=0}^{\infty}\left(\sum_{X_{k_1\cdots k_m}\in\mathscr{D}_m}\left(\int_{T^m X_{k_1\cdots k_m}\cap X_{a_1\cdots a_{n+k}}}|\det D\Psi_{k_1\cdots k_m}(x)|d\lambda(x)\right)\right)\end{aligned}$$

$$\leqq G\sum_{m=0}^{\infty}\Big(\sum_{X_{k_1\cdots k_m}\in\mathscr{D}_m}\Big(\sup_{x\in T^m X_{k_1\cdots k_m}\cap X_{a_1\cdots a_{n+k}}}|\det D\Psi_{k_1\cdots k_m}(x)|\cdot\lambda(T^m X_{k_1\cdots k_m}\cap X_{a_1\cdots a_{n+k}})\Big)\Big)$$

$$\leqq G\sum_{m=0}^{\infty}\Big(\sum_{X_{k_1\cdots k_m}\in\mathscr{D}_m}\Big(\sup_{x\in T^m X_{k_1\cdots k_m}\cap X_{a_1\cdots a_{n+k}}}|\det D\Psi_{k_1\cdots k_m}(x)|\cdot\lambda(X_{a_1\cdots a_{n+k}})\Big)\Big)\ .$$

If $X_{a_1\cdots a_{n+k}}\subset B_n$, then

$$\mu(X_{a_1\cdots a_{n+k}})\leqq G\cdot W_n\cdot\lambda(X_{a_1\cdots a_{n+k}})\ .$$

Taking $G\cdot W_n$ for $\hat{M}(n)$, we obtain the first statement. Note that $X_{a_1\cdots a_n}\in\mathscr{D}_n^c$ implies $X_{a_1\cdots a_n}\subset\bigcup_{j=1}^n B_j$. Then, from (C.6) the second assertion is verified immediately. □

The Conditions (C.7) and (C.8) allow us to have the following properties:

LEMMA 3.2. (3.2-a) $\log|\det DT(x)|\in\mathscr{L}^1(X,\mu)$,
(3.2-b) *for all* $n>0$ *and* $j\in\{0,\cdots,N\}$, put

$$c^j(n)\equiv-\sum_{\substack{(a_1\cdots a_n);\\ X_{a_1\cdots a_n}\cap U_j\neq\varnothing}}\lambda(X_{a_1\cdots a_n})\log\lambda(X_{a_1\cdots a_n}\cap U_j)\ .$$

Then $c^j(n)<+\infty$.
(3.2-c) $H(Q)=-\sum_a\mu(X_a)\log\mu(X_a)<+\infty$.

REMARK 4. Since (C.2) implies $X\in\{U_0,\cdots,U_N\}$, (3.2-b) allows us to have for all $n>0$ $-\sum\lambda(X_{a_1\cdots a_n})\log\lambda(X_{a_1\cdots a_n})<+\infty$.

PROOF. Here we denote $T|_{X_a}$ by T_a. Since

$$\int_X\max(0,\log|\det DT(x)|)d\mu(x)=\sum_a\int_{X_a}\max(0,\log|\det DT_a(x)|)d\mu(x)\ ,$$

and

$$\sup_{x\in X_a}(\log|\det DT_a(x)|)\leqq\log(\sup_{x\in X_a}|\det DT_a(x)|)\ ,$$

it follows from (C.7) and (3.1) that

$$\begin{aligned}\int_X\max(0,\log|\det DT(x)|)d\mu(x)&\leqq\sum_a\max(0,\mu(X_a)\log(\sup_{x\in X_a}|\det DT_a(x)|))\\ &\leqq\sum_a\max(0,\mu(X_a)\log(C_1\inf_{x\in X_a}|\det DT_a(x)|))\\ &\leqq\sum_a\max(0,M(1)\cdot\lambda(X_a)\log C_1+M(1)\lambda(X_a)\log(\inf_{x\in X_a}|\det DT_a(x)|))\\ &\leqq\sum_a M(1)\lambda(X_a)\log C_1+\sum_a\max(0,M(1)\lambda(X_a)\log(\inf_{x\in X_a}|\det DT_a(x)|))\ ,\end{aligned}$$

where C_1 is a constant such that

$$\frac{\sup_{x\in TX_a}|\det D\Psi_a(x)|}{\inf_{x\in TX_a}|\det D\Psi_a(x)|}<C_1$$

for all $X_a\in\mathscr{L}^{(1)}$.

On the other hand, using the inequality

$$\log(\inf_{x\in X_a}|\det DT_a(x)|)\leqq\inf_{x\in X_a}(\log|\det DT_a(x)|)\ ,$$

we have

$$\sum_a \max(0,\log(\inf_{x\in X_a}|\det DT_a(x)|)\cdot\lambda(X_a))\leqq\int_X \max(0,\log|\det DT(x)|)d\lambda\ .$$

From this and (C.8), we obtain

$$\begin{aligned}&\int_X \max(0,\log|\det DT(x)|)d\mu(x)\\&\quad<M(1)\log C_1(\sum_a\lambda(X_a))+M(1)\int_X\max(0,\log|\det DT(x)|)d\lambda\\&\quad<+\infty\ .\end{aligned}$$

Now we remark that $E\equiv\inf_{x\in X}|\det DT(x)|>0$. In fact, the relation

$$\lambda(T_aX_a)=\int_{X_a}|\det DT_a(x)|d\lambda\leqq C_1\cdot\inf_{x\in X_a}|\det DT_a(x)|\cdot 1$$

allows us to have

$$\inf_{x\in X_a}|\det DT_a(x)|\geqq\frac{L}{C_1}\quad\text{for all } X_a\in\mathscr{L}^{(1)}\ ,$$

where $L\equiv\min_{0\leq j\leq N}\lambda(U_j)$. This implies $\inf_{x\in X}|\det DT(x)|>0$. From this, we have immediately

$$\int_X \min(0,\log|\det DT(x)|)d\mu(x)\geqq\int_X\min(0,\log E)d\mu(x)>-\infty\ .$$

Therefore, combining the above results, we conclude (3.2-a).

By the condition (C.7), we have for all $n>0$ a constant C_n such that

$$\sup_{x\in T^nX_{a_1\cdots a_n}}|\det D\Psi_{a_1\cdots a_n}(x)|<C_n\cdot\inf_{x\in T^nX_{a_1\cdots a_n}}|\det D\Psi_{a_1\cdots a_n}(x)|$$

for any $X_{a_1\cdots a_n}\in\mathscr{L}^{(n)}$. Using this and the equality

$$\int_{X_{a_1\cdots a_n}} \log|\det DT^n(x)|d\lambda = \int_{X_{a_1\cdots a_n}} \log\left(\frac{1}{|\det D\Psi_{a_1\cdots a_n}(T^n x)|}\right)d\lambda ,$$

we obtain

$$\int_X \log|\det DT^n(x)|d\lambda(x)$$
$$\geqq \sum_{(a_1\cdots a_n)\in A(n)} \lambda(X_{a_1\cdots a_n})\log\left(\frac{1}{\sup_{x\in T^n X_{a_1\cdots a_n}} |\det D\Psi_{a_1\cdots a_n}(x)|}\right)$$
$$\geqq \sum_{(a_1\cdots a_n)\in A(n)} \lambda(X_{a_1\cdots a_n})\log\left(\frac{1}{C_n\cdot \inf_{x\in T^n X_{a_1\cdots a_n}} |\det D\Psi_{a_1\cdots a_n}(x)|}\right) .$$

Note that for all U_j, there exists $X_{b_1\cdots b_l}^{(j)}$ such that

$$T^l X_{b_1\cdots b_l}^{(j)} = U_j .$$

Therefore, if $X_{a_1\cdots a_n}\cap U_j\neq\varnothing$ then we have the following;

$$\lambda(X_{a_1\cdots a_n}\cap U_j) = \int_{X_{a_1\cdots a_n}\cap T^l X_{b_1\cdots b_l}^{(j)}} d\lambda(x)$$
$$= \int_{T^n(X_{a_1\cdots a_n}\cap T^l X_{b_1\cdots b_l}^{(j)})} |\det D\Psi_{a_1\cdots a_n}(x)|d\lambda(x)$$
$$= \int_{T^{n+l}(X_{b_1\cdots b_l a_1\cdots a_n})} |\det D\Psi_{a_1\cdots a_n}(x)|d\lambda(x)$$
$$\geqq \inf_{x\in T^{n+l}(X_{b_1\cdots b_l a_1\cdots a_n})} |\det D\Psi_{a_1\cdots a_n}(x)|\cdot\lambda(T^{n+l}X_{b_1\cdots b_l a_1\cdots a_n})$$
$$\geqq \inf_{x\in T^n(X_{a_1\cdots a_n})} |\det D\Psi_{a_1\cdots a_n}(x)|\cdot L .$$

In particular, for $U_j = X$, we have $\lambda(X_{a_1\cdots a_n})\geqq\inf|\det D\Psi_{a_1\cdots a_n}(x)|L$ similarly. Using the above inequality, we obtain

$$\int_X \log|\det DT^n(x)|\, d\lambda(x)$$
$$\geqq \log\frac{1}{C_n} + \sum_{\substack{X_{a_1\cdots a_n};\\ X_{a_1\cdots a_n}\cap U_j\neq\varnothing}} \lambda(X_{a_1\cdots a_n})\log\left(\frac{1}{\inf_{x\in T^n X_{a_1\cdots a_n}} |\det D\Psi_{a_1\cdots a_n}(x)|}\right)$$
$$+ \sum_{\substack{X_{a_1\cdots a_n};\\ X_{a_1\cdots a_n}\cap U_j=\varnothing}} \lambda(X_{a_1\cdots a_n})\log\left(\frac{1}{\inf|\det D\Psi_{a_1\cdots a_n}(x)|}\right)$$
$$\geqq \log\frac{1}{C_n} + \sum_{\substack{X_{a_1\cdots a_n};\\ X_{a_1\cdots a_n}\cap U_j\neq\varnothing}} \lambda(X_{a_1\cdots a_n})\log\left(\frac{L}{\lambda(X_{a_1\cdots a_n}\cap U_j)}\right)$$
$$+ \sum_{\substack{X_{a_1\cdots a_n};\\ X_{a_1\cdots a_n}\cap U_j=\varnothing}} \lambda(X_{a_1\cdots a_n})\log\left(\frac{L}{\lambda(X_{a_1\cdots a_n})}\right)$$

$$=\log\frac{L}{C_n}+\sum_{\substack{X_{a_1\cdots a_n};\\ X_{a_1\cdots a_n}\cap U_j\neq\emptyset}}\lambda(X_{a_1\cdots a_n})\log\left(\frac{1}{\lambda(X_{a_1\cdots a_n}\cap U_j)}\right)$$
$$+\sum_{\substack{X_{a_1\cdots a_n};\\ X_{a_1\cdots a_n}\cap U_j=\emptyset}}\lambda(X_{a_1\cdots a_n})\log\left(\frac{1}{\lambda(X_{a_1\cdots a_n})}\right).$$

Thus

$$-\sum_{\substack{X_{a_1\cdots a_n};\\ X_{a_1\cdots a_n}\cap U_j\neq\emptyset}}\lambda(X_{a_1\cdots a_n})\log\lambda(X_{a_1\cdots a_n}\cap U_j)$$
$$\leqq\int_X\log|\det DT^n(x)|d\lambda(x)+\log\frac{C_n}{L}+\sum_{\substack{X_{a_1\cdots a_n};\\ X_{a_1\cdots a_n}\cap U_j=\emptyset}}\lambda(X_{a_1\cdots a_n})\log\lambda(X_{a_1\cdots a_n})$$
$$<\int_X\log|\det DT^n(x)|\,d\lambda+\log\frac{C_n}{L}$$

for all U_j. Note that $(d\mu/d\lambda)\geqq G^{-1}$ and hence

$$\int_X\log|\det DT^n(x)|\,d\lambda\leqq G\cdot n\int_X\log|\det DT(x)|\,d\mu(x)\,.$$

This implies the statement of (3.2-b).

From (3.1), we have

$$\sum_a\lambda(X_a)\log\lambda(X_a)\leqq\sum_a\lambda(X_a)(\log G+\log\mu(X_a))$$
$$\leqq(\sum_a\lambda(X_a))\log G+\frac{1}{M(1)}\sum\mu(X_a)\log\mu(X_a)\,.$$

Therefore, (3.2-b) and the inequality

$$-\sum_a\lambda(X_a)\log\lambda(X_a)+\log G\geqq-\frac{1}{M(1)}\sum\mu(X_a)\log\mu(X_a)$$

imply (3.2-c). □

From now on, using these lemmas we prove Theorem 2. Since

$$\frac{1}{n}\log\left\{\frac{\inf|\det D\Psi_{a_1\cdots a_n(x)}(y)|}{\sup_{y\in T^nX_{a_1\cdots a_n}}|\det D\Psi_{a_1\cdots a_n(x)}(y)|}\right\}\leqq\frac{1}{n}\log\left\{\frac{|\det D\Psi_{a_1\cdots a_n(x)}(T^nx)|}{\lambda(X_{a_1\cdots a_n(x)})}\right\}$$
$$\leqq\frac{1}{n}\log\left\{\frac{\sup|\det D\Psi_{a_1\cdots a_n(x)}(y)|}{L\cdot\inf_{y\in T^nX_{a_1\cdots a_n}}|\det D\Psi_{a_1\cdots a_n(x)}(y)|}\right\}$$

(where $X_{a_1\cdots a_n(x)}$ denotes a cylinder of rank n containing x), by the Condition (C.7) we have

$$-\frac{1}{n}\log O(n)\leqq\frac{1}{n}\log\left(\frac{|\det D\Psi_{a_1\cdots a_n(x)}(T^n x)|}{\lambda(X_{a_1\cdots a_n(x)})}\right)\leqq\frac{1}{n}\log\frac{1}{L}+\frac{1}{n}\log O(n)\ ,$$

and hence

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{\lambda(X_{a_1\cdots a_n(x)})}=\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{|\det D\Psi_{a_1\cdots a_n(x)}(T^n x)|}\ .$$

Using the relation

$$\frac{1}{|\det D\Psi_{a_1\cdots a_n(x)}(T^n x)|}=|\det DT^n(x)|\ ,$$

we obtain

$$\lim_{n\to\infty}\frac{1}{\lambda(X_{a_1\cdots a_n(x)})}=\lim_{n\to\infty}\frac{1}{n}\log|\det DT^n(x)|\quad \text{for a.e. } x\ ,$$

therefore by the ergodic theorem

$$\lim_{n\to\infty}\frac{1}{\lambda(X_{a_1\cdots a_n(x)})}=\int_X\log|\det DT(x)|\,d\mu(x)\quad \text{for a.e. } x\ .$$

We remark that Lemma 3.1 allows us to have

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{\mu(X_{a_1\cdots a_n(x)})}{\lambda(X_{a_1\cdots a_n(x)})}=0\quad \text{for a.e. } x\in X\ .$$

This implies

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{\mu(X_{a_1\cdots a_n(x)})}=\int_X\log|\det DT(x)|\,d\mu(x)\quad \text{for a.e. } x\in X\ .$$

On the other hand, by Lemma 3.2, the Shannon-McMillan theorem ([1], [3]) allows us to conclude

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{\mu(X_{a_1\cdots a_n(x)})}=h(T)\quad \text{for a.e. } x\in X\ .$$

Therefore we have

$$h(T)=\int_X\log|\det DT(x)|\,d\mu(x)\ . \qquad \square$$

§4. Proof of Theorem 3.

In order to prove the main theorem, we construct an invertible extension of T as follows; let Z be the set of sequences of I, $z=(z_{-1}, z_{-2}, \cdots, z_{-n}, \cdots)$. We call the system $(Y, \bar{T})$ the extension of (X, T), where Y is the subset of $X \times Z$ composed of all pairs $(x; z)$ such that, for all $n>0$, there exists x_{-n} in X satisfying

$$Tx_{-1}=x , \qquad Tx_{-n}=x_{-n+1} , \qquad x_{-n} \in X_{z_{-n}} ,$$

and $\bar{T}$ is defined by

$$\bar{T}(x; z)=(x'; z') , \quad \text{with} \quad x'=Tx , \qquad z'_{-n}=z_{-n+1}$$

for all $n>1$ and z'_{-1} is the unique index for which $x \in X_{z'_{-1}}$.
It is easy to see that the projection π onto X commutes with the map T and that, for any invariant measure μ on X, there exists a unique invariant measure $\bar{\mu}$ on Y whose image by π is μ. If the measure μ is ergodic, so is $\bar{\mu}$. Many basic results about the natural extension of an endomorphism were stated in Rohlin's paper [13].

Now, we prepare some notations. Let $\bar{X}_a=\pi^{-1}X_a$, $\bar{Q}=\{\bar{X}_a\}$, $\xi=\vee_{i=1}^{\infty} \bar{T}^i\bar{Q}$, and $\eta=\bar{Q}^{\vee}\xi$. Then, we prove a property for conditional measures in the Rohlin decomposition, with respect to η, from which weak Bernoulli property for T follows. Throughout this section, we suppose the assumptions of Theorem 3 are valid.

LEMMA 4.1. *The measurable partition $\eta=\bar{Q}^{\vee}\xi$ has the following properties*:

(4.1-a) $\bar{T}^{-1}\eta \geqq \eta$,
(4.1-b) $\vee_{n=-\infty}^{\infty} \bar{T}^{-n}\eta=\varepsilon$,
(4.1-c) $\wedge_{n=-\infty}^{\infty} \bar{T}^{-n}\eta=\nu$,
(4.1-d) $h(\bar{T})=H(\bar{T}^{-1}\eta \mid \eta)$.

PROOF. From the definition of η it is immediate that η is a measurable partition, and (4.1-a) and (4.1-b) hold. By Theorem 1, we know that $\bar{T}$ is a Kolmogorov automorphism. From this and (3.2-c) of Lemma 3.2, we have (4.1-c). (4.1-d) is an immediate consequence of (4.1-b). □

Let

$$H_1=\min_{X_a \in \mathscr{D}_1} \left\{\frac{\lambda(X_a)}{C_1}\right\} \quad \text{and} \quad H_2=\frac{1}{\inf_{x \in X} |\det DT(x)|} .$$

Then we have, for any $X_a \in \mathscr{D}_1$,

$$\text{(4.1)} \qquad H_1 \leqq |\det D\Psi_a(x)| \leqq H_2 , \quad x \in TX_a .$$

Define for any $y=(x; z)$ and $y'=(x'; z)$

$$\Delta(y, y') \equiv \limsup_{j\to\infty} \prod_{n=1}^{j} \left| \frac{\det DT(\pi(\bar{T}^{-n}y))}{\det DT(\pi(\bar{T}^{-n}y'))} \right| .$$

Let

$$\mathscr{F}^{(n)} \equiv \{(a_{-1}, a_{-2}, \cdots a_{-n}): X_{a_{-n}, a_{-n+1}\cdots a_{-1}} \in \mathscr{D}_n\} ,$$

$$\alpha^{(n)} \equiv \{(a_{-1} \cdots a_{-n}): (a_{-1} \cdots a_{-n+1}) \in \mathscr{F}^{(n-1)}, (a_{-1} \cdots a_{-n}) \notin \mathscr{F}^{(n)}\} ,$$

$$F^n \equiv \bigcup_{(a_{-1}\cdots a_{-n}) \in \mathscr{F}^{(n)}} (\bar{T}\bar{X}_{a_{-1}} \cap \bar{T}^2\bar{X}_{a_{-2}} \cap \cdots \cap \bar{T}^n\bar{X}_{a_{-n}}) ,$$

$$A^n = \bigcup_{(a_{-1}\cdots a_{-n}) \in \alpha^{(n)}} (\bar{T}\bar{X}_{a_{-1}} \cap \cdots \cap \bar{T}^n\bar{X}_{a_{-n}}) .$$

We note that, from the Condition (C.3), $(a_{-1} \cdots a_{-n}) \in \mathscr{F}^{(n)}$ implies $(a_{-1} \cdots a_{-k}) \in \mathscr{F}^{(k)}$ for all $k(1 \leqq k \leqq n)$. With the above definitions, we show the following lemmas.

Lemma 4.2. *If $(a_{-1} \cdots a_{-n}) \in \alpha^{(n)}$, then there exists an integer $i(0 \leqq i \leqq k_0-1)$ such that $X_{a_{-n}a_{-n+1}\cdots a_{-n+i}} \in R(C.T)$.*

Proof. If $n \leqq k_0$, then it is trivial. Let $n > k_0$, and suppose that for all $i(0 \leqq i \leqq k_0-1)$

$$X_{a_{-n}\cdots a_{-n+i}} \notin R(C.T) . \tag{4.2.1}$$

Then we have $X_{a_{-n}\cdots a_{-n+k_0-1}} \in \mathscr{D}_{k_0}$. On the other hand, from (C.9) if $X_{a_{-n+1}\cdots a_{-1}} \in \mathscr{D}_{n-1}$ and $X_{a_{-n}\cdots a_{-1}} \in \mathscr{D}_n^c$, then

$$X_{a_{-n}\cdots a_{-1}} \subset \bigcup_{j=1}^{k_0} B_j .$$

This contradicts (4.2.1). □

Lemma 4.3. $\lim_{n\to\infty} \bar{\mu}(F^n)=0$, *and therefore* $Y=\bigcup_{n=1}^{\infty} A^n$ $(\bar{\mu} \bmod 0)$.

Proof. We note that $\bar{\mu}(F^n)=\mu(D_n)$. From (C.4) and relations

$$\mu(D_n) = \sum_{m=0}^{\infty} \Bigg(\sum_{X_{k(m)} \in \mathscr{D}_m} \Bigg(\sum_{X_{a(n)} \in \mathscr{D}_n} \nu(X_{k(m)a(n)}) \Bigg)\Bigg)$$
$$\leqq \sum_{m=0}^{\infty} \Bigg(\sum_{X_{k(m)} \in \mathscr{D}_m} \int_{D_n \cap T^m X_{k(m)}} |\det D\Psi_{k(m)}(x)| \, d\lambda(x) \Bigg) \cdot G ,$$

we have $\lim_{n\to\infty} \bar{\mu}(F^n)=0$. (Here $k(m)=(k_1 \cdots k_m)$, and $a(n)=(a_1 \cdots a_n)$). And hence, the equality

$$\bar{\mu}\Big(\Big(\bigcup_{n=1}^{\infty} A^n\Big)^c\Big)=\mu\Big(\Big(\bigcup_{n=1}^{\infty} B_n\Big)^c\Big)$$

implies the second assertion. □

For each k with $0\leqq k\leqq k_0-1$, we define for $n>k$

$$\alpha_k^{(n)}=\{(a_{-1}\cdots a_{-n})\in\alpha^{(n)}: X_{a_{-1}\cdots a_{-n+k}}\in R(C.T)\},$$

and

$$A_k^n=\bigcup_{(a_{-1}\cdots a_{-n})\in\alpha_k^{(n)}}(\bar{T}\bar{X}_{a_{-1}}\cap\cdots\cap\bar{T}^n\bar{X}_{a_{-n}}).$$

Then by Lemma 4.2 we can easily see that

$$\text{for}\quad n>k_0\ \ \alpha^{(n)}=\bigcup_{k=0}^{k_0-1}\alpha_k^{(n)},\quad A^n=\bigcup_{k=0}^{k_0-1}A_k^n$$

and

$$\text{for}\quad n\leqq k_0\ \ \alpha^{(n)}=\bigcup_{k=0}^{n-1}\alpha_k^{(n)},\quad A^n=\bigcup_{k=0}^{n-1}A_k^n,$$

where the above unions are disjoint.

LEMMA 4.4. *For $\bar{\mu}$ a.e. $y\in Y$, there exists a positive integer $K=K(y)$ such that*

for any $y'\in\xi(y)$ we have

$$\frac{1}{C}\cdot\Big(\frac{H_1}{H_2}\Big)^{K(y)}\leqq\Delta(y,y')\leqq C\cdot\Big(\frac{H_2}{H_1}\Big)^{K(y)},$$

and so

$$\frac{1}{C}\Big(\frac{H_1}{H_2}\Big)^{K(y)}\int_{\eta(y)}dy'\leqq\int_{\eta(y)}\Delta(y,y')dy'$$
$$\leqq C\Big(\frac{H_2}{H_1}\Big)^{K(y)}\int_{\eta(y)}dy',$$

where $\xi(y)$ and $\eta(y)$ denote the elements of ξ and η containing y respectively, and d denotes the natural Lebesgue measure on each element of η.

PROOF. For $y\in A_k^n$ from (C.3) and (4.1) we can easily see that for any $y'\in\xi(y)$

$$\frac{1}{C}\Big(\frac{H_1}{H_2}\Big)^{n-k-1}\leqq\Delta(y,y')\leqq C\Big(\frac{H_2}{H_1}\Big)^{n-k-1}.$$

Putting $K(y)=n-k-1$, we have the statement of Lemma 4.4 immediately. □

As we have announced in introduction, our main goal of this section is to prove

PROPOSITION 4.5. *The conditional measures of $\bar{\mu}$ with respect to the partition η are given by*

$$q(y, B)=\frac{\int_{B\cap\eta(y)} \Delta(y, y')dy'}{\int_{\eta(y)} \Delta(y, y')dy'}$$

for each $y \in Y$ and B a measurable subset.

To prove this proposition, we have to show the following:

LEMMA 4.6. *For each $n>0$, $\log q(y, [\bar{T}^{-n}\eta](y))$ is $\bar{\mu}$-integrable.*

REMARK 5. With this done, we can prove Proposition 4.5 as Ledrappier did in [6]. Therefore, we only give the outline of the argument used for this part. In fact, since the following equality is valid:

$$\log q(y, [\bar{T}^{-n}\eta](y))=-\log|\det DT^n(\pi y)|+\log\{k\circ \bar{T}^n(y)\}-\log k(y)\ ,$$

where $k(y)=\int_{\eta(y)} \Delta(y, y')dy'$, Lemma 4.6 allows us to apply the next classical lemma to our case;

LEMMA (cf. [6]). *Let $(Y, \bar{\mu}, \bar{T})$ be a dynamical system and g_i, $i=1, 2, 3$, be functions related by $g_1=g_2+g_3\circ\bar{T}-g_3$, with g_2 and g_1 integrable. Then we have*

$$\lim \frac{1}{n} g_3\circ \bar{T}^n=\hat{g}_1-\hat{g}_2=0 \quad \bar{\mu} \ a.e.\ ,$$

where $\hat{g}$ denotes the point-wise limit of ergodic averages of g.
Then we can see that the following relation is true:

$$-\int \log q(y, [\bar{T}^{-n}\eta](y))d\bar{\mu}(y)=\int \log|\det DT^n(x)|\, d\mu(x)$$
$$=n\int \log|\det DT(x)|\, d\mu(x)\ .$$

On the other hand, if $p(y, \cdot)$ is the conditional measure of $\bar{\mu}$ with respect to η, then by Lemma 4.1, (4.1-d) we have

$$n\cdot h(T)=H(\bar{T}^{-n}\eta\,|\,\eta)=-\int \log p(y, [\bar{T}^{-n}\eta](y))d\bar{\mu}(y)\ .$$

Therefore, by the Rohlin's formula we obtain

$$\int \log \frac{q(y, [\bar{T}^{-n}\eta](y))}{p(y, [\bar{T}^{-n}\eta](y))} d\bar{\mu}(y) = 0 .$$

By the concavity of the function log and from (4.1-b), the proposition follows by letting n go to infinity (cf. [7], [11]).

PROOF OF LEMMA 4.6. Let $y \in A_k^m$, $y = (x; a_{-1}, a_{-2}, \cdots a_{-m+k+1}, a_{-m+k} \cdots a_{-m} \cdots)$. Then a simple calculation gives

$$q(y, [\bar{T}^{-n}\eta](y)) = \frac{\int_{[\bar{T}^{-(m-k-1+n)}\eta](\bar{T}^{-(m-k-1)}y)} \Delta(\bar{T}^{-(m-k-1)}y, y')dy'}{\int_{[\bar{T}^{-(m-k-1)}\eta](\bar{T}^{-(m-k-1)}y)} \Delta(\bar{T}^{-(m-k-1)}y, y')dy'} .$$

Since $(1/C) \leqq \Delta(\bar{T}^{-(m-k-1)}y, y') \leqq C$ on $\eta(\bar{T}^{-(m-k-1)}y)$, this implies the following:

$$\begin{aligned} q(y, [\bar{T}^{-n}\eta](y)) &\geqq \frac{1}{C^2} \cdot \frac{\int_{[\bar{T}^{-(m-k-1+n)}\eta](\bar{T}^{-(m-k-1)}y)} dy'}{\int_{[\bar{T}^{-(m-k-1)}\eta](\bar{T}^{-(m-k-1)}y)} dy'} \\ &= \frac{1}{C^2} \cdot \frac{\lambda(X_{a_{-m+k+1}\cdots a_0 a_1 \cdots a_n(\pi(\bar{T}^{-(m-k-1)}y))} \cap U_{j(y)})}{\lambda(X_{a_{-m+k+1}\cdots a_0(\pi(\bar{T}^{-(m-k-1)}y))} \cap U_{j(y)})} \\ &\geqq \frac{1}{C^2} \cdot \frac{\lambda(X_{a_{-m+k+1}\cdots a_0 a_1 \cdots a_n(\pi(\bar{T}^{-(m-k-1)}y))} \cap U_{j(y)})}{\lambda(X_{a_{-m+k+1}\cdots a_0(\pi(\bar{T}^{-(m-k-1)}y))})} , \end{aligned}$$

where $\pi(\bar{T}^k y) \in X_{a_k}$ $(0 \leqq k \leqq n)$ and

$$U_{j(y)} \equiv \bigcap_{j=0}^{\infty} T^{j+1} X_{a_{-m+k-j}\cdots a_{-m+k}} .$$

Note that there exists $X_{b_1\cdots b_l}$ such that $U_{j(y)} = T^l X_{b_1\cdots b_l}$ and hence we can write $X_{a_{-m+k+1}\cdots a_{-1}} \cap U_{j(y)} = T^l(X_{b_1\cdots b_l a_{-m+k+1}\cdots a_{-1}})$.

Then we can see that

$$\begin{aligned} &T^{m-k-1}(X_{a_{-m+k+1}\cdots a_{-1}a_0\cdots a_n} \cap U_{j(y)}) \\ &\quad = T^{m-k-1}(T^l X_{b_1\cdots b_l a_{-m+k+1}\cdots a_{-1}} \cap T^{-(m-k-1)} X_{a_0\cdots a_n}) \\ &\quad = U_{t(y)} \cap X_{a_0\cdots a_n(y)} \end{aligned}$$

for some $t(y) \in \{0, 1, \cdots, N\}$. Using this, we have

$$\begin{aligned} &\lambda(X_{a_{-m+k+1}\cdots a_{-1}a_0\cdots a_n} \cap U_{j(y)}) \\ &\quad = \lambda(\Psi_{a_{-m+k+1}\cdots a_{-1}} \circ T^{m-k-1}(X_{a_{-m+k+1}\cdots a_n} \cap U_{j(y)})) \\ &\quad = \int_{U_{t(y)} \cap X_{a_0\cdots a_n(y)}} |\det D\Psi_{a_{-m+k+1}\cdots a_{-1}}(x')| \, d\lambda(x') \end{aligned}$$

$$\geqq \inf_{x' \in U_{t(y)} \cap X_{a_0 \cdots a_n(y)}} |\det D\Psi_{a_{-m+k+1} \cdots a_{-1}}(x')| \cdot \lambda(U_{t(y)} \cap X_{a_0 \cdots a_n(y)}) .$$

On the other hand, the following inequality is true:

$$\begin{aligned}\lambda(X_{a_{-m+k+1} \cdots a_{-1} a_0}) &= \int_{X_{a_0} \cap T^{m-k-1} X_{a_{-m+k+1} \cdots a_{-1}}} |\det D\Psi_{a_{-m+k+1} \cdots a_{-1}}(x')| d\lambda(x') \\ &\leqq \sup_{x' \in X_{a_0} \cap T^{m-k-1} X_{a_{-m+k+1} \cdots a_{-1}}} |\det D\Psi_{a_{-m+k+1} \cdots a_{-1}}(x')| \cdot 1 \\ &\leqq \sup_{x' \in T^{m-k-1} X_{a_{-m+k+1} \cdots a_{-1}}} |\det D\Psi_{a_{-m+k+1} \cdots a_{-1}}(x')| \cdot\end{aligned}$$

Combining the above results, we obtain

$$\begin{aligned}&q(y, [\bar{T}^{-n}\eta](y)) \\ &\geqq \frac{1}{C^2} \frac{\inf_{x' \in T^{m-k-1} X_{a_{-m+k+1} \cdots a_{-1}}} |\det D\psi_{a_{-m+k+1} \cdots a_{-1}}(x')|}{\sup_{x' \in T^{m-k-1} X_{a_{-m+k+1} \cdots a_{-1}}} |\det D\psi_{a_{-m+k+1} \cdots a_{-1}}(x')|} \lambda(U_{t(y)} \cap X_{a_0 \cdots a_n(y)}) ,\end{aligned}$$

and therefore by the Condition (C.7)

$$q(y, [\bar{T}^{-n}\eta](y)) \geqq \begin{cases} \dfrac{1}{C^2} \lambda(X_{a_0 \cdots a_n(y)} \cap U_{t(y)}) & \text{if } y \in A_0^1 \\ \dfrac{1}{C^2} \lambda(X_{a_0 \cdots a_n(y)} \cap U_{t(y)}) \cdot O\Big(\dfrac{1}{(m-k-1)^l}\Big) & \\ \qquad \geqq \dfrac{1}{C^2} \lambda(X_{a_0 \cdots a_n(y)} \cap U_{t(y)}) \cdot O\Big(\dfrac{1}{(m-1)^l}\Big) & \text{if } y \in A_k^m (m \geqq 2) . \end{cases}$$

From this and the equality

$$\int_Y \log q(y, [\bar{T}^{-n}\eta](y)) d\bar{\mu}(y) = \sum_{m=1}^{\infty} \int_{A^m} \log q(y, [\bar{T}^{-n}\eta](y)) d\bar{\mu}(y) ,$$

it follows that

$$\begin{aligned}&\int_Y \log q(y, [\bar{T}^{-n}\eta](y)) d\bar{\mu}(y) \\ &\geqq \int_{A_0^1} \Big\{\log \frac{1}{C^2} + \log \lambda(X_{a_0 \cdots a_n(y)} \cap U_{t(y)})\Big\} d\bar{\mu}(y) \\ &\quad + \sum_{m=2}^{\infty} \int_{A^m} \Big\{\log \frac{1}{C^2} + \log \lambda(X_{a_0 \cdots a_n(y)} \cap U_{t(y)}) + \log O\Big(\frac{1}{(m-1)^l}\Big)\Big\} d\bar{\mu}(y) \\ &\geqq 2\Big\{\Big(\log \frac{1}{C^2}\Big) \times \bar{\mu}(Y) + \int_Y \log \lambda(X_{a_0 \cdots a_n(y)} \cap U_{t(y)}) d\bar{\mu}(y)\Big\} \\ &\quad + \sum_{m=2}^{\infty} \bar{\mu}(A^m) \log\Big(O\Big(\frac{1}{(m-1)^l}\Big)\Big) .\end{aligned}$$

Now, we estimate $\int_Y \log \lambda(X_{a_0\cdots a_n(y)} \cap U_{t(y)})d\bar{\mu}(y)$. For each $i \in \{0, \cdots, N\}$, let $\mathscr{U}_i = \{y \in Y: U_{t(y)} = U_i\}$, where

$$U_{t(y)} = T^{m-k-1}(U_{j(\bar{T}^{-m+k+1}y)} \cap X_{a_{-m+k+1}\cdots a_{-1}}) \quad \text{for } y \in A_k^m .$$

Then we have

$$\begin{aligned}
&\int_Y \log \lambda(X_{a_0\cdots a_n(y)} \cap U_{t(y)})d\bar{\mu}(y)\\
&\quad = \sum_{i=0}^{N} \int_{\mathscr{U}_i} \log \lambda(X_{a_0\cdots a_n(y)} \cap U_i)d\bar{\mu}(y)\\
&\quad \geqq \sum_{i=0}^{N} \int_Y \log \lambda(X_{a_0\cdots a_n(y)} \cap U_i)d\bar{\mu}(y)\\
&\quad = \sum_{i=0}^{N} \Big(\sum_{(a_0\cdots a_n) \in A(n)} \mu(X_{a_0\cdots a_n})\log \lambda(X_{a_0\cdots a_n} \cap U_i)\Big)\\
&\quad \geqq M(n+1) \sum_{i=0}^{N} \Big(\sum_{(a_0\cdots a_n) \in A(n)} \lambda(X_{a_0\cdots a_n})\log \lambda(X_{a_0\cdots a_n} \cap U_i)\Big) ,
\end{aligned}$$

(the last inequality follows from Lemma 3.1). Therefore by (3.2-b) of Lemma 3.2 $\int_Y \log \lambda(X_{a_0\cdots a_n(y)} \cap U_{t(y)})d\bar{\mu}(y) > -\infty$.

Next, we estimate $\sum_{m=2}^{\infty} \bar{\mu}(A^m)\log(1/O\{(m-1)^l\})$. By Lemma 3.1, we obtain

$$\begin{aligned}
\bar{\mu}(A^m) &= \sum_{(a_{-1}\cdots a_{-m}) \in \alpha^{(m)}} \mu(X_{a_{-m}\cdots a_{-1}})\\
&\leqq \hat{M}(k_0) \sum_{(a_{-1}\cdots a_{-m}) \in \alpha^{(m)}} \lambda(X_{a_{-m}\cdots a_{-1}})\\
&= \hat{M}(k_0)\Big\{ \sum_{(a_{-1}\cdots a_{-m}) \in \alpha^{(m)}} \int_{TX_{a_{-m}} \cap X_{a_{-m+1}\cdots a_{-1}}} |\det D\Psi_{a_{-m}}(x)|d\lambda(x)\Big\}
\end{aligned}$$

and by (C.7)

$$\begin{aligned}
\bar{\mu}(A^m) &\leqq \hat{M}(k_0)\{ \sum_{(a_{-1}\cdots a_{-m}) \in \alpha^{(m)}} C_1 \cdot \inf_{x \in TX_{a_{-m}}} |\det D\Psi_{a_{-m}}(x)| \cdot \lambda(X_{a_{-m+1}\cdots a_{-1}})\}\\
&\leqq \hat{M}(k_0) \cdot C_1\Big\{ \sum_{(a_{-1}\cdots a_{-m}) \in \alpha^{(m)}} \int_{TX_{a_{-m}}} |\det D\Psi_{a_{-m}}(x)|d\lambda \cdot \frac{1}{L}\lambda(X_{a_{-m+1}\cdots a_{-1}})\Big\}\\
&= \frac{\hat{M}(k_0)}{L} \cdot C_1\{ \sum_{a_{-m}} \Big(\sum_{\substack{(a_{-m+1}\cdots a_{-1}):\\ (a_{-m}\cdots a_{-1}) \in \alpha^{(m)}}} \lambda(X_{a_{-m+1}\cdots a_{-1}})\Big)\lambda(X_{a_{-m}})\}\\
&\leqq \frac{\hat{M}(k_0)C_1}{L} \cdot \lambda(D_{m-1})\{ \sum_{a_{-m}} \lambda(X_{a_{-m}})\}\\
&\leqq \frac{\hat{M}(k_0)}{L} \cdot C_1 \cdot \lambda(D_{m-1}) .
\end{aligned}$$

Therefore by (C.4)* we have $\sum_{m=2}^{\infty} \bar{\mu}(A^m)\log\{1/O((m-1)^l)\} > -\infty$. Consequently we have the conclusion of Lemma 4.6. □

Proposition 4.5 implies that the conditional measures of $\bar{\mu}$ with respect to η are all absolutely continuous with respect to $\bar{\mu}$. Note that by Theorem 2 $\bar{T}$ is a Kolmogorov automorphism and hence the Pinsker partition $\wedge_{n=-\infty}^{\infty} \vee_{k=-\infty}^{n} \bar{T}^k\bar{Q}$ is trivial. Combining these results, we see that the sufficient condition for the weak Bernoulli partition is satisfied, i.e. the conditional measures of $\bar{\mu}$ with respect to η must all coincide on the remote past σ algebra $\sigma(\wedge_{n=-\infty}^{\infty} \vee_{k=-\infty}^{-n} \bar{T}^k\bar{Q})$, and this property of the conditional measures implies that Q is a weak Bernoulli partition. More detailed proof may be found for example in [6] [12] (cf. [8] [10]).

§5. Examples.

In this section, we present three examples to which we can apply our theorem. First we consider the following one parameter family of maps on an interval.

EXAMPLE 1. Let $X=[0, 1]$, and for α with $0<\alpha<1$ define

$$f_\alpha(x)=\begin{cases} \dfrac{x}{(1-x^\alpha)^{1/\alpha}}\,; & x<\left(\dfrac{1}{2}\right)^{1/\alpha} \\ \dfrac{1}{\left(\frac{1}{2}\right)^{1/\alpha}-1}x+\dfrac{1}{\left(1-\left(\frac{1}{2}\right)^{1/\alpha}\right)}\,; & x\geqq\left(\dfrac{1}{2}\right)^{1/\alpha} \end{cases} \quad \text{(see Figure 1)}\,.$$

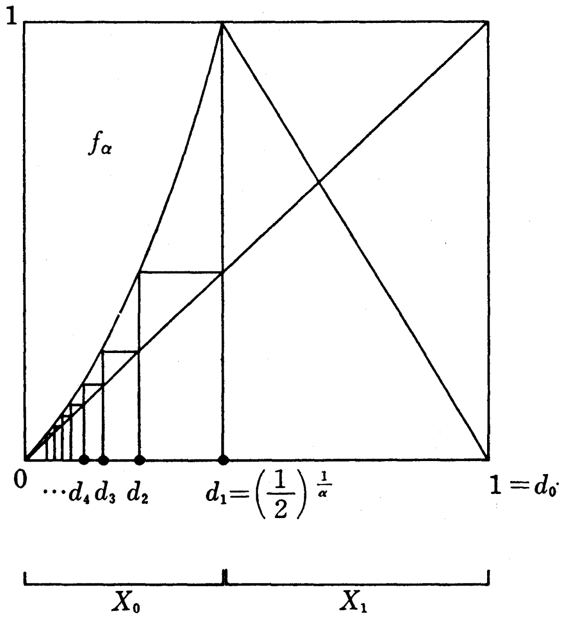


FIGURE 1

Let $X_0=[0, (1/2)^{1/\alpha})$, $X_1=[(1/2)^{1/\alpha}, 1]$, $\Psi_0(x)=x/((1+x^\alpha)^{1/\alpha})$, $\Psi_1(x)=\{(1/2)^{1/\alpha}-1\}x+1$. Then simple calculation gives the following properties:

(1) $(\Psi_0)'(x)$ is a positive and monotone decreasing function on [0, 1],

(2) $((\Psi_0)''/(\Psi_0)')(x)$ is a negative and monotone increasing function on [0, 1]. Let $X_{\underbrace{00\cdots01}_{n}}=[d_n, d_{n-1})$. From (1), we have

$$\Psi_0'(d_{k-1})\leqq\Psi_0'(x)\leqq\Psi_0'(d_k) \quad \text{for any} \quad x\in X_{\underbrace{00\cdots01}_{k}}.$$

Therefore

$$(3) \qquad \frac{\sup_{x\in X}|(\Psi_{00\cdots01})'(x)|}{\inf_{x\in X}|(\Psi_{\underbrace{00\cdots01}_{n}})'(x)|}\leqq\frac{\Psi_0'(d_n)}{\Psi_0'(1)}\leqq 2^{1+1/\alpha}.$$

We remark that

(4) if

$$n\geqq\left[\frac{2(2^{1/\alpha}-1)^{\alpha/(\alpha+1)}-2^{1/(\alpha+1)}}{2^{1/(\alpha+1)}-(2^{1/\alpha}-1)^{\alpha/(\alpha+1)}}+1\right]$$

(where [ ] denotes the integer part of a number), then

$$|\Psi_1'|<\inf_{x\in X_{\underbrace{00\cdots01}_{n}}}|\Psi_0'(x)|,$$

and

(5) for any sequence $(a_1\cdots a_n)\in A(n)$ such that $a_k=1$ for some $k\in\{1, 2, \cdots, n-1\}$, we have $d_{n-1}\leqq\inf_{x\in X_{a_1\cdots a_n}}\{x\}$.

From the above properties, we can show that there exists a constant $C>1$ such that for all $n>0$ and any $(a_1\cdots a_n)\in A(n)$ with $a_n=1$

$$\frac{\sup_{x\in X}|(\Psi_{a_1\cdots a_n})'(x)|}{\inf_{x\in X}|(\Psi_{a_1\cdots a_n})'(x)|}<C.$$

In fact, by the mean value theorem for any $\theta, \theta'\in[0, 1]$ such that $\theta'-\theta>0$, we have

(6)

$$\log\left|\frac{(\Psi_{00\cdots01})'(\theta')}{(\Psi_{\underbrace{00\cdots01}_{n}})'(\theta)}\right|=\sum_{j=1}^{n-1}\left|\left(\frac{\Psi_0''}{\Psi_0'}\right)(\xi_j(\theta, \theta'))\right||(\psi_{\underbrace{00\cdots01}_{n-j}})'(\eta_j(\theta, \theta')|\cdot(\theta'-\theta)$$

where $\xi_j(\theta, \theta')\in X_{\underbrace{00\cdots01}_{n-j}}$, $\eta_j(\theta, \theta')\in X$.

On the other hand, for any $x, x' \in [0, 1]$ Properties (1), (2), (3), (4), and (5) allow us to have the following:

(7)

$$\left|\log\left|\frac{(\Psi_{a_1\cdots a_n})'(x')}{(\Psi_{a_1\cdots a_n})'(x)}\right|\right| \leqq \sum_{j=1}^{n-1}\left|\left(\frac{\Psi''_{a_j}}{\Psi'_{a_j}}\right)(\xi_j(x, x'))\right|\cdot|(\Psi_{a_{j+1}\cdots a_n})'(\eta_j(x, x'))|\cdot|x'-x|$$

$$\leqq \sum_{j=1}^{n-1} \inf_{x\in X_{\underbrace{00\cdots 01}_{n-j}}}\left|\frac{\Psi''_0}{\Psi'_0}(x)\right| \inf_{x\in X_{\underbrace{00\cdots 01}_{n-j-1}}} \Psi'_0(x)\cdots \inf_{x\in X}\Psi'_0(x)\left(\frac{|\Psi'_1|}{\Psi'_0(1)}\right)^{n_0},$$

where

$$n_0=\left[\frac{2(2^{1/\alpha}-1)^{\alpha/(\alpha+1)}-2^{1/(\alpha+1)}}{2^{1/(\alpha+1)}-(2^{1/\alpha}-1)^{\alpha/(\alpha+1)}}+1\right].$$

Combining (5) and (6), we can take for C $2^{(1+1/\alpha)\cdot 2\cdot(2^{1/\alpha}-1)}$. It is easy to see that (C.3) is satisfied and $B_n=X_{\underbrace{00\cdots 01}_{n}}$, and $D_n=X_{\underbrace{00\cdots 0}_{n}}$. To verify (C.1), we remark that for all $n\geqq n_0$,

$$\lambda(X_{a_1\cdots a_n})\leqq C\left(\frac{|\Psi'_1|}{\Psi'_0(1)}\right)^{n_0}\cdot\lambda(X_{\underbrace{00\cdots 0}_{n}}).$$

In fact, for any $(a_1\cdots a_n)\in A(n)$, there exists k with $0\leqq k\leqq n$ such that $X_{a_1\cdots a_k}\in R(C.f_\alpha)$ and $X_{a_{k+1}\cdots a_n}=D_{n-k}$. Therefore

$$\lambda(X_{a_1\cdots a_n})=\int_{X_{\underbrace{00\cdots 0}_{n-k}}}|(\Psi_{a_1\cdots a_k})'(x)|dx$$

$$<C\cdot \inf_{x\in X_{\underbrace{00\cdots 0}_{n-k}}}|(\Psi_{a_1\cdots a_k})'(x)|\cdot\lambda(X_{\underbrace{00\cdots 0}_{n-k}})$$

$$<C\left(\frac{|\Psi'_1|}{\Psi'_0(1)}\right)^{n_0}\cdot\lambda(X_{\underbrace{00\cdots 0}_{n-k}})(\inf_{x\in X_{\underbrace{0\cdots 0}_{n-1}}}\Psi'_0(x)\cdot \inf_{x\in X_{\underbrace{0\cdots 0}_{n-2}}}\Psi'_0(x)\cdots \inf_{x\in X_{\underbrace{0\cdots 0}_{n-k}}}\Psi'_0(x)).$$

On the other hand

$$\lambda(X_{\underbrace{00\cdots 0}_{n}})=\int_{X_{\underbrace{0\cdots 0}_{n-k}}}|\Psi_{\underbrace{0\cdots 0}_{k}}{}'(x)|dx$$

$$\geqq(\inf_{x\in D_{n-1}}\Psi'_0(x)\cdot \inf_{x\in D_{n-2}}\Psi'_0(x)\cdots \inf_{x\in D_{n-k}}\Psi'_0(x))\cdot\lambda(X_{\underbrace{0\cdots 0}_{n-k}}),$$

consequently the above assertion is valid. From this and the fact $\lim_{n\to\infty}\lambda(X_{\underbrace{0\cdots 0}_{n}})=0$, we have (C.1). The Conditions (C.2), (C.4)*, (C.6) and

(C.9) are easily checked. Note that $\Psi_{\underbrace{0\cdots0}_{m}}(x)=x/(1+mx^{\alpha})^{1/\alpha}$ and hence $(\Psi_{0\cdots0})'(x)=1/(1+mx^{\alpha})^{1+1/\alpha}$. Then a direct calculation gives

$$W_n=\sum_{m=0}^{\infty}\frac{1}{(1+m(1/(n+1)^{1/\alpha})^{\alpha})^{1+1/\alpha}}$$
$$\leqq(n+1)^{1+1/\alpha}\left(\sum_{m=0}^{\infty}\frac{1}{(1+m)^{1+1/\alpha}}\right),$$

and

$$\frac{\sup_{x\in X}(\Psi_{0\cdots0})'(x)}{\inf_{x\in X}(\Psi_{\underbrace{0\cdots0}_{m}})'(x)}\leqq(1+m)^{1+1/\alpha}.$$

From these, (C.5) and (C.7) are satisfied. (C.8) and (C.9) are trivial. Therefore this example satisfies all of assumptions of Theorem 3.

Next, we consider a skew product transformation which is associated to Diophantine approximation in inhomogeneous linear class.

EXAMPLE 2. Let $X=\{(x, y)\in R^2: 0\leqq y\leqq 1,\ -y\leqq x<-y+1\}$ and define T on X by

$$T(x, y)=\left(\left(\frac{1}{x}\right)-\left[\frac{1-y}{x}\right]-\left[-\frac{y}{x}\right],\ -\left[-\frac{y}{x}\right]-\frac{y}{x}\right) \quad \text{(see Figure 2)}.$$

This transformation is a multi-dimensional mapping with a finite range structure and has a finite invariant measure whose density is unbounded

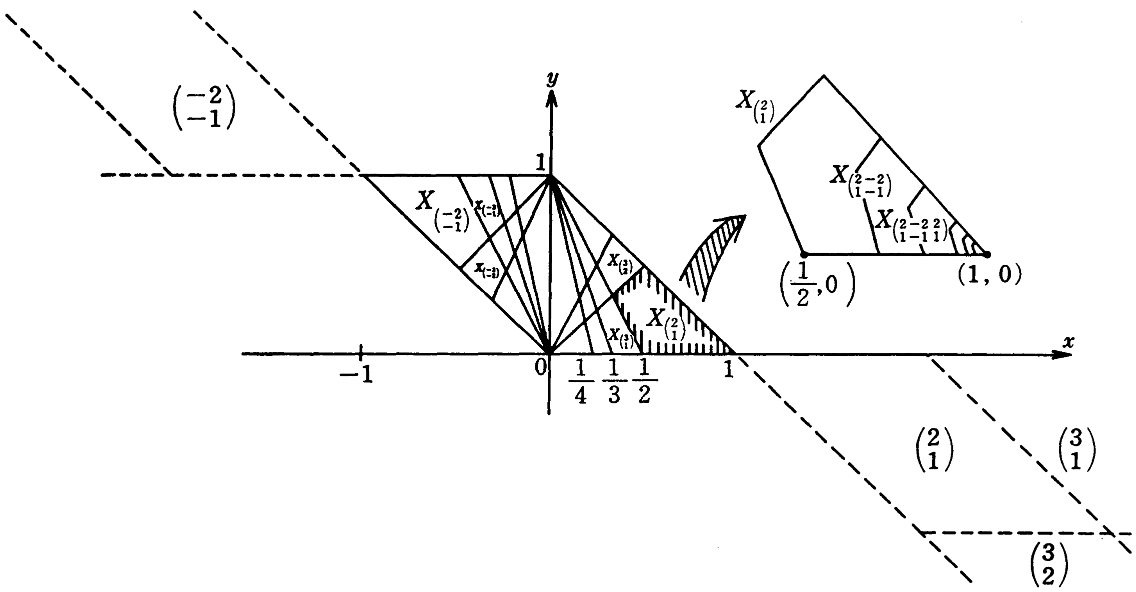


FIGURE 2

(see [4] and [5]).

Let

$$a(x, y)=\left[\frac{1-y}{x}\right]+\left[-\frac{y}{x}\right], \qquad b(x, y)=-\left[-\frac{y}{x}\right],$$

$$a_k(x, y)=a(T^{k-1}(x, y)), \qquad b_k(x, y)=b(T^{k-1}(x, y)),$$

and $q_n=a_nq_{n-1}+q_{n-2}$ ($q_0=1$, $q_1=0$). Then

$$\det D\Psi_{\binom{a_1\cdots a_n}{b_1\cdots b_n}}(x, y)=\frac{1}{(q_n+xq_{n-1})^3}.$$

Moreover, we can easily to see that

$$\mathscr{D}_n=\{X_{\binom{2\ -2\ 2\ -2\cdots}{1\ -1\ 1\ -1\cdots}}, X_{\binom{-2\ 2\ -2\ 2\cdots}{-1\ 1\ -1\ 1\cdots}}\},$$

$$\lambda(D_n)=O\left(\frac{1}{n^2}\right), \qquad |q_n|=O(n) \quad \text{for} \quad \begin{pmatrix}a_1 \cdots a_n\\ b_1 \cdots b_n\end{pmatrix}\in\mathscr{D}_n.$$

Therefore

$$\frac{\sup_{x\in T^nX_{\binom{a_1\cdots a_n}{b_1\cdots b_n}}}|\det D\Psi_{\binom{a_1\cdots a_n}{b_1\cdots b_n}}(x, y)|}{\inf_{x\in T^nX_{\binom{a_1\cdots a_n}{b_1\cdots b_n}}}|\det D\Psi_{\binom{a_1\cdots a_n}{b_1\cdots b_n}}(x, y)|}=O(n^3) \quad \text{for any} \quad X_{\binom{a_1\cdots a_n}{b_1\cdots b_n}}\in\mathscr{D}_n,$$

and if we put $\rho_n=\sup_{(x,y)\in D_n^c}\{|x|\}$, then $W_n\leqq\sum_{m=1}^{\infty}O(1/m^3)\cdot(1/(1-\rho_n)^3)$. Since $\det DT(x, y)=1/x^3$, a direct calculation allows us to verify (C.8). Consequently, all of conditions of Theorem 3 are satisfied.

Finally, we consider a complex continued fraction transformation considered by S. Tanaka [17].

EXAMPLE 3. Let $X=\{z=x\alpha+y\bar{\alpha}: -1/2\leqq x, y\leqq 1/2\}$ ($\alpha=1+i$) and define the transformation T on X by

$$Tz=\frac{1}{z}-\left[\frac{1}{z}\right]_1,$$

where $[z]_1$ denotes $[x+1/2]\alpha+[y+1/2]\bar{\alpha}$ for a complex number $z=x\alpha+y\bar{\alpha}$. Let $I=\{n\alpha+m\bar{\alpha}: m, n\in Z\}\backslash\{0\}$. The map T induces a continued fraction expansion of $z\in X$,

$$z=\frac{1|}{|a_1}+\frac{1|}{|a_2}+\cdots+\frac{1|}{|a_n}+\cdots$$

where each a_i is contained in I. In his paper [17], he obtained the density function of the invariant measure which is unbounded, the ergodicity, and some limiting values by his own method. His method cannot apply to general case. Here, applying our theory we obtain further results, i.e. exactness, Rohlin's formula, and a weak Bernoulli property.

Define U_j $(0\leqq j\leqq 4)$ by

$$U_0=X\,,\qquad U_1=\left\{z\in X:\left|z+\frac{\alpha}{2}\right|\geqq\frac{1}{\sqrt{2}}\right\}\,,\qquad U_2=-i\times U_1\,,$$
$$U_3=-i\times U_2\qquad U_4=-i\times U_3\,.$$

From the above definitions it is easy to see that T is a multidimensional map with a finite range structure, and Conditions (C.2) and (C.8) are satisfied. For (C.1), the detailed proof may be found in [17]. Define $q_n\in I$ $(n\geqq -1)$ inductively by

$$q_{-1}=0\,,\qquad q_0=\alpha\,,\qquad q_n=a_nq_{n-1}+q_{n-2}\quad(n\geqq 1)\,.$$

Then a simple calculation gives the relation

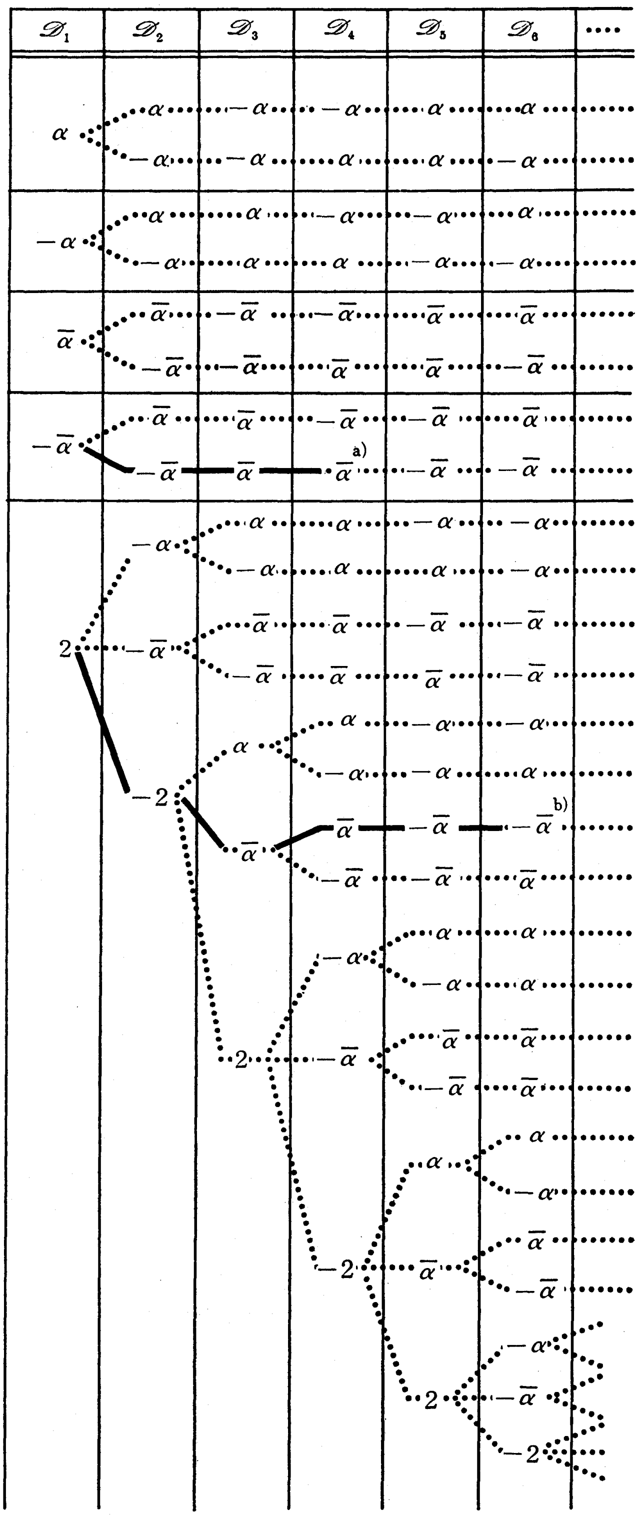
$$|\det D\Psi_{a_1\cdots a_n}(z)|=\frac{1}{|q_n|^4|1+(q_{n-1}/q_n)z|^4}$$

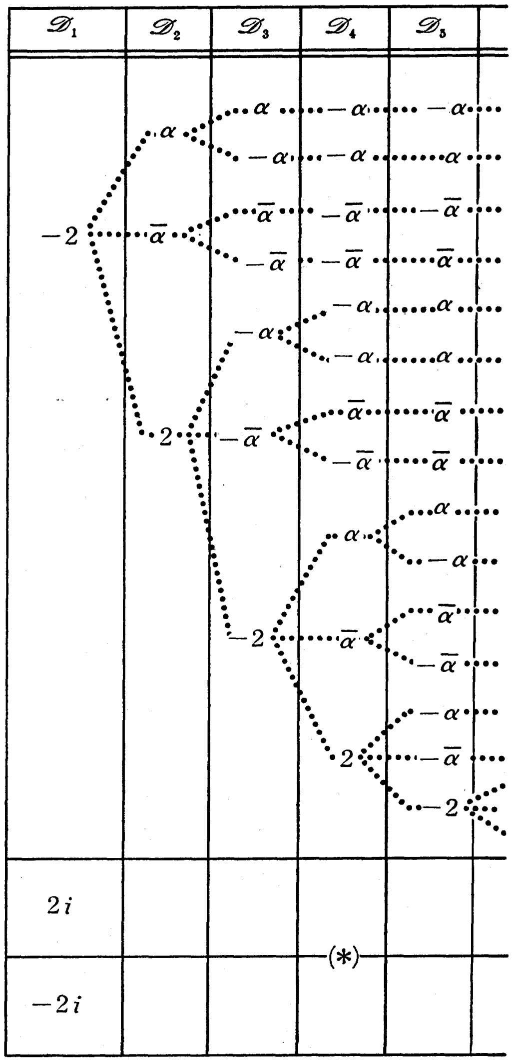
so that we can take Renyi's constant, $C=5^4$, and immediately we also verify (C.3) and (C.6) (see [17]). To examine (C.9), we show in Table 1, the admissibility of sequences $(a_1\cdots a_n)$ for which $X_{a_1\cdots a_n}\in\mathscr{D}_n$. This table shows that $k_0=3$ and $^\sharp\mathscr{D}_n=4(4n-3)$.

To estimate $\lambda(D_n)$, we remark that it is sufficient to estimate $\lambda(D_n')$, where

$$\mathscr{D}_n'=\left\{\begin{array}{l}(-2i,\ -2i,\ \cdots\cdots\cdots\cdots\cdots,\ -2i)\\(-2i,\ -2i,\ \cdots\cdots\cdots\cdots,\ -2i,\ \bar{\alpha})\\(-2i,\ -2i,\ \cdots\cdots\cdots\cdots,-2i,\ \bar{\alpha}\ \bar{\alpha})\\(-2i,\ -2i,\ \cdots\cdots\cdots,\ -2i,\ \bar{\alpha}\ -\bar{\alpha})\\(-2i,\ -2i,\ \cdots\cdots\cdots,-2i,\ \bar{\alpha}\ \bar{\alpha}\ -\bar{\alpha})\\(-2i,\ -2i,\ \cdots\cdots,-2i,\ \bar{\alpha}\ -\bar{\alpha}\ -\bar{\alpha})\\(-2i,\ -2i,\ \cdots,\ -2i,\ \bar{\alpha}\ \bar{\alpha}\ -\bar{\alpha}\ -\bar{\alpha})\\(-2i,\ -2i,\ \cdots,\ -2i,\ \bar{\alpha}\ -\bar{\alpha}\ -\bar{\alpha}\ \bar{\alpha})\\\qquad\vdots\\(-2i,\ \bar{\alpha},\ \bar{\alpha},\ -\bar{\alpha},\ \cdots\cdots\cdots\cdots\cdots)\\(-2i,\ \bar{\alpha},\ -\bar{\alpha},\ -\bar{\alpha},\ \cdots\cdots\cdots\cdots)\\(\bar{\alpha},\ \bar{\alpha},\ -\bar{\alpha},\ -\bar{\alpha}),\ \cdots\cdots\cdots\cdots\cdots)\\(\bar{\alpha},\ -\bar{\alpha},\ -\bar{\alpha},\ \bar{\alpha},\ \cdots\cdots\cdots\cdots\cdots)\end{array}\right.$$

TABLE 1





(1) In this table 1, the sequence of symbols $(-\bar{\alpha}, -\bar{\alpha}, \bar{\alpha}, \bar{\alpha})^{a)}$ means $X_{-\bar{\alpha},-\bar{\alpha},\bar{\alpha},\bar{\alpha}} \in \mathscr{D}_4$, in the same way, the sequence $(2, -2, \bar{\alpha}, \bar{\alpha}, -\bar{\alpha}, -\bar{\alpha})^{b)}$ means $X_{2,-2,\bar{\alpha},\bar{\alpha},-\bar{\alpha},-\bar{\alpha}} \in \mathscr{D}_6$.

(2) (*)The trees for $2i$ and $-2i$ are quite similar to trees for 2 and -2, and hence are omitted.

and

$$D'_n = \bigcup_{X_{a_1\cdots a_n}\in\mathscr{D}'_n} X_{a_1\cdots a_n}\,.$$

Let $O(a, b, c)$ be a disc centered at (a, b) with the radius c. Since

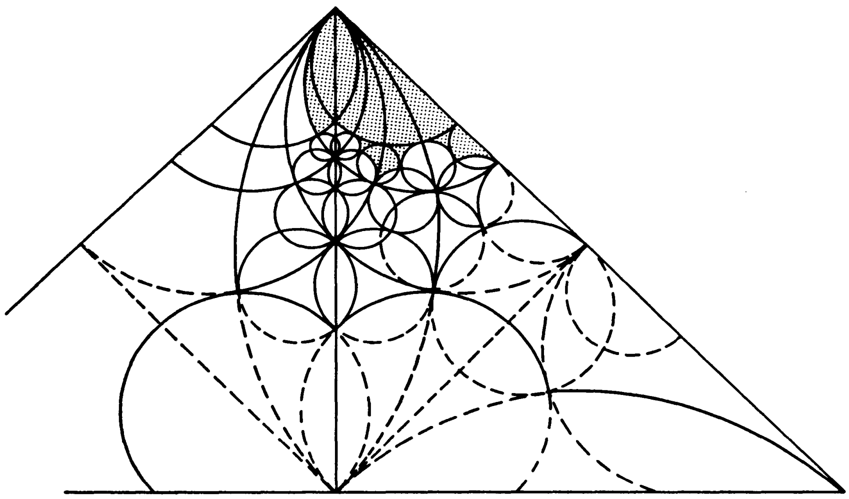
$$D'_n \subset O\left(\frac{1}{2n},\ \frac{2n-1}{2n},\ \frac{\sqrt{2}}{2n}\right) \cup O\left(\frac{2n-1}{2n},\ \frac{1}{2n},\ \frac{\sqrt{2}}{2n}\right) \quad \text{(See Figure 3.)}\,,$$

we have $\lambda(D'_n) \lessapprox O(1/n^2)$ and so $\lambda(D_n) \lessapprox O(1/n^2)$. Therefore (C.4)* is satisfied.

From the inequality

$$\frac{\sup_{z\in T^n X_{a_1\cdots a_n}} |\det D\Psi_{a_1\cdots a_n}(z)|}{\inf_{z\in T^n X_{a_1\cdots a_n}} |\det D\Psi_{a_1\cdots a_n}(z)|} \leq \frac{(1+|q_{n-1}/q_n|)^4}{(1-|q_{n-1}/q_n|)^4}\,,$$

to verify $(C.7)$, it is sufficient to estimate $|q_{n-1}/q_n|$. A direct calculation of $|q_n|$ seems to be very complicated, but the author has achieved to obtain



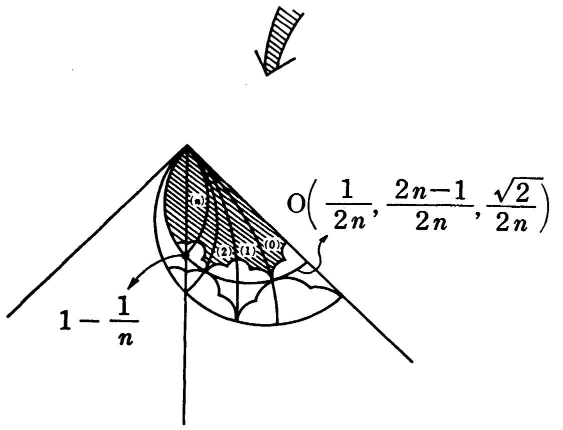


FIGURE 3

Here typical cylinders of $\mathscr{D}_n$ are shown; $(0) = X_{(\bar{\alpha}, -\bar{\alpha}, -\bar{\alpha}\cdots)}$

$(1) = X_{(-2i, \bar{\alpha}, -\bar{\alpha}, -\bar{\alpha}\cdots)}$

$(2) = X_{(-2i, -2i, \bar{\alpha}, -\bar{\alpha}\cdots)}$

$\vdots$

$(k) = X_{(\underbrace{-2i, -2i\cdots -2i}_{k}, \bar{\alpha}, -\bar{\alpha}\cdots)}$

$\vdots$

$(n) = X_{(-2i, -2i\cdots -2i)}$

Other types have the same forms as these types and so are omitted.

the following table:

TABLE 2

type	$\lvert q_n \rvert$
$(\alpha, \alpha, -\alpha, -\alpha, \cdots)$	$\lvert q_{2k-1}\rvert = 2k$, $\lvert q_{2k}\rvert = \sqrt{4k^2+4k+2}$
$(\bar{\alpha}, \bar{\alpha}, -\bar{\alpha}, -\bar{\alpha}, \cdots)$	$\lvert q_{2k-1}\rvert = 2\sqrt{k^2-2k+2}$, $\lvert q_{2k}\rvert = \sqrt{2k^2-2k+10}$
$(2i, 2i, \cdots)$	
$(2, -2, 2, -2, \cdots)$	$\lvert q_n\rvert = \sqrt{2}(n+1)$
$(-2i, -2i, \cdots)$	

Since the type $(\cdots, 2i, \alpha, \alpha, -\alpha, -\alpha, \cdots)$ is complicated, before we estimate $\lvert q_n\rvert$ we give a table of q_n as follows:

TABLE 3

type	$n=4l$	$n=4l+1$	$n=4l+2$	$n=4l+3$
$2i \cdots 2i\, \alpha$	$(4l+1) + i(-4l+1)$	$4l+i(4l+2)$	$(-4l-3) + i(4l+1)$	$(-4l-2) + i(-4l-4)$
	$(l \geqq 1)$	$(l \geqq 1)$	$(l \geqq 0)$	$(l \geqq 0)$
$2i \cdots 2i\, \alpha\, \alpha$	$(-4l+3) + i(-12l+3)$	$12l+i(-4l+2)$	$(4l-1) + i(12l+3)$	$(-12l-6)+i\,4l$
$2i \cdots 2i\, \alpha\, -\alpha$	$(-4l-1) + i(4l-1)$	$(-4l) + i(-4l-2)$	$(4l+3) + i(-4l-1)$	$(4l+2) + i(4l+4)$
	$(l \geqq 1)$	$(l \geqq 1)$	$(l \geqq 1)$	$(l \geqq 0)$
$2i \cdots 2i\, \alpha\, \alpha\, -\alpha$	$(12l-9) + i(12l-5)$	$(-12l+2) + i(12l-6)$	$(-12l+3) + i(-12l-1)$	$(12l+4) + i(-12l)$
$2i \cdots 2i\, \alpha\, -\alpha\, -\alpha$	$(-4l+3) + i(-4l-1)$	$(4l+2) + i(-4l+2)$	$(4l-1) + i(4l+3)$	$(-4l-4)+i\,4l$
	$(l \geqq 1)$	$(l \geqq 1)$	$(l \geqq 1)$	$(l \geqq 1)$
$2i \cdots 2i\, \alpha\, \alpha\, -\alpha\, -\alpha$	$(-20l+15) + i(12l-13)$	$(-12l+10) + i(-20l+10)$	$(20l-5) + i(-12l+7)$	$(12l-4)+i\,20l$
$2i \cdots 2i\, \alpha\, -\alpha\, -\alpha\, \alpha$	$(-4l+3) + i(-4l-1)$	$(4l+2) + i(-4l+2)$	$(4l-1) + i(4l+3)$	$(-4l-4)+i\,4l$
	$(l \geqq 2)$	$(l \geqq 1)$	$(l \geqq 1)$	$(l \geqq 1)$
$2i \cdots 2i\, \alpha\, \alpha\, -\alpha\, -\alpha\, \alpha$	$(-20l+19) + i(20l-25)$	$(-20l+20) + i(-20l+4)$	$(20l-9) + i-(20l+15)$	$(20l-10) + i(20l-4)$
$2i \cdots 2i\, \alpha\, -\alpha\, -\alpha\, \alpha\, \alpha$	$(-4l-1) + i(4l-5)$	$(-4l+4) + i(-4l-2)$	$(4l+3) + i(-4l+3)$	$(4l-2) + i(4l+4)$
$\vdots$	$(l \geqq 2)$ $\vdots$	$(l \geqq 2)$ $\vdots$	$(l \geqq 1)$ $\vdots$	$(l \geqq 1)$ $\vdots$

Therefore, we have the following:

$(2i, 2i, \cdots, 2i, \underbrace{\alpha, \alpha, \cdots}_{m})$	$m=2k \quad \lvert q_{4l+j}\rvert=\sqrt{\{32(4m^2+1)\}l^2+O(l)}$ $m=2k+1 \quad \lvert q_{4l+j}\rvert=\sqrt{\{32(2m+1)^2\}l^2+O(l)}$ $(j=0, 1, 2, 3)$
$(2i, 2i, \cdots, 2i, \underbrace{\alpha, -\alpha, \cdots}_{m})$	$m=2k \quad \lvert q_n\rvert=\sqrt{(n+1)^2+(n-m+1)^2}$ $m=2k+1 \quad \lvert q_n\rvert=\sqrt{(n+1)^2+(n-m)^2}$

Other types are quite similar to one of the above types, and so are omitted. These allow us to verify (C.7).

Let $\rho_n=\sup_{z\in D_n^c}\{|z|\}$. Then it is easy to see that

$$W_n \leqq \left\{\sum_{m=0}^{\infty} {}^{\sharp}\mathscr{D}_m O\left(\frac{1}{m^4}\right)\right\}\cdot\frac{1}{(1-\rho_n)^4}=\left\{\sum_{m=0}^{\infty} O\left(\frac{1}{m^3}\right)\right\}\cdot\frac{1}{(1-\rho_n)^4}\,.$$

This implies (C.5).

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