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Hilbert Transforms on One Parameter Groups of Operators II

Shiro ISHIKAWA

Keio University

Introduction

In [2], we studied about the existence theorems of a Hilbert transform on a complete locally convex space. In this paper, we shall consider some properties of the Hilbert transform. For this, we define several terms, some of which were already defined in [2].

DEFINITION 1. Let $\boldsymbol{R}$ be a real field. Let X be a complete locally convex space and let $\{U_t: t\in\boldsymbol{R}\}$ be a one parameter group of operators on X, that is,

(i) $U_t: X\to X$ is a continuous linear operator for all $t\in\boldsymbol{R}$, and U_0 is an identity operator on X,

(ii) $U_tU_s=U_{t+s}$ for all $t, s\in\boldsymbol{R}$,

(iii) for any $t\in\boldsymbol{R}$ and any $x\in X$, $(U_{t+h}-U_t)x$ converges to 0 as $h\to 0$ in the topology of X (for short, in X).

Moreover, the following condition (iv) is assumed in this paper:

(iv) $U_t: X\to X$ is continuous uniformly for $t\in\boldsymbol{R}$, that is, for any neighborhood V of 0 in X, there exists a neighborhood W of 0 in X such that

$$U_tW\subset V \quad \text{for all } t\in\boldsymbol{R}\ .$$

If $\lim_{T\to\infty}(1/2T)\int_{-T}^{T} U_txdt$ exists in X, then we denote it by $\bar{x}$.

DEFINITION 2. A continuous linear operator $H_{\varepsilon,N}$ $(0<\varepsilon<N<\infty)$ on X is defined as follows;

$$H_{\varepsilon,N}x=\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{U_tx}{t}dt \quad (x\in X)\ ,$$

(this integral can be well defined since a mapping $t\in\boldsymbol{R}\to(U_tx)/t\in X$ is continuous on a compact set $\{t\in\boldsymbol{R}: \varepsilon\leqq|t|\leqq N\}$). Also, if $\lim_{\varepsilon\to0+,N\to\infty} H_{\varepsilon,N}x$

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exists in X, we denote it by Hx and call it a Hilbert transform of x. And the domain of H (i.e. $\{x \in X: Hx$ exists$\}$) is denoted by $D(H)$.

§1. Special case (in Hilbert space).

In this section, we shall show several results in a Hilbert space, which will be generalized in a complete locally convex space in the following section.

THEOREM 1. *Let $\{U_t: t \in \boldsymbol{R}\}$ be a one parameter group of unitary operators on a Hilbert space X (i.e. $U_t^* = U_{-t}$ for all $t \in \boldsymbol{R}$). Then, for any element x in X, Hx exists in X. Moreover it is seen that*

$$||Hx||^2 = ||x - \bar{x}||^2 \leqq ||x||^2$$

for all $x \in X$.

PROOF. Let x be any element in X. Since $\{U_t: t \in \boldsymbol{R}\}$ be a one parameter group of unitary operators on a Hilbert space X, we, by Stone's theorem, see the following spectral representation of $U_t x$;

$$U_t x = \int_{-\infty}^{\infty} e^{i\lambda t}\, dE(\lambda)x$$

where $\{E(\lambda): \lambda \in \boldsymbol{R}\}$ is a spectral family of a one parameter group of unitary operators $\{U_t: t \in \boldsymbol{R}\}$. In order to show the first part of Theorem, it is sufficient to prove that $||H_{\varepsilon,N}x - H_{\varepsilon',N'}x||$ converges to 0 as $\varepsilon, \varepsilon' \to 0+$ and $N, N' \to \infty$. From the spectral represenation of $U_t x$, we see that

$$\begin{aligned}
&||H_{\varepsilon,N}x - H_{\varepsilon',N'}x||^2 \\
&= \left\| \frac{1}{\pi}\int_{\varepsilon<|t|<N} \frac{U_t x}{t} dt - \frac{1}{\pi}\int_{\varepsilon'<|t|<N} \frac{U_t x}{t} dt \right\|^2 \\
&= \left\| \int_{-\infty}^{\infty} \left[\frac{1}{\pi}\int_{\varepsilon<|t|<N} \frac{e^{i\lambda t}}{t} dt - \frac{1}{\pi}\int_{\varepsilon'<|t|<N'} \frac{e^{i\lambda t}}{t} dt \right] dE(\lambda)x \right\|^2 \\
&= \left\| \int_{-\infty}^{\infty} [g_{\varepsilon,N}(\lambda) - g_{\varepsilon',N'}(\lambda)] dE(\lambda)x \right\|^2 \\
&= \int_{-\infty}^{\infty} |g_{\varepsilon,N}(\lambda) - g_{\varepsilon',N'}(\lambda)|^2 d||E(\lambda)x||^2 , \qquad (1)
\end{aligned}$$

where

$$g_{\varepsilon,N}(\lambda) = \frac{1}{\pi}\int_{\varepsilon<|t|<N} \frac{e^{i\lambda t}}{t} dt .$$

It is clear that $g_{\varepsilon,N}(\lambda)$ has the following properties:

(i) $g_{\varepsilon,N}(\lambda)$ is a continuous function on R such that $|g_{\varepsilon,N}(\lambda)|\leqq 1$ for all $\lambda\in \boldsymbol{R}$,

(ii) if α and β are real numbers such that $0<\alpha<\beta<\infty$, then $g_{\varepsilon,N}$ uniformly converges to 1 (-1) for the closed interval $[\alpha, \beta]$ $([-\beta, -\alpha])$ as $\varepsilon\to 0+$, $N\to\infty$

and

(iii) $g_{\varepsilon,N}(0)=0$.

From (1) and these properties of $g_{\varepsilon,N}$, we can easily see that $\|H_{\varepsilon,N}x-H_{\varepsilon',N'}x\|$ converges to 0 as ε, $\varepsilon'\to 0+$ and N, $N'\to\infty$. Hence $H_{\varepsilon,N}x$ converges to a certain element Hx in X as $\varepsilon\to 0+$ and $N\to\infty$.

Now we shall prove the second part of theorem. We see that

$$\begin{aligned}\|Hx\|^2&=\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}}\|H_{\varepsilon,N}x\|^2\\&=\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}}\left\|\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{1}{t}\left(\int_{-\infty}^{\infty}e^{i\lambda t}dE(\lambda)x\right)dt\right\|^2\\&=\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}}\int_{\infty}^{\infty}|g_{\varepsilon,N}(\lambda)|^2d\|E(\lambda)x\|^2\\&=\|x\|^2-\|E(0+)x\|^2+\|E(0-)x\|^2 .\end{aligned}$$

Also we see that $\|E(0+)x\|^2-\|E(0-)x\|^2=\|\bar{x}\|^2$ and $\|\bar{x}\|^2+\|x-\bar{x}\|^2=\|x\|^2$. From this, the second part of theorem immediately follows. This completes the proof.

THEOREM 2. *Let $\{U_t: t\in \boldsymbol{R}\}$ be a one parameter group of unitary operators on a Hilbert space X. Then, for any x, y in X,*

(i) $(Hx, y)=-(x, Hy)$,

(ii) $(Hx, Hy)=(x-\bar{x}, y-\bar{y})$.

PROOF. Let x and y be any elements in X. Then we see, from the unitarity of $\{U_t: t\in \boldsymbol{R}\}$, that for any x, $y\in X$

$$\begin{aligned}(Hx, y)&=\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}}(H_{\varepsilon,N}x, y)\\&=\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}}\left(\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{U_tx}{t}dt, y\right)\\&=\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}}\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{1}{t}(U_tx, y)dt\\&=\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}}\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{1}{t}(x, U_t^*y)dt\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\substack{\varepsilon \to 0+ \\ N \to \infty}} \frac{1}{\pi} \int_{\varepsilon < |t| < N} \frac{1}{t} (x, U_{-t} y) dt \\
&= \lim_{\substack{\varepsilon \to 0+ \\ N \to \infty}} \left(x, \frac{1}{\pi} \int_{\varepsilon < |t| < N} \frac{U_{-t} y}{t} dt \right) \\
&= -\lim_{\substack{\varepsilon \to 0+ \\ N \to \infty}} (x, H_{\varepsilon, N} y) \\
&= -(x, Hy) ,
\end{aligned}
$$

which proves (i).
Also we see, from Theorem 1, that

$$
\begin{aligned}
&4(Hx, Hy) \\
&= \|Hx + Hy\|^2 - \|Hx - Hy\|^2 + i\|Hx + iHy\|^2 - i\|Hx - iHy\|^2 \\
&= \|(x+y) - (x+y)^-\|^2 - \|(x-y) - (x-y)^-\|^2 \\
&\quad + i\|(x+iy) - (x+iy)^-\|^2 - i\|(x-iy) - (x-iy)^-\|^2 \\
&= \|(x-\bar{x}) + (y+\bar{y})\|^2 - \|(x-\bar{x}) - (y-\bar{y})\|^2 \\
&\quad + i\|(x-\bar{x}) + i(y-\bar{y})\|^2 - i\|(x-\bar{x}) - i(y-\bar{y})\|^2 \\
&= 4(x-\bar{x}, y-\bar{y}) ,
\end{aligned}
$$

which is (ii). This completes the proof.

THEOREM 3. *Let $\{U_t : t \in \boldsymbol{R}\}$ be a one parameter group of unitary operators on a Hilbert space X. Then it follows that*

$$H(Hx) = -(x-\bar{x}) \quad \text{for all } x \in X .$$

PROOF. Let x be any elements in X. Then we see, from Theorem 1 and Theorem 2, that, for any $y \in X$,

$$(H(Hx), y) = -(Hx, Hy) = -(x-\bar{x}, y-\bar{y}) = -(x-\bar{x}, y) + (x-\bar{x}, \bar{y}) .$$

Since $(x-\bar{x}, \bar{y}) = 0$, we obtain that

$$(H(Hx), y) = (-(x-\bar{x}), y) \quad \text{for all } y \in X ,$$

which gives us $H(Hx) = -(x-\bar{x})$. This completes the proof.

§2. General case (in a complete locally convex space).

In this section, we shall generalize the theorems in the section 2. The following lemma is fundumental for our theory.

LEMMA 1. *Let $\{U_t : t \in \boldsymbol{R}\}$ be a one parameter group of operators on* *a complete locally convex space X. Let x be any element in X and let* η, ε, *N and M be positive numbers such that* $0<\eta<\varepsilon<(1/2)<2<N<2N+1<M<\infty$ *(more precisely,* $0<\varepsilon-\eta<\varepsilon+\eta<N-\eta<N+\eta<M-N<M-\varepsilon<M+\varepsilon<M+N$*). Then it follows that*

$$H_{\eta,M}H_{\varepsilon,N}x=-\frac{1}{\pi^2}\Big[\int_{-(M-\varepsilon)/\varepsilon}^{(M-\varepsilon)/\varepsilon}\frac{U_{\varepsilon t}x}{t}\log\left|\frac{t+1}{t-1}\right|dt$$

$$-\int_{-(M-N)/N}^{(M-N)/N}\frac{U_{Nt}x}{t}\log\left|\frac{t+1}{t-1}\right|dt+R(\eta,\varepsilon,N,M;x)\Big]$$

where

$$R(\eta,\varepsilon,N,M;x)$$

$$=\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{U_ax+U_{-a}x}{a}\log\left|\frac{(a+\eta)(a-\varepsilon)}{\varepsilon\eta}\right|da+\int_{\varepsilon+\eta}^{N-\eta}\frac{U_ax+U_{-a}x}{a}\log\left|\frac{a+\eta}{a-\eta}\right|da$$

$$+\int_{N-\eta}^{N+\eta}\frac{U_ax+U_{-a}x}{a}\log\left|\frac{N\eta}{(a-N)(a-\eta)}\right|da$$

$$+\int_{M-N}^{M-\varepsilon}\frac{U_ax+U_{-a}x}{a}\log\left|\frac{(a-M)(a-N)}{MN}\right|da$$

$$+\int_{M-\varepsilon}^{M+\varepsilon}\frac{U_ax+U_{-a}x}{a}\log\left|\frac{\varepsilon(a-N)}{N(a-\varepsilon)}\right|da$$

$$+\int_{M+\varepsilon}^{M+N}\frac{U_ax+U_{-a}x}{a}\log\left|\frac{(a-M)(a-N)}{MN}\right|da\,.$$

PROOF. Let x be any element in X, and let η, ε, N and M be a positive numbers such that

$$0<\eta<\varepsilon<\frac{1}{2}<2<N<2N+1<M<\infty\,.$$

Then we see that,

$$(1)\quad H_{\eta,M}(H_{\varepsilon,N}x)$$

$$=\frac{1}{\pi}\int_{\eta<|s|<M}\frac{U_s}{s}\Big(\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{U_tx}{t}dt\Big)ds$$

(change variable $t\longrightarrow -t$)

$$=\frac{-1}{\pi^2}\iint_{\substack{\eta<|s|<M\\ \varepsilon<|t|<N}}\frac{U_{s-t}x}{st}dsdt$$

(change variable $s-t\longrightarrow a$, $t\longrightarrow v$)

$$=\frac{-1}{\pi^2}\iint_{\substack{\eta<|a+v|<M\\ \varepsilon<|v|<N}}\frac{U_ax}{v(a+v)}dadv$$

$$
\begin{aligned}
&=-\frac{1}{\pi^2}\left[\int_{-(\varepsilon-\eta)}^{\varepsilon-\eta}\left\{\left(\int_{-N}^{-\varepsilon}+\int_{\varepsilon}^{N}\right)\frac{U_a x}{v(a+v)}dv\right\}da\right] \\
&\quad-\frac{1}{\pi^2}\left[\int_{\varepsilon-\eta}^{\varepsilon+\eta}\left\{\left(\int_{-N}^{-(a+\eta)}+\int_{\varepsilon}^{N}\right)\frac{U_a x}{v(a+v)}dv\right\}da\right. \\
&\qquad\qquad\left.+\int_{-(\varepsilon+\eta)}^{-(\varepsilon-\eta)}+\left\{\left(\int_{-N}^{-\varepsilon}+\int_{-a+\eta}^{N}\right)\frac{U_a x}{v(a+v)}dv\right\}da\right] \\
&\quad-\frac{1}{\pi^2}\left[\int_{\varepsilon+\eta}^{N-\eta}\left\{\left(\int_{-N}^{-(a+\eta)}+\int_{-(a-\eta)}^{-\varepsilon}+\int_{\varepsilon}^{N}\right)\frac{U_a x}{v(a+v)}dv\right\}da\right. \\
&\qquad\qquad\left.+\int_{-(N-\eta)}^{-(\varepsilon+\eta)}\left\{\left(\int_{-N}^{-\varepsilon}+\int_{\varepsilon}^{-a-\eta}+\int_{-a+\eta}^{N}\right)\frac{U_a x}{v(a+v)}dv\right\}da\right] \\
&\quad-\frac{1}{\pi^2}\left[\int_{N-\eta}^{N+\eta}\left\{\left(\int_{-(a-\eta)}^{-\varepsilon}+\int_{\varepsilon}^{N}\right)\frac{U_a x}{v(a+v)}dv\right\}da\right. \\
&\qquad\qquad\left.+\int_{-(N+\eta)}^{-(N-\eta)}\left\{\left(\int_{-N}^{-\varepsilon}+\int_{\varepsilon}^{-a-\eta}\right)\frac{U_a x}{v(a+v)}dv\right\}da\right] \\
&\quad-\frac{1}{\pi^2}\left[\int_{N+\eta}^{M-N}\left\{\left(\int_{-N}^{-\varepsilon}+\int_{\varepsilon}^{N}\right)\frac{U_a x}{v(a+v)}dv\right\}da\right. \\
&\qquad\qquad\left.+\int_{-(M-N)}^{-(N+\eta)}\left\{\left(\int_{-N}^{-\varepsilon}+\int_{\varepsilon}^{N}\right)\frac{U_a x}{v(a+v)}dv\right\}da\right] \\
&\quad-\frac{1}{\pi^2}\left[\int_{M-N}^{M-\varepsilon}\left\{\left(\int_{-N}^{-\varepsilon}+\int_{\varepsilon}^{M-a}\right)\frac{U_a x}{v(a+v)}dv\right\}da\right. \\
&\qquad\qquad\left.+\int_{-(M-\varepsilon)}^{-(M-N)}\left\{\left(\int_{-(M+a)}^{-\varepsilon}+\int_{\varepsilon}^{N}\right)\frac{U_a x}{v(a+v)}dv\right\}da\right] \\
&\quad-\frac{1}{\pi^2}\left[\int_{M-\varepsilon}^{M+\varepsilon}\left\{\int_{-N}^{-\varepsilon}\frac{U_a x}{v(a+v)}dv\right\}da+\int_{-(M+\varepsilon)}^{-(M-\varepsilon)}\left\{\int_{\varepsilon}^{N}\frac{U_a x}{v(a+v)}dv\right\}da\right] \\
&\quad-\frac{1}{\pi^2}\left[\int_{M+\varepsilon}^{M+N}\left\{\int_{-N}^{-(a-M)}\frac{U_a x}{v(a+v)}dv\right\}da+\int_{-(M+N)}^{-(M+\varepsilon)}\left\{\int_{-a-M}^{N}\frac{U_a x}{v(a+v)}dv\right\}da\right] \\
&=-\frac{1}{\pi^2}I_1-\frac{1}{\pi^2}I_2-\frac{1}{\pi^2}I_3-\cdots-\frac{1}{\pi^2}I_8\,,\quad\text{say.}
\end{aligned}
$$

Now we shall calculate $I_1, I_2, I_3 \cdots I_7$ and I_8, and successively have the followings;

(2) $$
\begin{aligned}
I_1&=\int_{-(\varepsilon-\eta)}^{\varepsilon-\eta}\frac{U_a x}{a}\left[\left(\int_{-N}^{-\varepsilon}+\int_{\varepsilon}^{N}\right)\left(\frac{1}{v}-\frac{1}{a+v}\right)dv\right]da \\
&=\int_{-(\varepsilon-\eta)}^{\varepsilon-\eta}\frac{U_a x}{a}\left[\log\left|\frac{v}{a+v}\right|\Big|_{-N}^{-\varepsilon}+\log\left|\frac{v}{a+v}\right|\Big|_{\varepsilon}^{N}\right]da \\
&=\int_{-(\varepsilon-\eta)}^{\varepsilon-\eta}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da-\int_{-(\varepsilon-\eta)}^{\varepsilon-\eta}\frac{U_a x}{a}\log\left|\frac{a+N}{a-N}\right|da\,,
\end{aligned}
$$

(3)
$$I_2=\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{U_a x}{a}\left(\log\left|\frac{v}{a+v}\right|\Bigg|_{-N}^{-(a+\eta)}+\log\left|\frac{v}{a+v}\right|\Bigg|_{\varepsilon}^{N}\right)da$$
$$+\int_{-(\varepsilon+\eta)}^{-(\varepsilon-\eta)}\frac{U_a x}{a}\left(\log\left|\frac{v}{a+v}\right|\Bigg|_{-N}^{-\varepsilon}+\log\left|\frac{v}{a+v}\right|\Bigg|_{-a+\eta}^{N}\right)da$$
$$=\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{U_a x}{a}\log\left|\frac{(a+\varepsilon)(a+\eta)}{\varepsilon\eta}\right|da+\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{U_a x}{a}\log\left|\frac{a-N}{a+N}\right|da$$
$$+\int_{-(\varepsilon+\eta)}^{-(\varepsilon-\eta)}\frac{U_a x}{a}\log\left|\frac{\varepsilon\eta}{(a-\varepsilon)(a-\eta)}\right|da+\int_{-(\varepsilon+\eta)}^{-(\varepsilon-\eta)}\frac{U_a x}{a}\log\left|\frac{a-N}{a+N}\right|da$$
$$=\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da-\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{U_a x}{a}\log\left|\frac{a+N}{a-N}\right|da$$
$$+\int_{-(\varepsilon+\eta)}^{-(\varepsilon-\eta)}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da-\int_{-(\varepsilon+\eta)}^{-(\varepsilon-\eta)}\frac{U_a x}{a}\log\left|\frac{a+N}{a-N}\right|da$$
$$+\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{U_a x+U_{-a} x}{a}\log\left|\frac{(a-\varepsilon)(a+\eta)}{\varepsilon\eta}\right|da\,,$$

(4)
$$I_3=\int_{\varepsilon+\eta}^{N-\eta}\frac{U_a x}{a}\left(\log\left|\frac{v}{a+v}\right|\Bigg|_{-N}^{-(a+\eta)}+\log\left|\frac{v}{a+v}\right|\Bigg|_{-(a-\eta)}^{-\varepsilon}+\log\left|\frac{v}{a+v}\right|\Bigg|_{\varepsilon}^{N}\right)da$$
$$+\int_{-(N-\eta)}^{-(\varepsilon+\eta)}\frac{U_a x}{a}\left(\log\left|\frac{v}{a+v}\right|\Bigg|_{-N}^{-\varepsilon}+\log\left|\frac{v}{a+v}\right|\Bigg|_{\varepsilon}^{-a-\eta}+\log\left|\frac{v}{a+v}\right|\Bigg|_{-a+\eta}^{N}\right)da$$
$$=\int_{\varepsilon+\eta}^{N-\eta}\frac{U_a x}{a}\log\left|\frac{a+\eta}{a-\eta}\right|da+\int_{\varepsilon+\eta}^{N-\eta}\frac{U_a x}{a}\log\left|\frac{a-N}{a+N}\right|da$$
$$+\int_{\varepsilon+\eta}^{N-\eta}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da+\int_{-(N-\eta)}^{-(\varepsilon+\eta)}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a+\varepsilon}\right|da$$
$$+\int_{-(N-\eta)}^{-(\varepsilon+\eta)}\frac{U_a x}{a}\log\left|\frac{a-N}{a+N}\right|da+\int_{-(N-\eta)}^{-(\varepsilon+\eta)}\frac{U_a x}{a}\log\left|\frac{a+\eta}{a-\eta}\right|da$$
$$=\int_{\varepsilon+\eta}^{N-\eta}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da-\int_{\varepsilon+\eta}^{N-\eta}\frac{U_a x}{a}\log\left|\frac{a+N}{a-N}\right|da$$
$$+\int_{-(N-\eta)}^{-(\varepsilon+\eta)}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da-\int_{-(N-\eta)}^{-(\varepsilon+\eta)}\frac{U_a x}{a}\log\left|\frac{a+N}{a-N}\right|da$$
$$+\int_{\varepsilon+\eta}^{N-\eta}\frac{U_a x+U_{-a} x}{a}\log\left|\frac{a+\eta}{a-\eta}\right|da\,,$$

(5)
$$I_4=\int_{N-\eta}^{N+\eta}\frac{U_a x}{a}\left(\log\left|\frac{v}{a+v}\right|\Bigg|_{-(a-\eta)}^{-\varepsilon}+\log\left|\frac{v}{a+v}\right|\Bigg|_{\varepsilon}^{N}\right)da$$
$$+\int_{-(N+\eta)}^{-(N-\eta)}\frac{U_a x}{a}\left(\log\left|\frac{v}{a+v}\right|\Bigg|_{-N}^{-\varepsilon}+\log\left|\frac{v}{a+v}\right|\Bigg|_{\varepsilon}^{-a-\eta}\right)da$$
$$=\int_{N-\eta}^{N+\eta}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da+\int_{N-\eta}^{N+\eta}\frac{U_a x}{a}\log\left|\frac{N}{a+N}\right|da+\int_{N-\eta}^{N+\eta}\frac{U_a x}{a}\log\left|\frac{\eta}{a-\eta}\right|da$$

$$+\int_{-(N+\eta)}^{-(N-\eta)}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da-\int_{-(N+\eta)}^{-(N-\eta)}\frac{U_a x}{a}\log\left|\frac{N}{a-N}\right|da$$

$$+\int_{-(N+\eta)}^{-(N-\eta)}\frac{U_a x}{a}\log\left|\frac{a+\eta}{\eta}\right|da$$

$$=\int_{N-\eta}^{N+\eta}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da-\int_{N-\eta}^{N+\eta}\frac{U_a x}{a}\log\left|\frac{a+N}{a-N}\right|da$$

$$+\int_{-(N+\eta)}^{-(N-\eta)}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da-\int_{-(N+\eta)}^{-(N-\eta)}\frac{U_a x}{a}\log\left|\frac{a+N}{a-N}\right|da$$

$$+\int_{N-\eta}^{N+\eta}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{N\eta}{(a-\eta)(a-N)}\right|da\ ,$$

(6) $$I_5=\int_{N+\eta}^{M-N}\frac{U_a x}{a}\left(\log\left|\frac{v}{a+v}\right|\Big|_{-N}^{-\varepsilon}+\log\left|\frac{v}{a+v}\right|\Big|_{\varepsilon}^{N}\right)da$$

$$+\int_{-(M-N)}^{-(N+\eta)}\frac{U_a x}{a}\left(\log\left|\frac{v}{a+v}\right|\Big|_{-N}^{-\varepsilon}+\log\left|\frac{v}{a+v}\right|\Big|_{\varepsilon}^{N}\right)da$$

$$=\int_{N+\eta}^{M-N}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da-\int_{N+\eta}^{M-N}\frac{U_a x}{a}\log\left|\frac{a+N}{a-N}\right|da$$

$$+\int_{-(M-N)}^{-(N+\eta)}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da-\int_{-(M-N)}^{-(N+\eta)}\frac{U_a x}{a}\log\left|\frac{a+N}{a-N}\right|da\ ,$$

(7) $$I_6=\int_{M-N}^{M-\varepsilon}\frac{U_a x}{a}\left(\log\left|\frac{v}{a+v}\right|\Big|_{-N}^{-\varepsilon}+\log\left|\frac{v}{a+v}\right|\Big|_{\varepsilon}^{M-a}\right)da$$

$$+\int_{-(M-\varepsilon)}^{-(M-N)}\frac{U_a x}{a}\left(\log\left|\frac{v}{a+v}\right|\Big|_{-(M+a)}^{-\varepsilon}+\log\left|\frac{v}{a+v}\right|\Big|_{\varepsilon}^{N}\right)da$$

$$=\int_{M-N}^{M-\varepsilon}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da+\int_{-(M-\varepsilon)}^{-(M-N)}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da$$

$$+\int_{M-N}^{M-\varepsilon}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{(a-M)(a-N)}{MN}\right|da\ ,$$

(8) $$I_7=\int_{M-\varepsilon}^{M+\varepsilon}\frac{U_a x}{a}\log\left|\frac{v}{a+v}\right|\Big|_{-N}^{-\varepsilon}da+\int_{-(M+\varepsilon)}^{-(M-\varepsilon)}\frac{U_a x}{a}\log\left|\frac{v}{a+v}\right|\Big|_{\varepsilon}^{N}da$$

$$=\int_{M-\varepsilon}^{M+\varepsilon}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{\varepsilon(a-N)}{N(a-\varepsilon)}\right|da$$

and finaly

(9) $$I_8=\int_{M+\varepsilon}^{M+N}\frac{U_a x}{a}\log\left|\frac{v}{a+v}\right|\Big|_{-N}^{-(a-M)}da+\int_{-(M+N)}^{-(M+\varepsilon)}\frac{U_a x}{a}\log\left|\frac{v}{a+v}\right|\Big|_{-a-M}^{N}da$$

$$=\int_{M+\varepsilon}^{M+N}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{(a-M)(a-N)}{MN}\right|da\ .$$

Therefore we obtain, from (1), (2), $\cdots$, (8) and (9), that

$$-\pi^2 H_{\eta,M}H_{\varepsilon,N}x$$
$$=\int_{-(M-\varepsilon)}^{M-\varepsilon}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da-\int_{-M+N}^{M-N}\frac{U_a x}{a}\log\left|\frac{a+N}{a-N}\right|da+R(\eta,\varepsilon,N,M;x)$$

(change variables $a\longrightarrow\varepsilon t$ and $a\longrightarrow Nt$ respectively)

$$=\int_{-(M-\varepsilon)/\varepsilon}^{(M-\varepsilon)/\varepsilon}\frac{U_{\varepsilon t}x}{t}\log\left|\frac{t+1}{t-1}\right|dt-\int_{-(M-N)/N}^{(M-N)/N}\frac{U_{Nt}x}{t}\log\left|\frac{t+1}{t-1}\right|dt$$
$$+R(\eta,\varepsilon,N,M;x)\ .$$

This completes the proof.

LEMMA 2. *Let $\{U_t: t\in R\}$ be a one parameter group of operators on a complete locally convex space X. Let $R(\eta,\varepsilon,N,M;x)$ be defined as in Lemma 1. Then it follows that*

(i) $\lim_{\eta\to 0+,M\to\infty}R(\eta,\varepsilon,N,M;x)=0$ *for all $x\in X$ and all ε, N such that* $0<\varepsilon<(1/2)<2<N<\infty$,

and

(ii) $\lim_{x\to 0}R(\eta,\varepsilon,N,M;x)=0$ *uniformly for ε, η, M, N such that* $0<\eta<\varepsilon<(1/2)<2<N<2N+1<M<\infty$.

PROOF. We can easily find a constant $K>0$ such that

$$(1)\qquad \int_{\alpha-1}^{\alpha+1}\frac{1}{t}|\log|t\pm 1||\,dt<K \quad\text{and}\quad \int_{1-\beta}^{1+\beta}\frac{1}{t}|\log|t\pm1||\,dt<K\ ,$$

for all $1<\alpha<\infty$ and all $0<\beta<1$. Also, we see that

$$(2)\qquad \lim_{\alpha\to\infty}\int_{\alpha-1}^{\alpha+1}\frac{1}{t}|\log|t\pm1||\,dt=0 \quad\text{and}\quad \lim_{\alpha\to 0}\int_{1-\alpha}^{1+\alpha}\frac{1}{t}|\log|t\pm1||\,dt=0\ .$$

Let $R(\eta,\varepsilon,N,M;x)$ be defined as in Lemma 1, that is,

$$R(\eta,\varepsilon,N,M;x)$$
$$=\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{(a+\eta)(a-\varepsilon)}{\varepsilon\eta}\right|da+\int_{\varepsilon+\eta}^{N-\eta}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{a+\eta}{a-\eta}\right|da$$
$$+\int_{N-\eta}^{N+\eta}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{N\eta}{(a-N)(a-\eta)}\right|da$$
$$+\int_{M-N}^{M-\varepsilon}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{(a-M)(a-N)}{MN}\right|da$$
$$+\int_{M-\varepsilon}^{M+\varepsilon}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{\varepsilon(a-N)}{N(a-\varepsilon)}\right|da$$

$$+\int_{M+\varepsilon}^{M+N}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{(a-M)(a-N)}{MN}\right|da$$
$$=J_1+J_2+\cdots+J_6\,,\quad\text{say}\,.$$

Now we are going to estimate $R(\eta, \varepsilon, N, M; x)$. Let x be any element in X. Let q be any semi-norm from the system $\{q\}$ semi-norms defining the topology of X.

Let θ be any positive number. Since $U_t: X\to X$ is continuous uniformly for $t\in \boldsymbol{R}$, we take a balanced convex neighborhood W of 0 in X such that

$$q(U_t x)<\theta \quad\text{for all } x\in W \text{ and } t\in \boldsymbol{R}\,.$$

First we see that

$$\begin{aligned} q(J_1)&=q\left(\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{(a+\eta)(a-\varepsilon)}{\varepsilon\eta}\right|da\right)\\ &\leqq 2\theta\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{1}{a}\left|\log\left|\frac{a-\varepsilon}{\varepsilon}\right|\right|da+2\theta\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{1}{a}\left|\log\left|\frac{a+\eta}{\eta}\right|\right|da\\ &\leqq 2\theta\int_{1-(\eta/\varepsilon)}^{1+(\eta/\varepsilon)}\frac{1}{t}|\log|t-1||\,dt+2\theta\int_{(\varepsilon/\eta)-1}^{(\varepsilon/\eta)+1}\frac{1}{t}|\log|t+1||\,dt \end{aligned}$$

which implies, by (1) and (2), that

(3) $\lim\limits_{\substack{\eta\to 0+\\ M\to\infty}} J_1=0$ for all $x\in X$ and all $0<\varepsilon<N<\infty$

and

(3)′ $\lim\limits_{x\to 0} J_1=0$ uniformly for $0<\eta<\varepsilon<1/2<2<N<2N+1<M<\infty$.

Next we see that

$$\begin{aligned} q(J_2)&=q\left(\int_{\varepsilon+\eta}^{N-\eta}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{a+\eta}{a-\eta}\right|da\right)\\ &\leqq 2\theta\int_{(\varepsilon/\eta)-1}^{(\varepsilon/\eta)+1}\frac{1}{t}\left|\log\left|\frac{t+1}{t-1}\right|\right|dt\,. \end{aligned}$$

This implies that

(4) $\lim\limits_{\substack{\eta\to 0+\\ M\to\infty}} J_2=0$ for all $x\in X$ and all $0<\varepsilon<N<\infty$

and

(4)′ $\lim\limits_{x\to 0} J_2=0$ uniformly for $0<\eta<\varepsilon<1/2<2<N<2N+1<M<\infty$,

since $\int_{-\infty}^{\infty}(1/t)|\log|(t+1)/(t-1)||dt=\pi^2$.
Similarly we see that

$$q(J_3)=q\left(\int_{N-\eta}^{N+\eta}\frac{U_ax+U_{-a}x}{a}\log\left|\frac{N\eta}{(a-N)(a-\eta)}\right|da\right)$$
$$\leqq 2\theta\int_{1-(\eta/N)}^{1+(\eta/N)}\frac{1}{t}|\log|t-1||\,dt+2\theta\int_{(N/\eta)-1}^{(N/\eta)+1}\frac{1}{t}|\log|t-1||\,dt\,,$$

which implies, by (1) and (2), that

(5) $\lim_{\substack{\eta\to 0+\\ M\to\infty}} J_3=0$ for all $x\in X$ and all $0<\varepsilon<N<\infty$

and

(5)′ $\lim_{\alpha\to 0} J_3=0$ uniformly for $0<\eta<\varepsilon<1/2<2<N<2N+1<M<\infty$,

and we see that

$$q(\{J_4+J_6\})=q\left(\int_{M-N}^{M-\varepsilon}\frac{U_ax+U_{-a}x}{a}\log\left|\frac{(a-M)(a-N)}{MN}\right|da\right.$$
$$\left.+\int_{M+\varepsilon}^{M+N}\frac{U_ax+U_{-a}x}{a}\log\left|\frac{(a-M)(a-N)}{MN}\right|da\right)$$
$$\leqq 2\theta\int_{M-N}^{M+N}\frac{1}{a}\log\left|\frac{(a-M)(a-N)}{MN}\right|da$$
$$\leqq 2\theta\int_{1-(N/M)}^{1+(N/M)}\frac{1}{t}|\log|t-1||\,dt+2\theta\int_{(M/N)-1}^{(M/N)+1}\frac{1}{t}|\log|t-1||\,dt\,,$$

which implies, by (1) and (2), that

(6) $\lim_{\substack{\eta\to 0+\\ M\to\infty}}\{J_4+J_6\}=0$ for all $x\in X$ and all $0<\varepsilon<N<\infty$

and

(6)′ $\lim_{\alpha\to 0}\{J_4+J_6\}=0$ uniformly for $0<\eta<\varepsilon<1/2<2<N<2N+1<M<\infty$.

Lastly we see that

$$q(J_5)=q\left(\int_{M-\varepsilon}^{M+\varepsilon}\frac{U_ax+U_{-a}x}{a}\log\left|\frac{\varepsilon(a-N)}{N(a-\varepsilon)}\right|da\right)$$
$$\leqq 2\theta\int_{M-\varepsilon}^{M+\varepsilon}\frac{1}{a}\log\left|\frac{(a-N)}{N}\right|da+2\theta\int_{M-\varepsilon}^{M+\varepsilon}\left|\frac{1}{a}\log\left|\frac{\varepsilon}{a-\varepsilon}\right|\right|da$$
$$\leqq 2\theta\int_{(M/N)-1}^{(M/N)+1}\frac{1}{t}|\log|t-1||\,dt+2\theta\int_{(M/\varepsilon)-1}^{(M/\varepsilon)+1}\frac{1}{t}|\log|t-1||\,dt\,,$$

which implies, by (1) and (2), that

(7) $\lim_{\substack{\eta\to 0+\\ M\to\infty}} J_5=0$ for all $x\in X$ and $0<\varepsilon<N<\infty$

and

(7)′ $\lim_{x\to 0} J_5=0$ uniformly for $0<\eta<\varepsilon<1/2<2<N<2N+1<M<\infty$.

Putting above estimates (3), $\cdots$, (7) and (3)′, $\cdots$, (7)′ together, we see that (i) and (ii) are true. The proof of the lemma is now complete.

THEOREM 4. *Let $\{U_t; t\in \boldsymbol{R}\}$ be a one parameter group of operators on a complete locally convex space X. Then, for any $x\in X$, $HH_{\varepsilon,N}\circ x$ $(0<\varepsilon<N<\infty)$ exists in X, and*

$$HH_{\varepsilon,N}x=-\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{U_{\varepsilon t}x}{t}\log\left|\frac{t+1}{t-1}\right|dt+\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{U_{Nt}x}{t}\log\left|\frac{t+1}{t-1}\right|dt\ .$$

PROOF. Let x be any element in X. By Lemma 1 and (i) in Lemma 2, we see that

$$\begin{aligned}HH_{\varepsilon,N}x&=\lim_{\substack{\eta\to 0+\\ M\to\infty}} H_{\eta,M}H_{\varepsilon,N}x\\ &=\lim_{\substack{\eta\to 0+\\ M\to\infty}}\Big[-\frac{1}{\pi^2}\Big[\int_{-(M-\varepsilon)/\varepsilon}^{(M-\varepsilon)/\varepsilon}\frac{U_{\varepsilon t}x}{t}\log\left|\frac{t+1}{t-1}\right|dt\\ &\qquad -\int_{-(M-N)/N}^{(M-N)/N}\frac{U_{Nt}x}{t}\log\left|\frac{t+1}{t-1}\right|dt+R(\eta,\varepsilon,N,M;x)\Big]\\ &=-\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{U_{\varepsilon t}x}{t}\log\left|\frac{t+1}{t-1}\right|dt+\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{U_{Nt}x}{t}\log\left|\frac{t+1}{t-1}\right|dt\ .\end{aligned}$$

This completes the proof.
The following lemmas are useful to prove Theorem 5.

LEMMA 3. *Let $\{U_t: t\in\boldsymbol{R}\}$ be a one parameter group of operators on a complete locally convex space X. Let x be any element in X such that $\bar{x}$ exists. Then, for any $\phi\in L^1(\boldsymbol{R})$ such that $\int_{-\infty}^{\infty}\phi(t)dt=1$, $\lim_{N\to\infty}\int_{-\infty}^{\infty}U_{Nt}x\phi(t)dt$ exists in X and is equal to $\bar{x}$.*

PROOF. We define a characteristic function $\chi_{(a,b]}\colon \boldsymbol{R}\to\{0, 1\}$ such that

$$\chi_{(a,b]}(t)=1\ (\text{for } t\in(a, b])\text{ and } 0\ (\text{elsewhere})\ .$$

First we assume that ϕ is represented by a linear combination of above characteristic functions i.e.

$$\phi(t)=\sum_{i=1}^{m} c_i \chi_{(a_i,b_i]}(t)$$

where $(a_i, b_i]$, $i=1,2,\cdots,m$, are disjoint intervals, and hence $\sum_{i=1}^{m} c_i(b_i-a_i)=\int_{-\infty}^{\infty}\phi(t)dt=1$. Then we see that

$$\begin{aligned}\lim_{N\to\infty}\int_{-\infty}^{\infty} U_{Nt}x\phi(t)dt&=\lim_{N\to\infty}\sum_{i=1}^{m} c_i\int_{a_i}^{b_i} U_{Nt}xdt=\sum_{i=1}^{m}\left[c_i(b_i-a_i)\lim_{N\to\infty}\frac{1}{(b_i-a_i)N}\int_{a_iN}^{b_iN} U_t x dt\right]\\&=\bar{x}\sum_{i=1}^{m} c_i(b_i-a_i)=\bar{x}\ .\end{aligned}$$

Next we shall consider the case of a general ϕ. Let ϕ be any element in $L^1(\boldsymbol{R})$ such that $\int_{-\infty}^{\infty}\phi(t)dt=1$. Let ε be any positive real and let q be any semi-norm from the system of semi-norms defining the topology of X. Then we can find an $L>0$ and a linear combination $\phi_0(t)=\sum_{i=1}^{m} c_i\chi_{(a_i,b_i]}(t)$ such that

$$q(U_t x)\leqq L \quad \text{for all } t\in\boldsymbol{R}$$

and

$$\|\phi-\phi_0\|_1<\varepsilon\ .$$

Therefore, we see that

$$\begin{aligned}&q\Big(\int_{-\infty}^{\infty} U_{Nt}x\phi(t)dt-\int_{-\infty}^{\infty} U_{Mt}x\phi(t)dt\Big)\\&\quad\leqq q\Big(\int_{-\infty}^{\infty} U_{Nt}x(\phi(t)-\phi_0(t))dt\Big)+q\Big(\int_{-\infty}^{\infty} U_{Nt}x\phi_0(t)dt-\int_{-\infty}^{\infty} U_{Mt}x\phi_0(t)dt\Big)\\&\qquad+q\Big(\int_{-\infty}^{\infty} U_{Mt}x(\phi(t)-\phi_0(t))dt\Big)\\&\quad\leqq 2L\varepsilon+q\Big(\int_{-\infty}^{\infty} U_{Nt}x\phi_0(t)dt-\int_{-\infty}^{\infty} U_{Mt}x\phi_0(t)dt\Big)\ ,\end{aligned}$$

which implies that $\left\{\int_{-\infty}^{\infty} U_{Nt}x\phi(t)dt\right\}_{N=1}^{\infty}$ is a Cauchy sequence in X,and has a certain limit y in X, since as was shown above $\left\{\int_{-\infty}^{\infty} U_{Nt}x\phi_0(t)dt\right\}_{N=1}^{\infty}$ is a convergent sequence in X. Moreover, it is clear that $y=\bar{x}$, and this completes the proof.

LEMMA 4. *Let $\{U_t;\, t\in\boldsymbol{R}\}$ be a one parameter group of operators on a complete locally convex space X. Let x be an element in X such that $\lim_{N\to\infty}\int_{-\infty}^{\infty} U_{Nt}x\phi(t)dt$ exists for some $\phi\in L^1(\boldsymbol{R})$ with $\int_{-\infty}^{\infty}\phi(t)dt=1$. Then*

$\bar{x}$ exists in X and $\bar{x}=\lim_{N\to\infty}\int_{-\infty}^{\infty} U_{Nt}x\phi(t)dt$.

PROOF. Let $\lim_{N\to\infty}\int_{-\infty}^{\infty} U_{Nt}x\phi(t)dt$ be denoted by x^*. Firstly, we shall prove that $U_s x^*=x^*$ for all $s\in \boldsymbol{R}$. Let s be any fixed real number. Then we see that

$$
\begin{aligned}
(1)\qquad & U_s x^*-x^* \\
&=U_s\Big[\lim_{N\to\infty}\int_{-\infty}^{\infty} U_{Nt}x\phi(t)dt\Big]-x^* \\
&=U_s\Big[\lim_{N\to\infty}\frac{1}{N}\int_{-\infty}^{\infty} U_t x\phi\Big(\frac{t}{N}\Big)dt\Big]-x^* \\
&=\lim_{N\to\infty}\frac{1}{N}\int_{-\infty}^{\infty} U_{t+s}x\phi\Big(\frac{t}{N}\Big)dt-x^* \\
&=\lim_{N\to\infty}\frac{1}{N}\int_{-\infty}^{\infty} U_t x\phi\Big(\frac{t-s}{N}\Big)dt-x^* \\
&=\lim_{N\to\infty}\int_{-\infty}^{\infty} U_{Nt}x\phi\Big(t-\frac{s}{N}\Big)dt-\lim_{N\to\infty}\int_{-\infty}^{\infty} U_{Nt}x\phi(t)dt \\
&=\lim_{N\to\infty}\int_{-\infty}^{\infty} U_{Nt}x\Big(\phi\Big(t-\frac{s}{N}\Big)-\phi(t)\Big)dt \\
&=0\,,
\end{aligned}
$$

since U_t: $X\to X$ is continuous uniformly for $t\in\boldsymbol{R}$ and $\phi(t-(s/N))-\phi(t)\to 0$ in $L^1(\boldsymbol{R})$ as $N\to\infty$. Hence we get that $U_s x^*=x^*$ for all $s\in\boldsymbol{R}$.

Now let $D(t)$ be a function on $\boldsymbol{R}$ such that

$$D(t)=1/2\quad (t\in[-1,1])\ \text{ and } 0\ \text{(elsewhere)}\,.$$

Let V be any balanced convex neighborhood of 0 in X. By the continuity of U_t: $X\to X$ uniformly for $t\in\boldsymbol{R}$, there exists a balanced convex neighborhood W of 0 in X such that

$$(2)\qquad U_t W\subset\frac{V}{3}\quad \text{for all } t\in\boldsymbol{R}\,.$$

Also, there exists an $\eta>0$ such that

$$(3)\qquad \int_{-\infty}^{\infty} U_{Nt}xD(t)dt-\int_{-\infty}^{\infty} U_{Nt}x\Big(\int_{-\infty}^{\infty} D(t+s)\frac{1}{\eta}\phi\Big(\frac{s}{\eta}\Big)ds\Big)dt\in\frac{V}{3}\quad \text{for all } N\geqq 0\,,$$

since $\int_{-\infty}^{\infty} D(t+s)(1/\eta)\phi(s/\eta)ds\to D(t)$ $(\eta\to 0+)$ in $L^1(\boldsymbol{R})$.
And there exists an $N_0=N_0(\eta)>0$ such that

$$(4) \qquad \int_{-\infty}^{\infty} U_{N\eta t}x\phi(t)dt - x^* \in W \quad (N \geqq N_0)$$

and

$$(5) \qquad x^* - \int_{-\infty}^{\infty} U_{Nt}x\phi(dt) \in \frac{V}{3} \quad (N \geqq N_0) .$$

Then we see, by (1), (4) and (2), that

$$\begin{aligned}(6) \qquad & \int_{-\infty}^{\infty} U_{Nt}x\left(\int_{-\infty}^{\infty} D(t+s)\frac{1}{\eta}\phi\left(\frac{s}{\eta}\right)ds\right)dt - x^* \\ &= \int_{-\infty}^{\infty} D(s)\left[\int_{-\infty}^{\infty} U_{Nt}x\frac{1}{\eta}\phi\left(\frac{t+s}{\eta}\right)dt - x^*\right]ds \\ &= \frac{1}{2}\int_{-1}^{1}\left[U_{-Ns}\left(\int_{-\infty}^{\infty} U_{N\eta t}x\phi(t)dt - x^*\right)\right]ds \\ &\in V/3 \quad (N \geqq N_0) .\end{aligned}$$

Therefore we, by (3), (6) and (5), find that

$$\begin{aligned} & \frac{1}{2N}\int_{-N}^{N} U_t x dt - \int_{-\infty}^{\infty} U_{Nt}x\phi(t)dt \\ &= \left[\int_{-\infty}^{\infty} U_{Nt}xD(t)dt - \int_{-\infty}^{\infty} U_{Nt}x\left(\int_{-\infty}^{\infty} D(t+s)\frac{1}{\eta}\phi\left(\frac{s}{\eta}\right)ds\right)dt\right] \\ &\quad + \left[\int_{-\infty}^{\infty} U_{Nt}x\left(\int_{-\infty}^{\infty} D(t+s)\frac{1}{\eta}\phi\left(\frac{s}{\eta}\right)ds\right)dt - x^*\right] \\ &\quad + \left[x^* - \int_{-\infty}^{\infty} U_{Nt}x\phi(t)dt\right] \\ &\in \frac{V}{3} + \frac{V}{3} + \frac{V}{3} = V \quad (N \geqq N_0) .\end{aligned}$$

Since V is arbitrary convex balanced neighborhood of 0 in X, this implies that $\bar{x}$ exists and $\bar{x} = \lim_{N\to\infty}\int_{-\infty}^{\infty} U_{Nt}x\phi(t)dt$. This completes the proof.

THEOREM 5. *Let $\{U_t; t \in \boldsymbol{R}\}$ be a one parameter group of operators on a complete locally convex space X. Let x be any element in $D(H)$. Then the following two statements are equivalent.*

(i) *$\bar{x}$ exists in X,*

(ii) *Hx belongs to $D(H)$.*

Moreover, if $\bar{x}$ exists in X, then $H^2x = -(x - \bar{x})$.

PROOF. Let x be any element in $D(H)$. Since $H_{\epsilon,N}$ is continuous, we see, by Theorem 4, that

$$(1)\quad \lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} H_{\varepsilon,N}Hx=\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} H_{\varepsilon,N}(\lim_{\substack{\eta\to 0+\\ M\to\infty}} H_{\eta,M}x)=\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}}\lim_{\substack{\eta\to 0+\\ M\to\infty}} H_{\eta,M}H_{\varepsilon,N}x=\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} HH_{\varepsilon,N}x$$

$$=-\frac{1}{\pi^2}\lim_{\varepsilon\to 0+}\int_{-\infty}^{\infty}\frac{U_{\varepsilon t}x}{t}\log\left|\frac{t+1}{t-1}\right|dt+\frac{1}{\pi^2}\lim_{N\to\infty}\int_{-\infty}^{\infty}\frac{U_{Nt}x}{t}\log\left|\frac{t+1}{t-1}\right|dt$$

$$=-x+\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}}\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{U_{Nt}x}{t}\log\left|\frac{t+1}{t-1}\right|dt\,.$$

Hence we get, by (1) and Lemma 3, that (i) implies (ii). Moreover, it immediately follows that (i) implies that $H^2x=-(\bar{x}-x)$.
Also, we get, by (1) and Lemma 4, that (ii) implies (i). This completes the proof.

LEMMA 5. *Let $\{U_t; t\in \boldsymbol{R}\}$ be a one parameter group of operators on a complete locally convex space X. Then, it follows that $\lim_{x\to 0} H_{\eta,M}H_{\varepsilon,N}x=0$ uniformly for ε, η, M, N such that $0<\eta<\varepsilon<\frac{1}{2}<2<N<2N+1<M<\infty$.*

PROOF. Let θ be any positive number. Let q be any semi-norm from the system $\{q\}$ of semi-norms defining the topology of X. Since U_t: $X\to X$ is continuous uniformly for $t\in \boldsymbol{R}$, we can take a neighborhood W of 0 in X such that

$$q(U_tx)\leqq\theta \quad \text{for all } x\in W \text{ and all } t\in \boldsymbol{R}\,.$$

Then, we see, by Lemma 2, that, for any $x\in W$ and any ε, η, M, N such that $0<\eta<\varepsilon<(1/2)<2<N<2N+1<M<\infty$,

$$\begin{aligned}
&q(H_{\eta,M}H_{\varepsilon,N}x)\\
&\leqq q\Big(-\frac{1}{\pi^2}\Big[\int_{-(M-\varepsilon)/\varepsilon}^{(M-\varepsilon)/\varepsilon}\frac{U_{\varepsilon t}x}{t}\log\left|\frac{t+1}{t-1}\right|dt\\
&\quad-\int_{-(M-N)/N}^{(M-N)/N}\frac{U_{Nt}x}{t}\log\left|\frac{t+1}{t-1}\right|dt+R(\eta,\varepsilon,N,M;x)\Big]\Big)\\
&\leqq\frac{\theta}{\pi^2}\Big[\int_{-\infty}^{\infty}\frac{1}{t}\log\left|\frac{t+1}{t-1}\right|dt+\int_{-\infty}^{\infty}\frac{1}{t}\log\left|\frac{t+1}{t-1}\right|dt+K\Big]\\
&\leqq\theta\Big(2+\frac{K}{\pi^2}\Big)\,,
\end{aligned}$$

where K is a positive constant independent of η, ε, N and M (such K exists by Lemma 2, (ii)). This completes the proof.

LEMMA 6. *Let $\{U_t; t\in \boldsymbol{R}\}$ be a one parameter group of operators on* *a complete locally convex space X. Then, it follows that*

(i) *for any $x \in X$ and any $0<\varepsilon<N<\infty$, $(H_{\varepsilon,N}x)^{-}=0$,*

and

(ii) *for any $x \in D(H)$, $(Hx)^{-}=0$.*

PROOF. Firstly we shall prove the first part of lemma. Let x be any element in X. We see that, for large $T>0$,

$$(1)\qquad \begin{aligned} I&=\frac{1}{2T}\int_{-T}^{T} U_t H_{\varepsilon,N}x dt \\ &=\frac{1}{2T}\int_{-T}^{T} U_t\left[\frac{1}{\pi}\int_{\varepsilon<|s|<N}\frac{U_s x}{s}ds\right]dt \\ &=\frac{1}{2T}\int_{-T}^{T}\left[\frac{1}{\pi}\int_{\varepsilon<s<N}\frac{U_{t+s}x-U_{t-s}x}{s}ds\right]dt \\ &=\frac{1}{2\pi}\int_{\varepsilon<s<N}\frac{1}{s}\left[\frac{1}{2T}\int_{-T}^{T}(U_{t+s}x-U_{t-s}x)dt\right]ds \\ &=\frac{1}{2\pi}\int_{\varepsilon<s<N}\frac{1}{s}\left[-\frac{1}{2T}\int_{-T-s}^{-T+s}U_t x dt+\frac{1}{2T}\int_{T-s}^{T+s}U_t x dt\right]ds\ . \end{aligned}$$

Let q be and semi-nom from the system $\{q\}$ of semi-norms defining the topology of X. By the uniform continuity of $\{U_t: t \in \boldsymbol{R}\}$, we can take $C>0$ such that

$$q(U_t x)\leqq C \quad \text{for all } t \in \boldsymbol{R}\ .$$

Hence we get, by (1), that

$$q(I)\leqq\frac{1}{2\pi}\int_{\varepsilon<s<N}\frac{1}{s}\left[\frac{2Cs}{T}\right]ds=\frac{C(N-\varepsilon)}{\pi T}\longrightarrow 0 \quad \text{as } T\longrightarrow+\infty\ .$$

This implies that $(H_{\varepsilon,N}x)^{-}=0$.

Next we shall prove the second part of lemma. Let x be any element in $D(H)$. Let V be any balanced convex neighborhood of 0 in X. By the uniform continuity of U_t, there exists a balanced convex neighborhood W of 0 in X such that

$$(2)\qquad U_t W\subset\frac{V}{2} \quad \text{for all } t \in \boldsymbol{R}\ .$$

Since $x \in D(H)$, there exist positive number ε_0 and N_0 such that

$$(3)\qquad Hx-H_{\varepsilon_0,N_0}x \in W\ .$$

And, by the first part of theorem, we take $T_0>0$ such that

$$(4)\qquad \frac{1}{2T}\int_{-T}^{T} U_t H_{\varepsilon_0,N_0} x dt \in \frac{V}{2} \quad \text{for all } T \geqq T_0 .$$

Hence we see, by (2), (3) and (4), that, for any $T \geqq T_0$,

$$\begin{aligned}&\frac{1}{2T}\int_{-T}^{T} U_t H x dt \\ &\quad = \frac{1}{2T}\int_{-T}^{T} U_t(H - H_{\varepsilon_0,N_0}) x dt + \frac{1}{2T}\int_{-T}^{T} U_t H_{\varepsilon_0,N_0} x dt \\ &\quad \in \frac{V}{2} + \frac{V}{2} = V\end{aligned}$$

which implies that $(Hx)^-$ exists in X and $(Hx)^- = 0$. This completes the proof.

THEOREM 6. *Let $\{U_t; t \in R\}$ be a one parameter group of operators on a complete locally convex space X. Then, the Hilbert transform H is a closed operator on X (though $D(H)$ is not always dense in X).*

PROOF. Assume that $\{x_k\}_{k \in K}$ is any generalized sequence in $D(H)$ such that

$$(1)\qquad x_k \longrightarrow x \quad \text{and} \quad Hx_k \longrightarrow y \text{ in } X .$$

It is sufficient to prove that $x \in D(H)$ and $Hx = y$. Let V be any balanced convex neighborhood of 0 in X. By Lemma 5, we can take a balanced convex neighborhood W of 0 in X such that

$$(2)\qquad H_{\eta,M} H_{\varepsilon,N} W \subset \frac{V}{4} ,$$

for all ε, η, M, N such that $0 < \eta < \varepsilon < (1/2) < 2 < N < 2N+1 < M < \infty$.
And we take $k_0 \in K$ such that

$$(3)\qquad Hx_k - y \in W \quad \text{for all } k \geqq k_0 .$$

By (2) and (3), we see that

$$(4)\qquad H_{\eta,M} H_{\varepsilon,N}(Hx_k - y) \in \frac{V}{4} ,$$

for all $k \geqq k_0$ and for all ε, η, M, N such that $0 < \eta < \varepsilon < (1/2) < 2 < N < 2N+1 < M < \infty$.
Letting $\eta \to 0+$, $M \to \infty$ in (4) and noting $\lim_{\eta \to 0+, M \to \infty} H_{\eta,M} H_{\varepsilon,N} H x_k = H^2 H_{\varepsilon,N} x_k$, we see, by Theorem 4, Theorem 5 and Lemma 6, that

$$(5)\qquad -H_{\varepsilon,N}x_k - HH_{\varepsilon,N}y \in \frac{V}{3},$$

for all $k \geqq k_0$ and for all $0<\varepsilon<(1/2)<2<N<\infty$.
Letting $k \to \infty$ in (5), we find that

$$(6)\qquad -H_{\varepsilon,N}x - HH_{\varepsilon,N}y \in \frac{V}{2},$$

for all $0<\varepsilon<(1/2)<2<N<\infty$.

Next we shall prove that $\bar{y}=0$. Let G be any balanced convex neighborhood of 0 in X. We can, by the uniform continuity of U_t, take $k_0 \in K$ such that

$$(7)\qquad \frac{1}{2T}\int_{-T}^{T} U_t(y-Hx_{k_0})dt \in \frac{G}{2} \quad \text{for all } T>0,$$

and we can, from Lemma 6, take $T_0>0$ such that

$$(8)\qquad \frac{1}{2T}\int_{-T}^{T} U_t(Hx_{k_0})dt \in \frac{G}{2} \quad \text{for all } T \geqq T_0.$$

By (7) and (8), we see that, for large T such that $T \geqq T_0$,

$$\frac{1}{2T}\int_{-T}^{T} U_t y\,dt = \left[\frac{1}{2T}\int_{-T}^{T} U_t(y-Hx_k)dt\right] + \left[\frac{1}{2T}\int_{-T}^{T} U_t(Hx_{k_0})dt\right] \in \frac{G}{2}+\frac{G}{2}=G,$$

which implies that $\bar{y}=0$.
From this, Theorem 4 and Theorem 5, we see that, for small ε and large N,

$$HH_{\varepsilon,N}y-(-y) \in \frac{V}{2}$$

Then if follows, from this and (6), that

$$H_{\varepsilon,N}y - y \in V$$

for small ε and large N, which implies that $x \in D(H)$ and $Hx=y$. This completes the proof.

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Present Address:
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCES AND ENGINEERINGS
KEIO UNIVERSITY
HIYOSHI, KOHOKU-KU
YOKOHAMA 223