TOKYO J. MATH.
VOL. 9, NO. 2, 1986

Einstein Parallel Kaehler Submanifolds in a Complex Projective Space

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(Communicated by K. Ogiue)

Introduction

Submanifolds with parallel second fundamental form (which are simply called parallel submanifolds) have been studied by many differential geometers. In particular, parallel Kaehler submanifolds in a complex projective space are completely determined (see [1]).

In this paper, we give some characterizations of Einstein parallel Kaehler submanifolds in a complex projective space.

Let $X: M\to E^N$ be an isometric immersion of an n-dimensional compact Riemannian manifold into an N-dimensional Euclidean space. We denote by Δ and $\mathrm{Spec}(M)=\{0<\lambda_1<\lambda_2<\cdots\}$, the Laplacian acting on differentiable functions of M and the spectrum of Δ, respectively. Then, X can be decomposed as $X=X_0+\sum_{k\in N}X_k$, where X_k is a k-th eigenfunction of Δ of M, X_0 is a constant mapping, and the addition is convergent componentwise for the L^2-topology on $C^\infty(M)$. We say that the immersion is of *order* $\{l\}$ (or mono-order) if $X=X_0+X_l$, $l\in \boldsymbol{N}$, $X_l\neq 0$, and of *order* $\{k, l\}$ (or bi-order) if $X=X_0+X_k+X_l$, $k, l\in \boldsymbol{N}$, $l>k$, $X_k, X_l\neq 0, \cdots$ (see [4]).

Let $F: \boldsymbol{CP}^m\to E^N$ be the first standard imbedding of an m-dimensional complex projective space of constant holomorphic sectional curvature 1 into an N-dimensional Euclidean space, and $i: M^n\to \boldsymbol{CP}^m$ be a Kaehler immersion of an n-dimensional compact Kaehler manifold. We consider $\phi=F\circ i: M^n\to E^N$. Then, ϕ is mono-order if and only if M is totally geodesic (See [3].), and totally geodesic Kaehler submanifolds are of order 1. Let A be the shape operator of the immersion i, and define the tensor T by

$$T(\xi, \eta)=\mathrm{tr}\, A_\xi A_\eta \qquad \text{for} \quad \xi, \eta\in NM\,,$$

where NM is the normal bundle of M. Then T is a symmetric bilinear mapping from $NM\times NM$ into $\boldsymbol{R}$. A. Ros [3] has proved that M is bi-order

Received June 8, 1985

if and only if M is an Einstein Kaehler submanifold with $T=kg|_{NM\times NM}$, where k is some real number and g is the Kaehler metric of CP^m. With reference to this fact, we have the following

THEOREM 1. *Let M be an n-dimensional compact Kaehler submanifold fully immersed in CP^m which is not totally geodesic. Then, the following conditions are mutually equivalent.*

(i) *M is an Einstein parallel submanifold,*

(ii) *M is of order* $\{1, 2\}$,

(iii) *M is bi-order,*

(iv) *M is an Einstein submanifold with* $T=kg|_{NM\times NM}$,

(v) *M is an Einstein submanifold and NM admits an Einstein Kaehler metric.*

It is already shown in [4] that Einstein parallel Kaehler submanifolds which are not totally geodesic are of order $\{1, 2\}$. It is trivial that (ii) implies (iii). The equivalence between (iii) and (iv) is proved by A. Ros [3]. We will show that the condition (iv) implies (i) in §3. We will explain what the condition (v) means and prove the equivalence between (iv) and (v) in §2.

REMARK 1. The equivalence between (ii) and (iii) implies that there exist no bi-order immersions other than immersions of order $\{1, 2\}$.

Throughout this paper, we use the following convention on the range of indices:

$$A, B, \cdots=1, \cdots, n, n+1, \cdots, m\ ; \quad a, b\cdots=1, \cdots, n\ ;$$
$$\alpha, \beta, \cdots=n+1, \cdots, m\ .$$

The author wishes to thank Professors K. Ogiue, N. Ejiri and A. Ros for many valuable comments and suggestions, and also thank Professor K. Tsukada for many valuable advices, in particular, the method of using non-negativity of $\|\bar{\nabla}\overset{+}{\nabla}H\|^2$ in (3.4) is due to him. The author is also grateful to the referee for many valuable suggestions, thanks to which the original paper is improved.

§1. Preliminaries.

In this section, we give some basic formulas for Kaehler submanifolds in CP^m. For details, see [1] and [2]. Let M be an n-dimensional Kaehler submanifold immersed in CP^m. Let TM^c be the complexification of the tangent bundle TM of M. Then we have $TM^c=TM^+ + TM^-$ (orthogonal sum), where the fibre $T_pM^{\pm}$ at $p\in M$ is the $\pm\sqrt{-1}$ eigenspace of the complex structure tensor on T_pM^C. In the same way, we have $NM^C=NM^+ + NM^-$ (orthogonal sum) for the complexification NM^C of the normal bundle NM of the immersion. We denote by $x\to\bar{x}$ the complex conjugation, so that $\overline{T_pM^{\pm}}=T_pM^{\mp}$ and $\overline{N_pM^{\pm}}=N_pM^{\mp}$. We choose a local field of unitary frames $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_m\}$ on CP^m in such a way that, restricted to M, $e_1, \cdots, e_n$ are tangent to M. With respect to the frame field on CP^m, let $\{\omega^1, \cdots, \omega^n, \omega^{n+1}, \cdots, \omega^m\}$ be the field of dual frames. Then, the Kaehler metric of CP^m is given by $\sum_A \omega^A\cdot\bar{\omega}^A$ (in [1], the Kaehler metric of CP^m is given by $2\sum_A \omega^A\cdot\bar{\omega}^A$) and the structure equations of CP^m are given by

$$(1.1)\qquad d\omega^A+\sum_B \omega^A_B\wedge\omega^B=0\ , \quad \omega^A_B+\bar{\omega}^B_A=0\ ,$$

$$(1.2)\qquad d\omega^A_B+\sum_C\omega^A_C\wedge\omega^C_B=\tilde{\Omega}^A_B\ , \quad \tilde{\Omega}^A_B=\sum_{C,D}\tilde{R}^A_{BC\bar{D}}\omega^C\wedge\bar{\omega}^D$$

Since CP^m is a complex space form of constant holomorphic sectional curvature 1, we have

$$(1.3)\qquad \tilde{R}^A_{BC\bar{D}}=(1/4)(\delta^A_B\delta_{CD}+\delta^A_C\delta_{BD})$$

Restricting these forms to M^n, we have

$$(1.4)\qquad \omega^\alpha=0\ ,$$

and the Kaehler metric g of M^n is given by $g=\sum_a \omega^a\cdot\bar{\omega}^a$. Moreover we obtain

$$(1.5)\qquad \omega^\alpha_a=\sum_b k^\alpha_{ab}\omega^b\ , \quad k^\alpha_{ab}=k^\alpha_{ba}\ ,$$

$$(1.6)\qquad d\omega^a+\sum_b \omega^a_b\wedge\omega^b=0\ , \quad \omega^a_b+\bar{\omega}^b_a=0\ ,$$

$$(1.7)\qquad d\omega^a_b+\sum_c \omega^a_c\wedge\omega^c_b=\Omega^a_b\ , \quad \Omega^a_b=\sum_{c,d} R^a_{bc\bar{d}}\omega^c\wedge\bar{\omega}^d\ ,$$

$$(1.8)\qquad d\omega^\alpha_\beta+\sum_\gamma \omega^\alpha_\gamma\wedge\omega^\gamma_\beta=\Omega^\alpha_\beta\ , \quad \Omega^\alpha_\beta=\sum_{c,d} R^\alpha_{\beta c\bar{d}}\omega^c\wedge\bar{\omega}^d\ ,$$

From (1.5) and (1.7), we have the equation of Gauss

$$(1.9)\qquad R^a_{bc\bar{d}}=(1/4)(\delta^a_b\delta_{cd}+\delta^a_c\delta_{bd})-\sum_\alpha k^\alpha_{bc}\bar{k}^\alpha_{ad}\ ,$$

and from (1.5), (1.6) and (1.8), we have

$$(1.10)\qquad R^\alpha_{\beta c\bar{d}}=(1/4)\delta^\alpha_\beta\delta_{cd}+\sum_a k^\beta_{ac}\bar{k}^\alpha_{ad}\ .$$

The Ricci tensor $S_{c\bar{d}}$ and the scalar curvature τ of M^n are given by

$$(1.11)\qquad S_{c\bar{d}}=(n+1)/2\delta_{cd}-2\sum_{\alpha,a} k^\alpha_{ac}\bar{k}^\alpha_{ad}\ ,$$

(1.12) $$\tau=n(n+1)-4\sum_{\alpha,c,d}k^{\alpha}_{cd}\bar{k}^{\alpha}_{cd}\ .$$

Now, we define the covariant derivatives k^{α}_{abc} and $k^{\alpha}_{ab\bar{c}}$ of k^{α}_{ab} by

$$\sum_c k^{\alpha}_{abc}\omega^c+\sum_c k^{\alpha}_{ab\bar{c}}\bar{\omega}^c=dk^{\alpha}_{ab}-\sum_c k^{\alpha}_{bc}\omega^c_a-\sum_c k^{\alpha}_{ac}\omega^c_b+\sum_\beta k^{\beta}_{ab}\omega^{\alpha}_{\beta}\ .$$

Then we have

(1.13) $$k^{\alpha}_{abc}=k^{\alpha}_{bac}=k^{\alpha}_{acb}\ ,\quad k^{\alpha}_{ab\bar{c}}=0\ .$$

We can define inductively the covariant derivatives $k^{\alpha}_{a_1\cdots a_m a_{m+1}}$ and $k^{\alpha}_{a_1\cdots a_m\bar{a}_{m+1}}$ of $k^{\alpha}_{a_1\cdots a_m}$ for $m\geqq 2$. It is clear that $(\bar{k}^{\alpha}_{a_1\cdots a_m})_b=\bar{k}^{\alpha}_{a_1\cdots a_m\bar{b}}$ and $(\bar{k}^{\alpha}_{a_1\cdots a_m})_{\bar{b}}=\bar{k}^{\alpha}_{a_1\cdots a_m b}$.

We see that $k^{\alpha}_{a_1\cdots a_m}$ is symmetric with respect to $a_1,\cdots,a_m$. The following formula is proved in [1]:

LEMMA 1.

(1.14) $$\begin{aligned}k^{\alpha}_{a_1\cdots a_m\bar{b}}=&(m-2)/4\sum_{r=1}^{m}k^{\alpha}_{a_1\cdots\hat{a}_r\cdots a_m}\delta_{a_r b}\\&-\sum_{r=1}^{m-2}1/r!(m-r)!\sum_{\sigma,\beta,c}k^{\alpha}_{ca_{\sigma(1)}\cdots a_{\sigma(r)}}k^{\beta}_{a_{\sigma(r+1)}\cdots a_{\sigma(m)}}\bar{k}^{\beta}_{cb}\end{aligned}$$

for $m\geqq 3$, *where the summation on* σ *is taken over all permutations of* $(1,\cdots,m)$.

§2. Normal Einstein metric.

First, we state the following.

DEFINITION. We put $R^{\alpha}_{\beta}=\sum_c R^{\alpha}_{\beta c\bar{c}}$. Then, we call this tensor on NM the normal Ricci tensor. If $R^{\alpha}_{\beta}=\lambda\delta^{\alpha}_{\beta}$ for some real function λ on M, we say that NM admits an Einstein Kaehler metric.

Let J be the complex structure of CP^m. Since $A_{J\xi}=JA_\xi$, $A_\xi J=-JA_\xi$ for any $\xi\in NM$ (See [2].), we have

$$T(J\xi,J\eta)=T(\xi,\eta)\qquad\text{for any}\quad \xi,\eta\in NM\ .$$

Next, we extend T to the complex bilinear mapping from $NM^c\times NM^c$ into C. Then, we have

$$T(N_pM^+,N_pM^+)=0\ ,\quad T(N_pM^-,N_pM^-)=0\qquad\text{for any}\quad p\in M\ .$$

Therefore, we can see that

$$T=kg|_{NM\times NM}\quad\text{if and only if}\quad \sum_{a,b}k^{\alpha}_{ab}\bar{k}^{\beta}_{ab}=(k/2)\delta_{\alpha\beta}\ ,$$

and k is given by $(n(n+1)-\tau)/2(m-n)$. On the other hand, it follows from (1.10) that

$$R^{\alpha}_{\beta}=(n/4)\delta^{\alpha}_{\beta}+\sum_{a,b} k^{\beta}_{ab}\bar{k}^{\alpha}_{ab} .$$

Thus, we have

PROPOSITION 1. *Let M be a Kaehler submanifold in CP^m. Then, the following conditions are mutually equivalent.*

(i) $T=kg|_{NM\times NM}$,

(ii) *NM admits an Einstein Kaehler metric.*

§3. Proof of Theorem 1.

To complete the proof of Theorem 1, it is enough to prove the following

PROPOSITION 2. *Let M be an n-dimensional Einstein Kaehler submanifold in CP^m.*

If M is a submanifold with $T=kg|_{NM\times NM}$, then M is a parallel submanifold.

PROOF. Since M is Einstein, we have $\sum_{\alpha,a} k^{\alpha}_{ac}\bar{k}^{\alpha}_{ad}=\{\|k\|^2/4n\}\delta_{cd}$ by (1.11) and it follows that

$$\sum_{\alpha,a} k^{\alpha}_{abc}\bar{k}^{\alpha}_{ad}=0 \quad \text{(see (1.13))} . \tag{3.1}$$

Using Lemma 1, we obtain

$$k^{\alpha}_{abc\bar{d}}=(1/4)\{k^{\alpha}_{bc}\delta_{ad}+k^{\alpha}_{ac}\delta_{bd}+k^{\alpha}_{ab}\delta_{cd}\} \\ -\sum_{\beta,e}\{k^{\alpha}_{ea}k^{\beta}_{bc}+k^{\alpha}_{eb}k^{\beta}_{ca}+k^{\alpha}_{ec}k^{\beta}_{ab}\}\bar{k}^{\beta}_{ed} .$$

This, together with (1.13) and (3.1), implies

$$\sum_{\alpha,a,b,c} k^{\alpha}_{abc}(\overline{k^{\alpha}_{abc\bar{d}}})=0 . \tag{3.2}$$

Then, we have

$$\begin{aligned} 0&=\sum_{\alpha,a,b,c,d} k^{\alpha}_{abc\bar{d}}(\overline{k^{\alpha}_{abc\bar{d}}})+\sum_{\alpha,a,b,c,d} k^{\alpha}_{abc}(\overline{k^{\alpha}_{abc\bar{d}d}}) \\ &=\sum k^{\alpha}_{abc\bar{d}}(\overline{k^{\alpha}_{abc\bar{d}}})+(3/4)\sum k^{\alpha}_{abc}\bar{k}^{\alpha}_{abc} \\ &\quad -3\sum k^{\alpha}_{abc}\bar{k}^{\beta}_{bc}k^{\beta}_{ed}\bar{k}^{\alpha}_{eda} , \end{aligned} \tag{3.3}$$

where we have used (1.13) and (3.1). If we put $H=(k^{\alpha}_{ab})$, $\tilde{T}=(\tilde{T}_{\alpha\bar{\beta}})=(\sum_{a,b} k^{\alpha}_{ab}\bar{k}^{\beta}_{ab})$ and denote by $\overset{+}{\nabla}$, $\overset{-}{\nabla}$ and $\nabla^{\perp}$, the (1, 0)-type covariant derivative, the (0, 1)-type covariant derivative, and the normal connection of M, respectively, then (3.3) implies

$$3\|\overset{+}{\nabla}{}^{\perp}\tilde{T}\|^2=\|\overset{-}{\nabla}\overset{+}{\nabla}H\|^2+(3/4)\|\overset{+}{\nabla}H\|^2 . \tag{3.4}$$

Since $\tilde{T}_{\alpha\bar{\beta}}=(k/2)\delta_{\alpha\beta}$, the left hand side of (3.4) vanishes, so does the right hand side. Since each term of the right hand side of (3.4) is non-negative, we obtain

$$\overset{+}{\nabla} H=0\ ,$$

that is, the second fundamental form is parallel. Q.E.D.

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