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Periodic Solutions on a Convex Energy Surface of a Hamiltonian System II

A Quantitative Estimate for Theorem by A. Weinstein Concerning Normal Modes

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Introduction

Let $p, q \in \boldsymbol{R}^n$ and $H=H(p, q) \in C^1(\boldsymbol{R}^{2n}, \boldsymbol{R})$. We consider a Hamiltonian system

$$(H) \qquad \dot{p}=-H_q\,, \qquad \dot{q}=H_p\,,$$

or concisely

$$(H) \qquad \dot{z}=JH'(z)\,,$$

where $z=(p, q)$ and $J=\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ with I being the identity of $\boldsymbol{R}^n$.

In 1973, A. Weinstein [10] obtained an interesting

THEOREM 1. *Let* $H \in C^2(\boldsymbol{R}^{2n}, \boldsymbol{R})$, $H(0)=0$, $H'(0)=0$ *and* $H''(0)>0$. *Then for sufficiently small* $\varepsilon>0$, *there exist* n *distinct periodic solutions of* (H) *on* $H^{-1}(\varepsilon)$.

For small $\varepsilon>0$, $H^{-1}(\varepsilon)$ is a convex manifold (manifold which bounds a usual convex set) close to an ellipsoid.

For ellipsoids, that is, for the system describing harmonic oscillators, there are exactly n periodic solutions on any energy surface if the angular frequencies are independent over $\boldsymbol{Q}$.

In 1978, A. Weinstein [11] also found at least *one* periodic solution on any convex Hamiltonian energy surface. Although that was covered by P. Rabinowitz [9] which gave the result for star-shaped one, an estimate obtained in [11] was used in [6].

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Since about that time, there seems to be the following speculation.

(*) There may be n periodic solutions on a convex or star-shaped Hamiltonian energy surface in $\boldsymbol{R}^{2n}$.

A plausibility of the problem for energy surfaces which are homeomorphic to S^{2n-1} of a *classical* Hamiltonian is given in [12].

A global result for the problem was obtained by Ekeland-Lasry [4], which asserted that there are at least n periodic solutions on any convex energy surface S with

$$r_1 B \subset C \subset r_2 B$$

where $r_2/r_1 < \sqrt{2}$, B is the unit ball in $\boldsymbol{R}^{2n}$ and $S=\partial C$.

In this note, we give a condition for convex energy surfaces to have n periodic solutions.

We identify $\boldsymbol{C}^n$ with $\boldsymbol{R}^{2n}$ by $z_j = p_j + iq_j$ $(j=1, 2, \cdots, n)$.

Let $\omega_1, \omega_2, \cdots, \omega_n$ be numbers with

$$0 < \omega_1 \leqq \omega_2 \leqq \cdots \leqq \omega_n = 1 .$$

We put $|z|_\omega^2 = \sum_{j=1}^n \omega_j |z_j|^2$ and $Q_\omega = \{z \in \boldsymbol{C}^n;\ |z|_\omega \leqq 1\}$.

THEOREM 2. *There is a number*

$$\rho(\omega_1, \omega_2, \cdots, \omega_n) > 1$$

satisfying the following:

Let C a compact, strictly convex subset of $\boldsymbol{C}^n = \boldsymbol{R}^{2n}$ with C^2 boundary S, and there are positive numbers r_1, r_2 with

$$r_2 < \rho(\omega_1, \cdots, \omega_n) r_1 \tag{0.1}$$

and C satisfies

$$r_1 Q_\omega \subset C \subset r_2 Q_\omega . \tag{0.2}$$

Then there are n *distinct periodic solutions of* (H) *on* S. *A candidate of $\rho(\omega_1, \cdots, \omega_n)$ is given later explicitly.*

Remark that the number of periodic solutions is independent of H as long as it has S as a regular energy surface. (cf. Lemma 1.5 in [9].)

This theorem can be considered as a quantitative estimate for ε in Theorem 1.

Now we give a candidate of $\rho(\omega_1, \cdots, \omega_n)$ explicitly by $\omega_1, \cdots, \omega_n$.

We put $\tau_j = 1/\omega_j$ and induce the equivalence relation $\sim$ in $\{\tau_j;\ j=$

$1, 2, \cdots, n\}$ defined by

$$\tau_j \sim \tau_k \quad \text{iff} \quad \tau_j/\tau_k \in \underset{\sim}{Q} .$$

We denote by $D_1, D_2, \cdots, D_A$ its equivalence classes.

For each $r=1, 2, \cdots, A$, there exists $a_r>0$ such that for some

$$b_{r1} \leqq b_{r2} \leqq \cdots \leqq b_{rB_r}$$

in $\boldsymbol{Z}_+=\{1, 2, \cdots\}$, we have

$$D_r=\{b_{rk}a_r;\ k=1, 2, \cdots, B_r\} , \qquad B_1+B_2+\cdots+B_A=n .$$

This gives

$$a_r/a_s \notin \boldsymbol{Q} \quad \text{if} \quad r \neq s . \tag{0.3}$$

Let σ_r be the least common multiple of $b_{r1}, b_{r2}, \cdots$, and b_{rB_r}.

Put $c_r=\sigma_r a_r$ for $r=1, 2, \cdots, A$.

We can assume

$$c_1<c_2<\cdots<c_A . \tag{0.4}$$

Remark that, if $r<s$, then $c_r/c_s \notin \boldsymbol{Q}$ from (0.3).

So, for $r, s=1, 2, \cdots, A$ with $r<s$,

$$\eta_{rs} \equiv \frac{1}{2} \operatorname{Min}\left\{\left|\frac{\mu}{\nu}c_s-c_r\right|;\ \nu=1, 2, \cdots, [c_s]+1,\ \mu=1, 2, \cdots, \nu\right\}>0 .$$

Put

$$\eta_1=\operatorname{Min}\{\eta_{rs};\ r, s=1, 2, \cdots, A,\ r<s\} .$$

For the case $A=1$, we put $\eta_1=1/2$.

For $r=1, 2, \cdots, A$, we define

$$P_r=\{(\mu, j) \in \boldsymbol{Z}_+ \times \{1, 2, \cdots, n\};\ \mu\tau_j<c_r\} .$$

We put

$$\eta_2=\operatorname{Min}\{c_r-\mu\tau_j;\ r=1, 2, \cdots, A,\ (\mu, j) \in P_r\}>0 .$$

Finally we put

$$\eta=\operatorname{Min}\{\eta_1, \eta_2, 1/2\} .$$

Then as a candidate, we can put

$$\rho(\omega_1, \cdots, \omega_n)=\sqrt{1+\eta/c_A} . \tag{0.5}$$

Recently [3] obtains a similar result for star-shaped Hamiltonian energy surfaces.

Our method of estimates, which was applied to the special case [7] and generalized using the estimates of [8], is slightly different from [3]. For example, if $n=2$ and $\omega_1=1/2$, $\omega_2=1$, [3] gives, if restricted to convex cases only, two periodic orbits on ∂C with

$$r_1 Q_\omega \subset C \subset r_2 Q_\omega\,, \qquad \frac{r_2}{r_1} < \sqrt{3/2}\,.$$

Our method replaces the left Q_ω with the unit ball as (ii) of Corollary of Theorem 2 in [8] shows.

The estimate (0.5) itself is not better than the one given in [3] restricted to convex cases.

These results of this note or [3] are considered as a step to give the answer to (*). The common method is the variational principle, corresponding to the Principle of Least Action, on the space of closed curves. When we obtain the solutions as critical points of the functional, it cannot be avoided that there appears multiple critical points, which are different as critical points but same geometrically. The conditions imposed on convex or star-shaped surfaces are the ones in order to eschew such difficulties. The same difficulty appears in the problem of closed geodesics [13].

The manifolds of closed curves and the situation of the functionals on them are different from those in [3], so there may be a meaning of proposing our method.

§ 1. Dual action principle.

This method was developed in [1]. A brief exposition is given in § 1 of [8] and we follow notations there.

Desired periodic solutions are obtained as critical points of a real valued function f, corresponding to the Principle of Least Action in a dual sence, on a Banach manifold M.

f satisfies Palais-Smale condition and

$$m=\inf_{u\in M} f(u)>0\,.$$

This gives automatically at least one periodic orbit on convex Hamiltonian energy surface, the result of [11].

M is a submanifold of

$$E=\left\{u \in L^\alpha(0,\, 2\pi;\, C^n);\ \int u \equiv \frac{1}{2\pi}\int_0^{2\pi} u(t)dt=0\right\}$$

with $1<\alpha<2$. And we have for $\mu \in \boldsymbol{Z}_+$

$$u \in M \Longrightarrow u^\mu \equiv \mu^\delta u(\mu\cdot) \in M \tag{1.1}$$

$$f(u^\mu)=\mu^\theta f(u) \quad \text{for} \quad u \in M \tag{1.2}$$

where $\delta=1/(2-\alpha)$ and $\theta=\alpha/(2-\alpha)=\alpha\delta$.

If $u \in M$ is a critical point of f, then so is u^μ, but these give the same periodic solution. To distinguish such multiple critical points is the main difficulty.

§ 2. Harmonic oscillators.

We freely refer the notations of [8] in particular § 1 and § 2.

In [8], starting from a convex set C, we constructed G, f, M, and Φ (written as ϕ in [8]). We attach the subindex i to G, f, etc. ($i=1, 2$) constructed from $r_i Q_\omega$. G_0, f_0, m_0 and M_0 mean the G, f, m, and M written in § 2 in [8]. Bare G, f, m and M are the one constructed from C in the introduction of this note with (0.2).

Section 2 in [8] gives

LEMMA 1.

$$m_0 \equiv \min_{u \in M_0} f_0(u)=f_0(v_n)=\frac{\pi}{\theta}\ ,$$

where v_j is defined in § 2 of [8].

Also (0.2) implies

$$G_1(u) \leqq G(u) \leqq G_2(u)\ . \tag{2.1}$$

And we have ($i=1, 2$)

$$G_i(u)=R_i^\alpha G_0(u)\ , \tag{2.2}$$

where

$$R_i=r_i\beta^{-1/\beta}\ . \tag{2.3}$$

Further we have

LEMMA 2. *For $u \in M_0$, $\Phi_i(u)=R_i^\theta u$ and $f_i(\Phi_i(u))=R_i^{2\theta} f_0(u)$.*

In particular $f_i(u)$ *attains the minimum* m_i *at* $u=R_i^\theta e^{it}a_n$ *and* $m_i=m_0R_i^{2\theta}$ $(i=1, 2)$.

PROOF. Let $u\in M_0$ and $\Phi_i(u)=\lambda u\in M_i$.

Then $\int u\cdot Lu=\alpha\int G_0(u)$, hence, by (1.6) and (2.13),

$$\begin{aligned}\lambda^{2-\alpha}&=\alpha\int G_i(u)\Big/\int u\cdot Lu\\ &=\alpha\int R_i^\alpha G_0(u)\Big/\alpha\int G_0(u)\\ &=R_i^\alpha\,,\end{aligned}$$

therefore $\lambda=R_i^\theta$.

And we have

$$\begin{aligned}f_i(\lambda u)&=m_0\alpha\int G_i(\lambda u)\\ &=\lambda^\alpha R_i^\alpha m_0\alpha\int G_0(u)\\ &=R_i^{\theta\alpha+\alpha}f_0(u)\\ &=R_i^{2\theta}f_0(u)\,.\end{aligned}$$

Q.E.D.

Also we have

LEMMA 3. $m_1\leqq m\leqq m_2$.

PROOF. Let $w=R_2^\theta e^{it}a_n\in M_2$ be the point such that $f_2(w)=m_2$ by Lemma 2. Then for some $\lambda>0$, $\lambda w\in M$.

$$\begin{aligned}\lambda^{2-\alpha}&=\alpha\int G(w)\Big/\int w\cdot Lw && \text{(by (5) in [2])}\\ &\leqq\alpha\int R_2^\alpha G_0(w)\Big/R_2^{2\theta} && \text{(by (2.1) and (2.2))}\\ &=R_2^\alpha\cdot R_2^{\theta\alpha}/R_2^{2\theta}\\ &=1\,.\end{aligned}$$

Hence

$$\begin{aligned}m&\leqq f(\lambda w)\\ &=m_0\alpha\int G(\lambda w) && \text{(by (6) in [2])}\\ &\leqq m_0\alpha\int R_2^\alpha G_0(\lambda w) && \text{(by (2.1))}\\ &=m_0R_2^\alpha\lambda^\alpha\alpha\int G_0(w)\end{aligned}$$

$$\leqq m_0 R_2^{\alpha} \cdot R_2^{\theta\alpha}$$
$$= m_0 R_2^{2\theta}$$
$$= m_2 , \qquad \text{(by Lemma 2).}$$

Now, since $R_1^{\alpha} G_0(u) \leqq G(u)$, we have

$$\min\{G(u);\ |u|=1\} \geqq \frac{1}{\alpha} R_1^{\alpha} ,$$

thus this R_1 plays the role of r in the proof of Lemma 3 of [2].

Because b in the lemma equals π, we have $m \geqq (\pi/\theta) R_1^{2\theta} = m_0 R_1^{2\theta} = m_1$ by (15) in the proof of the lemma. Q.E.D.

For given $K \geqq 1$, put $\nu_j = [K/\tau_j]$ for $j=1, 2, \cdots, n$ and $a = \nu_1 + \nu_2 + \cdots + \nu_n$.

For $\zeta = (\zeta_{j\mu})_{j=1,2,\cdots,n;\mu=1,2,\cdots,\nu_j} \in \boldsymbol{C}^a$, we define

$$u_\zeta = \sum_{j=1}^{n} \sum_{\mu=1}^{\nu_j} \zeta_{j\mu} v_j^{\mu} \in E .$$

Then, from (2.10), we have

$$\int u_\zeta \cdot L u_\zeta = \sum_{j=1}^{n} \sum_{\mu=1}^{\nu_j} (\mu \tau_j)^{\theta} |\zeta_{j\mu}|^2$$
$$\equiv \|\zeta\|^2 .$$

We put $\Sigma = \{\zeta \in \boldsymbol{C}^a;\ \|\zeta\| = 1\}$ and for $\zeta \in \Sigma$ we define

$$\lambda(\zeta) = \left[\alpha \int G_0(u_\zeta)\right]^{\delta} ,$$

then, (1.7) of [8] implies $\phi_0(\zeta) \equiv \lambda(\zeta) u_\zeta \in M_0$ and we have

$$f_0 \circ \phi_0(\zeta) = m_0 \left[\alpha \int G_0(u_\zeta)\right]^{2\delta} , \qquad \text{(by (1.9) in [8]).} \tag{2.4}$$

Then the following lemma is proved as Lemma 2 in [8].

LEMMA 4. *For any small $\varepsilon > 0$, choosing α sufficiently near* 2 *(choosing β in practice), we have*

$$\operatorname{Max} f_0 \circ \phi_0(\Sigma) \leqq (1+\varepsilon)^{\theta} K^{\theta} m_0 .$$

§ 3. Cohomological indices.

This concept was proposed in [5]. For the definition and the properties, we refer [4] or [9]. 1°, 2°, $\cdots$, 6° mean the ones of Lemma 1.13 in [9]. For a set B, on which an S^1 action is considered, the cohomological index of B is denoted by $i(B)$.

Let M, M_0, M_1, M_2 be the Banach submanifolds of E derived from C satisfying (0.2) with (0.1), $\beta^{1/\beta}Q_\omega$, r_1Q_ω and r_2Q_ω according to the procedure of § 1 respectively.

We attach i for the notations as G, f, m, Φ as above ($i=0, 1, 2$). We collect some relations about M, M_0, M_1, M_2, etc.

On E, we consider the usual S^1 action

$$A_s u(t)=u(t+s) \quad \text{for} \quad s\in S^1=\boldsymbol{R}/2\pi\boldsymbol{Z} .$$

M, M_0, M_1 and M_2 are invariant sets under the S^1 action. We put $M^c=\{u\in M; f(u)\leqq c\}$ and $M_i^c=\{u\in M_i; f_i(u)\leqq c\}$, then M^c and M_i^c are also invariant sets ($i=0, 1, 2$).

Now we put $p_r=\# P_r$ for $r=1, 2, \cdots, A$.

In Lemma 4 and the definitions above it, we denote Σ by Σ_r, for the case $K=c_r$, $r=1, 2, \cdots, A$.

We remark that

$$i(\Sigma_r)=p_r+B_r \tag{3.1}$$

from 2° and 6°.

We put $\rho=\sqrt{1+\eta/c_A}$ and choose $\varepsilon>0$ so small that

$$(1+\varepsilon)R_2/R_1<\rho , \tag{3.2}$$

and determine α_r to apply Lemma 4 for the above case.

We henceforth fix $\alpha=\mathrm{Min}\{\alpha_1, \alpha_2, \cdots, \alpha_A\}$.

Then Lemma 4 gives

$$\mathrm{Max}\, f_0\circ\phi_0(\Sigma_r)\leqq(1+\varepsilon)^\theta c_r^\theta m_0 . \tag{3.3}$$

Let ϕ: $\Sigma_r\to M$ be the mapping defined as ϕ_0: $\Sigma_r\to M_0$. Then we have

LEMMA 5. $\mathrm{Max}\, f\circ\phi(\Sigma_r)<(c_r+\eta)^\theta m$.

PROOF. For any $\zeta\in\Sigma_r$, we have

$$\begin{aligned} f\circ\phi(\zeta)&=m_0\Big[\alpha \int G(u_\zeta)\Big]^{2\delta} && \text{(by (1.9) in [8])}\\ &\leqq m_0\Big[\alpha \int G_2(u_\zeta)\Big]^{2\delta} && \text{(by (2.1))}\\ &=m_0\Big[\alpha \int G_0(u_\zeta)\Big]^{2\delta}\cdot R_2^{\alpha\cdot 2\delta} && \text{(by (2.2))}\end{aligned}$$

$$
\begin{aligned}
&= f_0 \circ \phi_0(\zeta) \cdot R_2^{2\theta} && \text{(by (1.9) in [8])} \\
&\leqq (1+\varepsilon)^\theta c_r^\theta m_0 R_2^{2\theta} && \text{(by (3.3))} \\
&= (1+\varepsilon)^{-\theta} c_r^\theta m_0 ((1+\varepsilon) R_2)^{2\theta} \\
&< c_r^\theta m_0 (\rho R_1)^{2\theta} && \text{(by (3.2))} \\
&= (c_r \rho^2)^\theta m_1 && \text{(by Lemma 2)} \\
&\leqq (c_r \cdot (1+\eta/c_A))^\theta m && \text{(by Lemma 3)} \\
&\leqq (c_r+\eta)^\theta m\,. && \text{(by (0.4))}
\end{aligned}
$$

Q.E.D.

LEMMA 6. *If* $c_r - \eta \leqq c < c_r$, *then* $i(M_1^{c^\theta m_1}) \leqq p_r$.

PROOF. We suppose

$$i(M_1^{c^\theta m_1}) \geqq p_r + 1\,. \tag{3.4}$$

For $k=1, 2, \cdots, p_r+1$, we put

$$\kappa_k = \inf_{B \in \Omega_k} \sup f_1(B)\,, \tag{3.5}$$

where $\Omega_k = \{B;$ invariant subset of M_1 with $i(B) \geqq k$ and $f_{1|B}$: bounded$\}$.

Then the usual argument of critical point theory asserts that

$$\kappa_1 \leqq \kappa_2 \leqq \cdots \leqq \kappa_{p_r+1}$$

are critical values of f_1: $M_1 \to [m_1, \infty)$ and from Remark 1.23 in [9], we have

$$\begin{aligned} &\textit{if} \quad \kappa_k = \kappa_{k+1} = \cdots = \kappa_{k+l}\,, \quad \textit{then} \\ &\qquad i(\text{the set of all critical points of the level } \kappa_k) \geqq l+1\,. \end{aligned} \tag{3.6}$$

From (3.4), we have

$$\kappa_{p_r+1} \leqq c^\theta m_1\,. \tag{3.7}$$

On the other hand, the critical points of f_1 in $M_1^{c^\theta m_1}$ are completely known.

By the definition of η_2, we remark that

$$P_r = \{(\mu, j);\ \mu\tau_j \leqq c\}\,. \tag{3.8}$$

First we have p_r critical points (mod. S^1 action)

$$w_j^\mu \equiv \Phi_1(v_j^\mu) = R_1^\theta v_j^\mu\ ; \quad (\mu, j) \in P_r\,.$$

If some of them are in the same level, say

$$f_1(w_{j_1}^{\mu_1})=f_1(w_{j_2}^{\mu_2})=\cdots=f_1(w_{j_l}^{\mu_l})=b^\theta m_1\ ,$$

then $\mu_1\tau_{j_1}=\mu_2\tau_{j_2}=\cdots=\mu_l\tau_{j_l}=b$.

We put $\Sigma'=\{\zeta_1 w_{j_1}^{\mu_1}+\zeta_2 w_{j_2}^{\mu_2}+\cdots+\zeta_l w_{j_l}^{\mu_l}\in E;\ (\zeta_1,\zeta_2,\cdots,\zeta_l)\in \boldsymbol{C}^l$ and $|\zeta_1|^2+|\zeta_2|^2+\cdots+|\zeta_l|^2=1\}$.

We claim that

(3.9) *Σ' is the set of all critical points in M_1 of level $b^\theta m_1$, and $i(\Sigma')=l$.*

First, for $w\in\Sigma'$, we have

$$\begin{aligned} f_1(w)&=m_0\int w\cdot Lw\\ &=m_0R_1^{2\theta}(|\zeta_1|^2(\mu_1\tau_{j_1})^\theta+\cdots+|\zeta_l|^2(\mu_l\tau_{j_l})^\theta) \qquad \text{(by (2.10) in [8])}\\ &=b^\theta m_1\ . \end{aligned}$$

Next, let $z(t)=(z_1(t),z_2(t),\cdots,z_n(t))$ be a nontrivial 2π-periodic solution of (H) with $H(z)=H_1(z)=r_1^{-\beta}|z|_\omega^\beta$.

Then $\dot z_j(t)=2i(\partial/\partial\bar z_j)H_1=ir_1^{-\beta}\cdot\beta|z(t)|_\omega^{\beta-2}\omega_j z_j$, $j=1,2,\cdots,n$. Remarking that $|z(t)|_\omega=r_1H_1(z(t))^{1/\beta}$ is constant, we put $b=r_1^{-\beta}\cdot\beta|z|_\omega^{\beta-2}$ and $\mu_j=b\omega_j$, then we have $\dot z_j(t)=i\mu_j z_j$, therefore

$$z(t)=\sum_{j\in J}\xi_j e^{\mu_j it}a_j\ ,\quad \xi_j\in\boldsymbol{C}$$

where $J=\{j\in\{1,2,\cdots,n\};\ \mu_j\in\boldsymbol{Z}_+\}$.

The function $u\in M_1$ corresponding to z is given by

$$\begin{aligned} u(t)&=-i\dot z\\ &=\sum_{j\in J}\xi_j\mu_j e^{\mu_j it}a_j\\ &=\sum_{j\in J}\omega_j^{1/2}\xi_j(\mu_j\tau_j)^{-1/(\beta-2)}v_j^{\mu_j}\ . \end{aligned}$$

Now

$$\begin{aligned} (\mu_j\tau_j)^{-1/(\beta-2)}&=b^{-1/(\beta-2)}\\ &=(r_1^{-\beta}\cdot\beta\cdot|z|_\omega^{\beta-2})^{-1/(\beta-2)}\\ &=R_1^\theta/|z|_\omega\ . \end{aligned}$$

Putting $\zeta_j=\omega_j^{1/2}\xi_j/|z|_\omega$, we have

$$u(t)=\sum_{j\in J}\zeta_j w_j^{\mu_j}\quad\text{and}\quad\sum_{j\in J}|\zeta_j|^2=1\ .$$

This means $u\in\Sigma'$, proving the first half of (3.9).

2° and 6° give the second half of (3.9).

Taking into account this argument and (3.6), the critical point of f_1 of the level κ_{p_r+1} cannot exist in $M_1^{c^\theta m_1}$.

This contradicts (3.7), proving the lemma. Q.E.D.

Also we have

LEMMA 7. $i(M^{((1+\varepsilon)^2(c_r-\eta))^\theta m})\leqq p_r$, $r=1, 2, \cdots, A$.

PROOF. For $u\in M^{((1+\varepsilon)^2(c_r-\eta))^\theta m}$, put $\lambda u=\Phi_1(u)$, then

$$\begin{aligned}\lambda^{2-\alpha}&=\alpha\int G_1(u)\Big/\int u\cdot Lu && \text{(by (5) in [2])}\\ &=\alpha\int G_1(u)\Big/\alpha\int G(u)\\ &\leqq 1\,, && \text{(by (2.1))}\end{aligned}$$

so $\lambda\leqq 1$, hence

$$\begin{aligned}f_1\circ\Phi_1(u)&=f_1(\lambda u)\\ &=m_0\alpha\int G_1(\lambda u) && \text{(by (6) in [2])}\\ &=\lambda^\alpha m_0\alpha\int G_1(u)\\ &\leqq m_0\alpha\int G(u)\\ &=f(u)\\ &\leqq(1+\varepsilon)^{2\theta}(c_r-\eta)^\theta m\\ &\leqq(1+\varepsilon)^{2\theta}(c_r-\eta)^\theta m_0R_2^{2\theta} && \text{(by Lemmas 2 and 3)}\\ &<(c_r-\eta)^\theta m_0(R_1\rho)^{2\theta}\\ &=((c_r-\eta)\rho^2)^\theta m_0R_1^{2\theta}\\ &\leqq(c_r-\eta^2/c_A)^\theta m_1\,.\end{aligned}\tag{3.10}$$

Put $c=c_r-\eta^2/c_A$, then $c_r-\eta<c<c_r$.

From (3.10), we have the continuous equivariant map

$$\Phi_1\colon M^{((1+\varepsilon)^2(c_r-\eta))^\theta m}\longrightarrow M_1^{c^\theta m_1}\,.$$

Therefore 2° and Lemma 6 yield the lemma. Q.E.D.

§ 4. Proof of Theorem 2.

We put

$$\gamma_k = \inf_{B\in\Gamma_k} \sup f(B) ,$$

where $\Gamma_k=\{B$; invariant subset of M with $i(B)\geqq k$ and $f|_B$: bounded$\}$.

Then for each $r=1, 2, \cdots, A$,

$$\gamma_{p_r+1}\leqq\gamma_{p_r+2}\leqq\cdots\leqq\gamma_{p_r+B_r}$$

are critical values of $f: M\to[m, \infty)$ as in the proof of Lemma 6.

Since $\phi: \Sigma_r\to M$ is a continuous equivariant map, (3.1) and 2° yield $\phi(\Sigma_r)\in\Gamma_{p_r+B_r}$. Hence we have

$$\gamma_{p_r+B_r}<(c_r+\eta)^\theta m , \tag{4.1}$$

by the definition of γ_k and Lemma 5.

Next we claim

$$\gamma_{p_r+1}>(c_r-\eta)^\theta m . \tag{4.2}$$

In fact, if $\gamma_{p_r+1}\leqq(c_r-\eta)^\theta m$, then there is a

$$B\in\Gamma_{p_r+1} \tag{4.3}$$

with $\sup f(B)\leqq((1+\varepsilon)^2(c_r-\eta))^\theta m$.

So 2° and Lemma 7 yield $i(B)\leqq p_r$, taking the inclusion as the equivariant map.

This contradicts (4.3), giving (4.2).

Now we put $I_r=(c_r-\eta, c_r+\eta)$ for $r=1, 2, \cdots, A$. Then

$$\text{if}\quad r<s\ ,\quad\text{then}\quad I_r\cap I_s=\varnothing\ , \tag{4.4}$$

because $\eta\leqq\eta_{rs}\leqq(1/2)(c_s-c_r)$.

We have

LEMMA 8. *Let $r, s\in\{1, 2, \cdots, A\}$ with $r<s$.*
If u and $v\in M$ satisfy

$$(c_r-\eta)^\theta m<f(u)<(c_r+\eta)^\theta m \tag{4.5}$$

$$(c_s-\eta)^\theta m<f(v)<(c_s+\eta)^\theta m , \tag{4.6}$$

then u and v are geometrically distinct.

PROOF. We assume u and v are geometrically equal, that is $u=w^\mu$ and $v=w^\nu$ for some $w\in M$ and $\mu, \nu\in \boldsymbol{Z}_+$ (mod. S^1 action).

Put $\lambda=(f(w)/m)^{1/\theta}$. Then we have $f(u)=\mu^\theta f(w)=(\mu\lambda)^\theta m$ and $f(v)=(\nu\lambda)^\theta m$.

(4.5) and (4.6) imply

$$c_r - \eta < \mu\lambda < c_r + \eta \tag{4.7}$$

$$c_s - \eta < \nu\lambda < c_s + \eta . \tag{4.8}$$

Since $\lambda \geqq 1$ and $\eta \leqq 1/2$, we have $\nu \leqq [c_s] + 1$ from (4.8).

We remark that $\mu < \nu$ from (4.4), in particular $\mu/\nu < 1$, so (4.8) gives

$$\frac{\mu}{\nu} c_s - \eta < \frac{\eta}{\nu}(c_s - \eta) < \frac{\mu}{\nu} \cdot \nu\lambda < \frac{\mu}{\nu}(c_s + \eta) < \frac{\mu}{\nu} c_s + \eta .$$

This implies $|\mu\lambda - (\mu/\nu)c_s| < \eta$.

On the other hand $|\mu\lambda - c_r| < \eta$, hence we have

$$\left| \frac{\mu}{\nu} c_s - c_r \right| < 2\eta .$$

This contradicts the definition of η_{rs}. Q.E.D.

Now we have B_r critical values

$$(c_r - \eta)^\theta m < \gamma_{p_r+1} \leqq \gamma_{p_r+2} \leqq \cdots \leqq \gamma_{p_r+B_r} < (c_r + \eta)^\theta m ,$$

from (4.1) and (4.2).

If $\gamma_{p_r+k} = \gamma_{p_r+k+1}$ for some $k = 1, 2, \cdots, B_r - 1$, then there exist infinitely many critical points (mod. S^1 action) in the level (cf. (3.6)). In any case, we have at least B_r critical points (mod. S^1 action) in the level

$$(c_r - \eta)^\theta m < f < (c_r + \eta)^\theta m .$$

These critical points are geometrically distinct, since $\eta \leqq 1/2$ (recall the method of distinguishing geometrically before).

Thus, from Lemma 8, we have at least geometrically distinct

$$B_1 + B_2 + \cdots + B_A = n$$

critical points altogether.

This completes the proof of Theorem 2.

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