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K-theory for the C^*-algebras of the Discrete Heisenberg Group

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Preliminaries

By the discrete Heisenberg group we mean the group G defined as that of the following matrices;

$$G=\left\{\begin{bmatrix}1 & m & l\\ 0 & 1 & k\\ 0 & 0 & 1\end{bmatrix}; k, l, m\in \boldsymbol{Z}\right\}.$$

We take two closed subgroups M and N of G as follows;

$$M=\left\{\begin{bmatrix}1 & m & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}; m\in \boldsymbol{Z}\right\}$$

and

$$N=\left\{\begin{bmatrix}1 & 0 & l\\ 0 & 1 & k\\ 0 & 0 & 1\end{bmatrix}; k, l\in \boldsymbol{Z}\right\}.$$

It is clear that $M\cong \boldsymbol{Z}$, $N\cong \boldsymbol{Z}^2$, so that we may identify M with $\boldsymbol{Z}$ and N with $\boldsymbol{Z}^2$. An action of M on N is defined by

$$m\cdot z=mzm^{-1}=(k, l+mk)$$

for $m\in M$ and $z=(k, l)\in N$. Then G is isomorphic to the semidirect product $N\times_s M$ of N by M with the multiplication

$$(z, m)(z', m')=(z+m\cdot z', m+m')$$

for (z, m) and $(z', m')\in N\times_s M$. Thererfore we identify G with $\boldsymbol{Z}^2\times_s \boldsymbol{Z}$

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and write the element of G as (k, l, m) where $(k, l) \in \boldsymbol{Z}^2 = N$ and $m \in \boldsymbol{Z} = M$. Further by definition of crossed products and the Fourier transformation we see that $C^*(G)$ is isomorphic to the crossed product $C(T^2) \times_\alpha \boldsymbol{Z}$ where α is the automorphism of $C(T^2)$ defined by

$$\alpha(f)(s, t) = f(s+t, t)$$
$$f \in C(T^2), \qquad (s, t) \in T^2$$

and T^2 is the two dimensional torus.

Let τ be the finite faithful trace on $C^*(G)$ defined by $\tau(x) = x(e)$ where $x \in l^1(G)$ and e is the unit element of G, and let σ be the trace on $C(T^2) \times_\alpha \boldsymbol{Z}$ by $\sigma(y) = \int_{T^2} y(0, s, t) ds dt$ where $y \in l^1(\boldsymbol{Z}, C(T^2))$. Then we see easily that $\tau = \sigma$ on $l^1(G)$. In what follows, we compute

$$K_j(C(T^2) \times_\alpha \boldsymbol{Z}) \quad (j=0, 1) \quad \text{and} \quad \sigma_*(K_0(C(T^2) \times_\alpha \boldsymbol{Z})) .$$

§ 1. Computation of $K_j(C(T^2) \times_\alpha \boldsymbol{Z})$ $j=0, 1$.

We use the following Pimsner-Voiculescu exact sequence of K-theory for crossed products;

$$\begin{array}{ccccc} K^0(T^2) & \xrightarrow{\mathrm{id}-\alpha_*^{-1}} & K^0(T^2) & \longrightarrow & K_0(C(T^2) \times_\alpha \boldsymbol{Z}) \\ \uparrow & & & & \downarrow \\ K_1(C(T^2) \times_\alpha \boldsymbol{Z}) & \longleftarrow & K^1(T^2) & \xleftarrow{\mathrm{id}-\alpha_*^{-1}} & K^1(T^2) . \end{array}$$

$K_j(C(T^2) \times_\alpha \boldsymbol{Z}) \cong K^j(T^2)/\mathrm{Im}(\mathrm{id} - \alpha_*^{-1}) \oplus \mathrm{Ker}(\mathrm{id} - \alpha_*^{-1})$ $(j=0, 1)$. We then compute $\mathrm{Im}(\mathrm{id} - \alpha_*^{-1})$ and $\mathrm{Ker}(\mathrm{id} - \alpha_*^{-1})$. Let $M_n(C(T^2))$ be the algebra of $n \times n$ matrices with entries in $C(T^2)$ and let $\mathrm{Proj}\, M_n(C(T^2))$ be the set of projections of $M_n(C(T^2))$ and let $U_n(C(T^2))$ be the unitary group of $M_n(C(T^2))$. We define p_j and q_j in $\cup_{n=1}^{\infty} \mathrm{Proj}\, M_n(C(T^2))$ $j=1, 2$ as follows;

$$p_1(s, t) = 1$$
$$q_1(s, t) = 0$$

and

$$p_2(s, t) = R(t) \begin{bmatrix} e^{-2\pi i s} & 0 \\ 0 & 1 \end{bmatrix} R(t)^* \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R(t) \begin{bmatrix} e^{2\pi i s} & 0 \\ 0 & 1 \end{bmatrix} R(t)^*$$

$$R(t) = \begin{bmatrix} \cos \frac{\pi}{2} t & -\sin \frac{\pi}{2} t \\ \sin \frac{\pi}{2} t & \cos \frac{\pi}{2} t \end{bmatrix}$$

$$0 \leqq s,\ t \leqq 1$$

$$q_2(s,\ t)=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}.$$

And we define u_j in $\cup_{n=1}^{\infty} U_n(C(T^2))$ $j=1, 2$ as follows;

$$u_1(s,\ t)=e^{2\pi it}$$
$$u_2(s,\ t)=e^{2\pi is}.$$

LEMMA 1. 1) *Two generators of* $K^0(T^2)$ *are* $[p_1]-[q_1]$ and $[p_2]-[q_2]$.
2) *Two generators of* $K^1(T^2)$ *are* $[u_1]$ *and* $[u_2]$.

REMARK. We identify $C(T^2)$ with all complex valued continuous functions on $[0, 1]\times[0, 1]$ such that $f(0, t)=f(1, t)$ and $f(s, 0)=f(s, 1)$ for $s, t \in [0, 1]$.

PROOF OF LEMMA 1. 1) $K^0(T^2)$ is isomorphic to $K^0(T^1)\oplus K^1(T^1)$. The isomorphism is the direct sum of i_* and Φ where i_* is the homomorphism of $K^0(T^1)$ into $K^0(T^2)$ induced by the inclusion map i; $C(T^1)\to C(T^2)$ and Φ is the composed map of the suspension map of $K^1(T^1)$ onto $K^0(T^1\times(0, 1))$ and the homomorphism of $K_0((T^1)\times(0, 1))$ into $K^0(T^2)$ induced by the inclusion map of $C_0(T^1\times(0, 1))$ into $C(T^2)$. And let $[1_{T^1}]$ be a generator of $K^0(T^1)$ where 1_{T^1} is the identity of $C(T^1)$ and let $[v]$ be a generator of $K^1(T^1)$ where v is defined by $v(s)=e^{2\pi is}$. Then $i_*([1_{T^1}])$ and $\Phi([v])$ are the generators of $K^0(T^2)$. Therefore we obtain 1).

2) We can prove 2) in the same manner as 1). Q.E.D.

LEMMA 2.

$$K_j(C(T^2)\times_\alpha \boldsymbol{Z})\cong \boldsymbol{Z}^3 \quad j=0, 1.$$

PROOF. We use the Pimsner-Voiculescu exact sequence. Clearly $\alpha_*^{-1}([p_1])=[p_1]$, $\alpha_*^{-1}([q_1])=[q_1]$, $\alpha_*^{-1}([q_2])=[q_2]$.

$$\alpha^{-1}(p_2)(s,\ t)=R(t)\begin{bmatrix}e^{-2\pi i(s-1)} & 0\\ 0 & 1\end{bmatrix}R(t)^*\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}R(t)\begin{bmatrix}e^{2\pi i(s-t)} & 0\\ 0 & 1\end{bmatrix}R(t)^*.$$

Let

$$V(s,\ t)=R(t)\begin{bmatrix}e^{2\pi it} & 0\\ 0 & 1\end{bmatrix}R(t)^*.$$

Then $V\in U_2(C(T^2))$ and $\alpha^{-1}(p_2)(s,\ t)=V(s,\ t)p_2(s,\ t)V(s,\ t)^*$.

Thus $\alpha_*^{-1}([p_2])=[p_2]$. Therefore the homomorphism $\mathrm{id}-\alpha_*^{-1}$ of $K^0(T^2)$

into $K^0(T^2)$ is a 0-map.

$$\alpha^{-1}(u_1)(s, t)=e^{2\pi it}=u_1(s, t)$$
$$\alpha^{-1}(u_2)(s, t)=e^{2\pi i(s-t)}=e^{2\pi is}e^{2\pi it}=u_2(s, t)u_1^*(s, t) .$$

Hence $\alpha_*^{-1}([u_1])=[u_1]$, $\alpha_*^{-1}([u_2])=-[u_1]+[u_2]$. Therefore the homomorphism $\mathrm{id}-\alpha_*^{-1}$ of $K^1(T^2)$ into $K^1(T^2)$ is given by the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It follows by the Pimsner-Voiculescu exact sequence that $K_j(C(T^2\times_\alpha \boldsymbol{Z})=\boldsymbol{Z}^3$ $(j=0, 1)$. Q.E.D.

COROLLARY 1.

$$K_j(C^*(G))\cong \boldsymbol{Z}^3 \quad j=0, 1 .$$

§ 2. Computation of $\sigma_*(K_0(C(T^2)\times_\alpha \boldsymbol{Z}))$.

Let $[e_j]-[f_j]$ $j=1, 2, 3$ be three generators of $K_0(C(T^2)\times_\alpha \boldsymbol{Z})$. The homomorphism i_* of $K^0(T^2)$ into $K_0(C(T^2)\times_\alpha \boldsymbol{Z}))$ is injective since

$$\mathrm{id}-\alpha_*^{-1};\ K^0(T^2)\longrightarrow K^0(T^2)$$

is a 0-map. Hence two generators are given as follows;

$$e_1(m, s, t)=\begin{cases} 1 & \text{if} \quad m=0 \\ 0 & \text{if} \quad m\neq 0 \end{cases}$$

$$f_1(m, s, t)=0$$

$$e_2(m, s, t)=\begin{cases} R(t)\begin{bmatrix} e^{-2\pi is} & 0 \\ 0 & 1 \end{bmatrix}R(t)^*\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}R(t)\begin{bmatrix} e^{2\pi is} & 0 \\ 0 & 1 \end{bmatrix}R(t)^* & \text{if} \quad m=0 \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \text{if} \quad m\neq 0 \end{cases}$$

$$f_2(m, s, t)=\begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \text{if} \quad m=0 \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \text{if} \quad m\neq 0 . \end{cases}$$

The generator $[e_3]-[f_3]$ is the element of $K_0(C(T^2)\times_\alpha \boldsymbol{Z})$ satisfying that

$$d_0([e_3]-[f_3])=[u_1]$$

where d_0 is the connecting map of $K_0(C(T^2)\times_\alpha \boldsymbol{Z})$ into $K^1(T^2)$.

Let g be the function on T^2 defined by $g(s, t)=\cos(\pi/2)t$ and let h be the function on T^2 defined by $h(s, t)=\sin(\pi/2)t$. We regard $C(T^2)$ as a C^*-subalgebra of $C(T^2)\times_\alpha \boldsymbol{Z}$. Then let

$$e_3=\begin{bmatrix}\delta_{-1} & 0\\ 0 & \delta_{-1}\end{bmatrix}\begin{bmatrix}g^2h^2 & -g^3h\\ gh^3 & -g^2h^2\end{bmatrix}+\begin{bmatrix}g^4+h^4 & g^3h-gh^3\\ g^3h-gh^3 & 2g^2h^2\end{bmatrix}+\begin{bmatrix}g^2h^2 & gh^3\\ -g^3h & -g^2h^2\end{bmatrix}\begin{bmatrix}\delta_1 & 0\\ 0 & \delta_1\end{bmatrix}$$

and

$$\delta_1(m)=\begin{cases}1_{T^2} & \text{if} \quad m=1\\ 0 & \text{if} \quad m\neq 1\end{cases}$$

$$\delta_{-1}(m)=\delta_1^*(m)=\begin{cases}1_{T^2} & \text{if} \quad m=-1\\ 0 & \text{if} \quad m\neq -1\,.\end{cases}$$

We see that e_3 is a *Rieffel projection* in $M_2(C(T^2)\times_\alpha \boldsymbol{Z})$ by computation.

Remark. Let A be a unital C^*-algebra and $(A, \boldsymbol{Z}, \beta)$ a C^*-dynamical system. A projection in $A\times_\alpha \boldsymbol{Z}$ satisfying the following condition is called a *Rieffel projection*;

1) $p=u^*x_1^*+x_0+x_1u \;\; x_0,\, x_1\in A$
2) u is a unitary element in A satisfying that $Adu=\beta$.

Lemma 3. *With the above notation let ε be the left support projection of x_1 in the enveloping von Neumann algebra of A. Then the unitary* $\exp(2\pi i x_0\varepsilon)$ *is in A and*

$$d_0([p])=[\exp(2\pi i x_0\varepsilon)]$$

where d_0 is the connecting map of $K_0(A\times_\beta \boldsymbol{Z})$ into $K_1(A)$.

Proof. See Pimsner-Voiculescu [4]. Q.E.D.

Lemma 4. *$[e_3]-[f_3]$ is a generator of $K_0(C(T^2)\times_\alpha \boldsymbol{Z})$ where*

$$f_3(m, s, t)=\begin{cases}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} & \text{if} \quad m=0\\ \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} & \text{if} \quad m\neq 0\,.\end{cases}$$

Proof. It is clear that $d_0([f_3])=0$. So we show that $d_0([e_3])=[u_1]$. Let

$$x_0=\begin{bmatrix}g^4+h^4 & g^3h-gh^3\\ g^3h-gh^3 & 2g^2h^2\end{bmatrix}$$

$$x_1=\begin{bmatrix} g^2h^2 & gh^3 \\ -g^3h & -g^2h^2 \end{bmatrix}.$$

Let ε be the left support projection of x_1 in the enveloping von Neumann algebra of $C(T^2)$. Since $\varepsilon=[x_1]=[x_1x_1^*]=s\text{-}\lim_{n\to\infty}(1/n+x_1x_1^*)^{-1}x_1x_1^*$, where $[x_1]$ and $[x_1x_1^*]$ are the range projections of x_1 and $x_1x_1^*$ respectively, by the trivial calculation we see that

$$\varepsilon(s, t)=\begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \text{if } t=0 \\ \begin{bmatrix} h^2(s, t) & -g(s, t)h(s, t) \\ -g(s, t)h(s, t) & g^2(s, t) \end{bmatrix} & \text{if } 0<t\leqq 1 . \end{cases}$$

Hence we obtain that

$$\exp(2\pi i x_0\varepsilon)=\exp\left(2\pi i h^2\begin{bmatrix} h^2 & -gh \\ -gh & g^2 \end{bmatrix}\right).$$

Let

$$F(c, s, t)=\exp\left(2\pi i h^2(s, t)\begin{bmatrix} h^2(s, ct) & -g(s, ct)h(s, ct) \\ -g(s, ct)h(s, ct) & g^2(s, ct) \end{bmatrix}\right) \quad 0\leqq c\leqq 1 .$$

Then

$$F(c, s, 0)=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} F(c, s, 1)&=\exp\left(2\pi i\begin{bmatrix} h^2(s, c) & -g(s, c)h(s, c) \\ -g(s, c)h(s, c) & g^2(s, c) \end{bmatrix}\right) \\ &=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+(e^{2\pi i}-1)\begin{bmatrix} h^2(s, c) & -g(s, c)h(s, c) \\ -g(s, c)h(s, c) & g^2(s, c) \end{bmatrix} \\ &=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

since $\begin{bmatrix} h^2(s, c) & -g(s, c)h(s, c) \\ -g(s, c)h(s, c) & g^2(s, c) \end{bmatrix}$ is a projection.

Therefore F is a continuous function of the interval $[0, 1]$ into $U_2(C(T^2))$. Hence

$$d_0([e_3])=[F(1)]=[F(0)]=[e^{2\pi i h^2}]=[u_1]$$

by Lemma 3. Thus we obtain Lemma 4. Q.E.D.

Theorem 1.

$$\sigma_*(K_0(C(T^2)\times_\alpha \boldsymbol{Z}))=\boldsymbol{Z}$$

where σ_* is the homomorphism of $K_0(C(T^2)\times_\alpha \boldsymbol{Z})$ into $\boldsymbol{R}$ induced by the trace σ defined in Introduction.

PROOF.

$$\sigma_*([e_1])=1$$
$$\sigma_*([f_1])=0$$
$$\sigma_*([e_2])=\int_0^1\int_0^1 \mathrm{Tr}(e_2(0, s, t))dsdt=1$$
$$\sigma_*([f_2])=1$$
$$\begin{aligned}\sigma_*([e_3])&=\int_0^1\int_0^1 \mathrm{Tr}(e_3(0, s, t))dsdt\\&=\int_0^1\int_0^1 (g^4(s, t)+h^4(s, t)+2g^2(s, t)h^2(s, t))dsdt=1\end{aligned}$$
$$\sigma_*([f_3])=1$$

where Tr is the canonical trace on the matrix algebra $M_2(\boldsymbol{C})$. Since σ_* is the homomorphism, we obtain that

$$\sigma_*(K_0(C(T^2)\times_\alpha \boldsymbol{Z}))=\boldsymbol{Z}\ . \qquad \text{Q.E.D.}$$

COROLLARY 2.

$$\tau_*(K_0(C^*(G)))=\boldsymbol{Z}$$

where τ_* is the homomorphism of $K_0(C^*(G))$ into $\boldsymbol{R}$ induced by the trace defined in Introduction.

REMARK. The above corollary shows that $C^*(G)$ has no nontrivial projection although it is not simple.

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ADDENDUM

After this paper had been typed out, I have received the following preprint of J. Anderson and W. Paschke whose results contain those of ours. I thank to them, but I had reached our results independently, so I will present here.

J. Anderson and W. Paschke, The rotation algebra, Preprint Series, M.S.R.I. Berkeley, February (1985).

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