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Kaehler Submanifolds of Complex Space Forms

Dedicated to Professor Morio Obata on his 60th birthday

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Introduction

E. Calabi [1] proved that a complex linear space, a complex hyperbolic space and a complex projective space can not be holomorphically and isometrically immersed in each other. In this paper we show that Kaehler submanifolds of complex space forms of different types are essentially different from each other. Namely we prove the following:

THEOREM. *Any two of complex space forms of different types have no Kaehler submanifold in common, that is,*

(1) *A Kaehler submanifold of C^N can not be a Kaehler submanifold of any complex hyperbolic space.*

(2) *A Kaehler submanifold of C^N can not be a Kaehler submanifold of any complex projective space.*

(3) *A Kaehler submanifold of a complex hyperbolic space can not be a Kaehler submanifold of any complex projective space.*

It should be remarked that no global assumption is made in the theorem, namely it is local in nature. In the proof of the theorem, the notion "diastasis" introduced by E. Calabi [1] plays an essential rôle. Though the diastasis depends only on the metric, it is compatible with that of an ambient space. Using this property of diastasis, a necessary and sufficient condition for a Kaehler manifold to be holomorphically and isometrically immersed into a complex space form has been obtained by E. Calabi [1]. Then the proof of our theorem reduces to the local rigidity theorem for isometric mappings of Kaehler submanifolds of C^N into the Hilbert space l^2.

The idea of the diastasis here can be applied to a wider class of

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complex manifold with a tensor field similar to a Kaehler metric, and a decomposition theorem on real analytic functions in some class will be obtained in [3]. Further applications will be made to Einstein Kaehler submanifolds of complex space forms. The author proves in [4] that any Einstein Kaehler submanifold is always totally geodesic in a complex linear or hyperbolic space.

In this paper, C^N denotes the complex linear N-space and $CH^N(b)$ (resp. $CP^N(b)$) the complex hyperbolic (resp. projective) N-space of holomorphic sectional curvature $b<0$ (resp. $b>0$). *Submanifolds are always assumed to be positive dimensional.*

§1. Preliminaries.

In this section we review several facts in the paper [1]. Let M be an analytic Kaehler n-manifold, that is, a complex manifold with a real analytic Kaehler metric g. Every Kaehler submanifold of complex space form is analytic. Then, locally, there exists a real analytic function f such that

$$\partial^2 f/\partial z^\alpha \partial \bar{z}^\beta = g_{\alpha\bar{\beta}} \quad (\alpha, \beta=1, \cdots, n),$$

where $(z^1, \cdots, z^n)$ is a local complex coordinate. Such a function f, which is said to be *primitive*, is not uniquely determined. In fact, for any holomorphic function h, $f+h+\bar{h}$ is primitive as well.

Now we recall the *diastasis* introduced by E. Calabi [1]. The diastasis $D_M(p, q)$ is a real analytic function defined on some neighborhood of the diagonal set $\{(p, p); p \in M\}$ in $M \times M$ and symmetric in p and q. It can be characterized as follows: (See Appendix.) For fixed $p \in M$, $\tilde{f}(q)=D_M(p, q)$ is a unique primitive function which satisfies

$$\partial^{|I|}\tilde{f}/\partial z^I=0 \quad \text{and} \quad \partial^{|I|}\tilde{f}/\partial \bar{z}^I=0 \quad \text{at} \quad p, \tag{1.1}$$

for each multi-index $I=\{i_1, \cdots, i_n\}$ $(i_1, \cdots, i_n \geqq 0)$.

EXAMPLE 1. Let $(\xi^1, \cdots, \xi^N)$ be the canonical complex coordinate in C^N. Then the diastasis of C^N is given by

$$D^N(p, q)=\sum_{\sigma=1}^{N} |\xi^\sigma(p)-\xi^\sigma(q)|^2 \quad (p, q \in C^N), \tag{1.2}$$

namely the square of the Euclidean distance.

EXAMPLE 2. The complex hyperbolic space $CH^N(2b)$ is a ball $\{q \in C^N; \sum_{\sigma=1}^N |\xi^\sigma(q)|^2<1\}$, whose diastasis is given by

$$D(p, q)=\frac{1}{b}\log\left(1-\sum_{\sigma=1}^{N}|\xi^{\sigma}(q)|^{2}\right), \tag{1.3}$$

where $p=(0, \cdots, 0)$.

Though the diastasis is globally defined in C^N or CH^N, it is not always so in general.

EXAMPLE 3. By using the homogeneous coordinate $(\zeta^0, \cdots, \zeta^N)$, the diastasis D of $CP^N(2b)$ is given by

$$D(p, q)=\frac{1}{b}\log\left(1+\sum_{\sigma=1}^{N}|\zeta^{\sigma}(q)|^{2}/|\zeta^{0}(q)|^{2}\right), \tag{1.4}$$

where $p=(1, 0, \cdots, 0)$ and $\zeta^0(q)\neq 0$.

The diastasis has the following property:

LEMMA 1.1 (E. Calabi). *Let ϕ be a holomorphic mapping of an analytic Kaehler manifold M into another analytic Kaehler manifold $\tilde{M}$. Then ϕ is isometric if and only if the diastasis of M is a restriction of that of $\tilde{M}$, i.e.,*

$$D_M(p, q)=D_{\tilde{M}}(\phi(p), \phi(q)), \tag{1.5}$$

for p and q in the region of definition, where D_M and $D_{\tilde{M}}$ are the diastases of M and $\tilde{M}$ respectively.

PROOF. Suppose that ϕ is isometric. For $p\in M$ fixed, $D_{\tilde{M}}(\phi(p), \phi(q))$ is obviously a primitive function of the metric g of M. Moreover since ϕ is holomorphic, $f\circ\phi$ also satisfies the condition (1.1) if so does f. Hence we have $D_M(p, q)=D_{\tilde{M}}(\phi(p), \phi(q))$. The converse is easily shown by differentiating (1.5) with respect to the variable q.

By using the lemma, we shall give criterions for a given Kaehler manifold to be holomorphically and isometrically immersed in complex space forms. We denote by $\tilde{M}^N(b)$ the complete and simply connected Kaehler N-manifold of constant holomorphic sectional curvature b.

LEMMA 1.2. *Let M be an analytic Kaehler manifold, and $p\in M$ an arbitrarily fixed point. Then a neighborhood U of p is a Kaehler submanifold of $\tilde{M}^N(2b)$ if and only if there exist holomorphic functions $\phi^1, \cdots, \phi^N$ defined on U such that $\phi^{\sigma}(p)=0$ $(\sigma=1, \cdots, N)$ and*

$$D_M(p, q)=\begin{cases}\sum_{\sigma=1}^{N} |\phi^\sigma(q)|^2 & \text{if } b=0 \\ \frac{1}{b}\log\left(1-\sum_{\sigma=1}^{N} |\phi^\sigma(q)|^2\right) & \text{if } b<0 \\ \frac{1}{b}\log\left(1+\sum_{\sigma=1}^{N} |\phi^\sigma(q)|^2\right) & \text{if } b>0 ,\end{cases} \tag{1.6}$$

for all $q \in U$.

PROOF. Suppose U is a Kaehler submanifold of $\tilde{M}^N(2b)$ immersed by ϕ. In case $b\leqq 0$, by a suitable motion in $\tilde{M}^N(2b)$, we may put $\phi(p)=0$. If we put $\phi=(\phi^1, \cdots, \phi^N)$, then by (1.2) and (1.3), we see that (1.6) holds if and only if the diastasis of U is the restriction of that of $\tilde{M}^N(2b)$. So we have (1.6) by Lemma 1.1. Similarly in case $b>0$, using the homogeneous coordinate, we may suppose $\phi(p)=(1, 0, \cdots, 0)$. Then $\phi=(1, \phi^1, \cdots, \phi^N)$ satisfies (1.6). The converse is now obvious.

The following lemma is easily proved.

LEMMA 1.3. *If there exist holomorphic functions* $\phi^1, \cdots, \phi^N$ *satisfying* (1.6) *and* $\phi^\sigma(p)=0$ $(\sigma=1, \cdots, N)$, *then a mapping of* U *into* C^N *defined by* $\phi=(\phi^1, \cdots, \phi^N)$ *is an immersion.*

Next we consider holomorphic mappings into an infinite dimensional space. The space l^2 consists of the points with coordinates $(\xi^1, \xi^2, \xi^3, \cdots)$ such that $\sum_{\sigma=1}^{\infty} |\xi^\sigma|^2<\infty$ $(\xi^\sigma \in C)$. The Hermitian inner product $\langle\ ,\ \rangle$ is given by

$$\langle p, q\rangle=\sum_{\sigma=1}^{\infty} \xi^\sigma(p)\overline{\xi^\sigma(q)} \quad (p, q \in l^2) .$$

DEFINITION (cf. [1; p. 5]). Let M be a complex manifold and ϕ a mapping of M into l^2. Then ϕ is said to be *holomorphic* if it satisfies the following two conditions:

(1) $\phi^\sigma=\xi^\sigma\circ\phi$ $(\sigma=1, 2, 3, \cdots)$ are all holomorphic.

(2) ϕ is locally bounded, that is, for every $p \in M$, there exists a neighborhood U of p and a positive number m such that $|\phi|=\langle\phi, \phi\rangle^{1/2}\leqq m$ on U.

The condition (2) is equivalent to the continuity of ϕ by the following lemma.

LEMMA 1.4. *Let* ϕ *and* ψ *be holomorphic mappings of a complex* n*-manifold* M *into* l^2. *If we define a function* f *by*

$$f(p, q)=\langle\phi(p), \psi(q)\rangle \quad (p, q\in M)\ ,$$

then f is a holomorphic function on $M\times\bar{M}$, where $\bar{M}$ denotes the conjugate manifold of M. In particular, if ψ is a constant function $\psi\equiv e$ $(e\in l^2)$, then the projection $\langle\phi, e\rangle$ is a holomorphic function on M.

PROOF. (This proof is suggested by H. Tasaki.) Let p_0 and q_0 be arbitrary points on M and $\bar{M}$ respectively. Then there exist complex coordinate systems $(U; z^1, \cdots, z^n)$ and $(V; \bar{w}^1, \cdots, \bar{w}^n)$ of $p_0\in M$ and $q_0\in\bar{M}$ so that

$$\begin{aligned}
&z^\alpha(p_0)=0\ , \quad \bar{w}^\beta(q_0)=0 \quad (\alpha, \beta=1, \cdots, n)\ ,\\
&U=\{(z^1, \cdots, z^n); |z^\alpha|<r_0 \ (\alpha=1, \cdots, n)\}\ ,\\
&V=\{(\bar{w}^1, \cdots, \bar{w}^n); |\bar{w}^\beta|<r_0 \ (\beta=1, \cdots, n)\}\ ,\\
&|\phi|<C_1 \text{ on } U \quad\text{and}\quad |\psi|<C_2 \text{ on } V\ .
\end{aligned}$$

We take numbers $0<r_1<r_2<r_0$. Let $U_1=\{|z^\alpha|<r_1\}$ and $V_1=\{|\bar{w}^\beta|<r_1\}$. On putting

$$f_m(p, q)=\sum_{\sigma=1}^m \phi^\sigma(p)\overline{\psi^\sigma(q)} \quad (m=1, 2, 3, \cdots)\ ,$$

f_m are holomorphic functions on $M\times\bar{M}$. We show that $\{f_m\}$ converges uniformly to f. By Cauchy's integral formula, we have

$$\phi^\sigma(p)\overline{\psi^\sigma(q)}=\frac{1}{(2\pi\sqrt{-1})^{2n}}\oint_\Delta \frac{\phi^\sigma(z)\overline{\psi^\sigma(\bar{w})}}{\prod_{\gamma=1}^n (z^\gamma-z^\gamma(p))(\bar{w}^\gamma-\bar{w}^\gamma(q))}d(z)d(\bar{w})\ ,$$

where $\Delta=\{|z^\alpha|=r_2, |\bar{w}^\beta|=r_2 \ (\alpha, \beta=1, \cdots, n)\}$. If we denote

$$L_\sigma=\frac{1}{\{2\pi(r_2-r_1)\}^{2n}}\oint_\Delta |\phi^\sigma(z)||\psi^\sigma(\bar{w})||d(z)||d(\bar{w})|\ ,$$

then $|\phi^\sigma(p)\overline{\psi^\sigma(q)}|<L_\sigma$ $(p\in U_1, q\in V_1)$. Since

$$\sum_{\sigma=1}^m|\phi^\sigma||\psi^\sigma|\leqq\Big(\sum_{\sigma=1}^m|\phi^\sigma|^2\Big)^{1/2}\Big(\sum_{\sigma=1}^m|\psi^\sigma|^2\Big)^{1/2}\leqq C_1C_2\ ,$$

we have

$$\begin{aligned}
\sum_{\sigma=1}^m L_\sigma&\leqq\frac{C_1C_2}{\{2\pi(r_2-r_1)\}^{2n}}\oint_\Delta |d(z)||d(\bar{w})|\\
&\leqq C_1C_2\Big\{\frac{r_2}{(r_2-r_1)}\Big\}^{2n}\ .
\end{aligned}$$

This estimate implies that the sequence $\{f_m\}_{m=1,2,3\cdots}$ converges uniformly to f in the wider sense on $M\times\bar{M}$. Hence f is a holomorphic function on $M\times\bar{M}$.

Now we define a diastasis of l^2 by $D^\infty(p, q)=|p-q|^2$ $(p, q\in l^2)$

DEFINITION. Let M be an analytic Kaehler manifold. Then a holomorphic mapping ϕ of M into l^2 is said to be *isometric* if the diastasis of M is the restriction of that of l^2, that is

$$D_M(p, q)=D^\infty(\phi(p), \phi(q)) \quad (p, q\in M) .$$

Then we have the following:

LEMMA 1.5. *Let ϕ be a holomorphic mapping of an analytic Kaehler manifold M into l^2. Then ϕ is isometric if and only if $|\phi|^2$ is a primitive function of the Kaehler metric of M.*

PROOF. We have

$$\begin{aligned}(1.7) \qquad D^\infty(\phi(p), \phi(q))=&\langle\phi(p), \phi(p)\rangle+\langle\phi(q), \phi(q)\rangle\\ &-\langle\phi(p), \phi(q)\rangle-\langle\phi(q), \phi(p)\rangle .\end{aligned}$$

If $|\phi|^2$ is a primitive function of g, then by the alternative definition of diastasis (See Appendix.) and Lemma 1.4, we have $D^\infty(\phi(p), \phi(q))=D_M(p, q)$. On the other hand, for $p\in M$ fixed, (1.7) implies that $D^\infty(\phi(p), \phi(q))$ and $|\phi|^2$ differ by the real part of some holomorphic function with respect to the variable q. Hence the converse is obvious.

The following lemma is easily proved by using the relation $\langle\overrightarrow{pq}, \overrightarrow{pq}\rangle= D^\infty(p, q)$ $(p, q\in l^2)$, where $\overrightarrow{pq}=q-p$.

LEMMA 1.6. *The diastasis of l^2 satisfies*

$$\frac{1}{2}\{D^\infty(p, q_1)+D^\infty(p, q_2)-D^\infty(q_1, q_2)\}=\mathrm{Re}\langle\overrightarrow{pq_1}, \overrightarrow{pq_2}\rangle ,$$

for all $p, q_1, q_2\in l^2$. Similarly, above identity holds for the diastasis D^N of C^N.

PROPOSITION 1.7. *Let M be a Kaehler n-submanifold of C^N immersed by ϕ, and ψ a holomorphic and isometric mapping of M into l^2. Then $\psi(M)$ lies in some complex N-plane in l^2, and hence ψ is rigid.*

PROOF. First we regard l^2 as a real vector space. Suppose that

$\psi(M)$ does not lie in any real $2N$-plane. Then there exist points $p, q_1, \cdots, q_{2N+1} \in M$ such that $\{\overrightarrow{\psi(p)\psi(q_i)}\}_{i=1,\cdots,2N+1}$ are $\boldsymbol{R}$-linearly independent. On the other hand, $\{\overrightarrow{\phi(p)\phi(q_i)}\}_{i=1,\cdots,2N+1}$ are $\boldsymbol{R}$-linearly dependent. So there exists a $(2N+1)$-tuple of real numbers $(a^1, \cdots, a^{2N+1}) \neq 0$ such that $\sum_{i=1}^{2N+1} a^i \overrightarrow{\phi(p)\phi(q_i)} = 0$. By Lemmas 1.1, 1.5 and 1.6, we have

$$\begin{aligned}
0 &= \mathrm{Re}\left\langle \sum_{i=1}^{2N+1} a^i \overrightarrow{\phi(p)\phi(q_i)}, \sum_{j=1}^{2N+1} a^j \overrightarrow{\phi(p)\phi(q_j)} \right\rangle \\
&= \sum_{i,j=1}^{2N+1} a^i a^j \,\mathrm{Re}\langle \overrightarrow{\phi(p)\phi(q_i)}, \overrightarrow{\phi(p)\phi(q_j)} \rangle \\
&= \frac{1}{2} \sum_{i,j=1}^{2N+1} a^i a^j \{D^N(\phi(p), \phi(q_i)) + D^N(\phi(p), \phi(q_j)) - D^N(\phi(q_i), \phi(q_j))\} \\
&= \frac{1}{2} \sum_{i,j=1}^{2N+1} a^i a^j \{D_M(p, q_i) + D_M(p, q_j) - D_M(q_i, q_j)\} \\
&= \frac{1}{2} \sum_{i,j=1}^{2N+1} a^i a^j \{D^\infty(\psi(p), \psi(q_i)) + D^\infty(\psi(p), \psi(q_j)) - D^\infty(\psi(q_i), \psi(q_j))\} \\
&= \mathrm{Re}\left\langle \sum_{i=1}^{2N+1} a^i \overrightarrow{\psi(p)\psi(q_i)}, \sum_{j=1}^{2N+1} a^j \overrightarrow{\psi(p)\psi(q_j)} \right\rangle .
\end{aligned}$$

Hence $\sum_{i=1}^{2N+1} a^i \overrightarrow{\psi(p)\psi(q_i)} = 0$, which yields a contradiction. So $\psi(M)$ lies in some real $2N$-plane. In particular $\psi(M)$ lies in some complex $2N$-plane. By applying Calabi's local rigidity theorem of finite dimensional version ([1; Theorem 2]), we conclude that $\psi(M)$ lies in some complex N-plane.

REMARK. E. Calabi [1] proved that a Kaehler submanifold immersed in a complex space form is locally rigid. Furthermore he asserted that this is valid for an infinite dimensional case without details.

§2. Kaehler submanifolds of complex space forms.

First of all, we prove the following:

PROPOSITION 2.1. *Let M be a Kaehler n-submanifold of C^N. Then any open subset of M can not be a Kaehler submanifold of $CP^{N'}(b')$ for any N' and $b'>0$.*

PROOF. We suppose that some open subset U of M is a Kaehler submanifold of $CP^{N'}(2b)$ $(b'=2b)$. Throughout this proof, we fix a point $p \in U$. By Lemma 1.2, there exist holomorphic functions $\phi^1, \cdots, \phi^{N'}$ defined on U such that

$$\exp\{bD_M(p, q)\} = 1 + \sum_{\sigma=1}^{N'} |\phi^\sigma(q)|^2 \quad (q \in U) ,$$

and

$$\phi^\sigma(p) = 0 \quad (\sigma = 1, \cdots, N') .$$

By Lemma 1.3, we can define a holomorphic immersion of U into $C^{N'}$ by $\phi = (\phi^1, \cdots, \phi^{N'})$. Then the new diastasis D_ϕ of the induced metric g_ϕ is given by

$$D_\phi(p, q) = \sum_{\sigma=1}^{N'} |\phi^\sigma(q)|^2 . \tag{2.1}$$

On the other hand, since $b>0$ and M is a Kaehler submanifold of C^N, by Lemma 1.2, there exist holomorphic functions $h^1, \cdots, h^N$ such that

$$bD_M(p, q) = \sum_{r=1}^{N} |h^r(q)|^2 \quad (q \in U) ,$$
$$h^r(p) = 0 \quad (r = 1, \cdots, N) .$$

Now we obtain the following expression:

$$\exp\{bD_M(p, q)\} = 1 + \sum_{l=1}^{\infty} |\psi^l(q)|^2 ,$$

where ψ^l $(l = 1, 2, 3, \cdots)$ are holomorphic functions which are determined as terms of the series expansion

$$\begin{aligned} &\sum_{m=1}^{\infty} \left(\sum_{r=1}^{N} |h^r|^2 \right)^m \Big/ m! \\ &= \sum_{i_1+\cdots+i_N=1}^{\infty} \left| \frac{1}{\sqrt{i_1! \cdots i_N!}} (h^1)^{i_1} \cdots (h^N)^{i_N} \right|^2 . \end{aligned} \tag{2.2}$$

So we can define a holomorphic mapping of U into l^2 by $\psi = (\psi^1, \psi^2, \psi^3, \cdots)$. Since

$$\sum_{\sigma=1}^{N'} |\phi^\sigma(q)|^2 = D_\phi(p, q) = \sum_{l=1}^{\infty} |\psi^l(q)|^2 ,$$

ψ is an isometric mapping of the Kaehler manifold (U, g_ϕ) into l^2. Hence by Proposition 1.7, $\psi(U)$ lies in some complex N-plane of l^2. On the other hand, $\{\psi^l\}_{l=1,2,3,\cdots}$ has the subsequence of the functions $\{(h^1)^m/\sqrt{m!}\}_{m=1,2,3,\cdots}$ which are linearly independent. This yields a contradiction.

PROPOSITION 2.2. *Let M be a Kaehler n-submanifold of C^N. Then* *any open subset of M can not be a Kaehler submanifold of $CH^{N'}(b')$ for any N' and $b'<0$.*

PROOF. We suppose that some open subset U of M is a Kaehler submanifold of $CH^{N'}(2b)$ $(b'=2b)$. Throughout this proof, we fix a point $p \in U$. Then by Lemma 1.2, there exist holomorphic functions $\phi^1, \cdots, \phi^{N'}$ defined on U such that

$$D_M(p, q)=\frac{1}{b}\log\Big(1-\sum_{\sigma=1}^{N'}|\phi^\sigma(q)|^2\Big) \quad (q \in U) ,$$
$$\phi^\sigma(p)=0 \quad (\sigma=1, \cdots, N') .$$

Since $b<0$, we have the following expression:

$$D_M(p, q)=\sum_{l=1}^{\infty}|\psi^l(q)|^2 \quad (q \in U) , \tag{2.3}$$

where ψ^l $(l=1, 2, 3, \cdots)$ are holomorphic functions which are determined as terms of the series expansion

$$\begin{aligned}&\frac{1}{b}\log\Big(1-\sum_{\sigma=1}^{N'}|\phi^\sigma|^2\Big)\\&=\sum_{i_1+\cdots+i_{N'}=1}^{\infty}\left|\sqrt{\frac{(i_1+\cdots+i_{N'}-1)!}{i_1!\cdots i_{N'}!(-b)}}(\phi^1)^{i_1}\cdots(\phi^{N'})^{i_{N'}}\right|^2 .\end{aligned} \tag{2.4}$$

So we can define an isometric mapping of U into l^2 by $\psi=(\psi^1, \psi^2, \psi^3, \cdots)$. Then by Proposition 1.7, $\psi(U)$ lies in some complex N-plane of l^2. On the other hand $\{\psi^l\}_{l=1,2,3,\cdots}$ has the subsequence of functions $\{(\phi^1)^m/\sqrt{-bm}\}_{m=1,2,3,\cdots}$ which are linearly independent. This makes a contradiction.

PROPOSITION 2.3. *Let M be a Kaehler n-submanifold of $CH^N(b')$. Then any open subset of M can not be a Kaehler submanifold of $CP^{N'}(c')$ for any N' and $c'>0$.*

PROOF. Since M is a Kaehler submanifold of $CH^N(2b)$ $(b'=2b)$, for fixed $p \in M$, there exist holomorphic functions $\phi^1, \cdots, \phi^N$ defined on some sufficiently small neighborhood U of $p \in M$ such that

$$D_M(p, q)=\frac{1}{b}\log\Big(1-\sum_{\sigma=1}^{N}|\phi^\sigma(q)|^2\Big) \quad (q \in U) ,$$
$$\phi^\sigma(p)=0 \quad (\sigma=1, \cdots, N) .$$

Now we assume that U is a Kaehler submanifold of $CP^{N'}(2c)$ $(c'=2c)$. By Lemma 1.2, there exist holomorphic functions $h^1, \cdots, h^{N'}$ defined on U such that

$$1+\sum_{r=1}^{N'}|h^r(q)|^2=\exp\{cD_M(p, q)\}$$
$$=\exp\left\{\frac{c}{b}\log\left(1-\sum_{\sigma=1}^{N}|\phi^\sigma(q)|^2\right)\right\}.$$

Let $\{\psi^l\}_{l=1,2,3,\cdots}$ be the system of holomorphic functions determined as terms of the series expansion (2.4) in which N' is replaced by N. Then

$$\sum_{r=1}^{N'}|h^r|^2=\exp\left(\sum_{l=1}^{\infty}c|\psi^l|^2\right)-1 \tag{2.5}$$
$$=\sum_{k=1}^{\infty}\sum_{i_1,\cdots,i_k=1}^{\infty}|c^{k/2}\psi^{i_1}\cdots\psi^{i_k}|^2 .$$

Using this we have the following expression:

$$\sum_{r=1}^{N'}|h^r|^2=\sum_{l=1}^{\infty}|\tilde{\psi}^l|^2 ,$$

where $\tilde{\psi}^l$ $(l=1, 2, 3, \cdots)$ are holomorphic functions determined by (2.5). By Lemma 1.3, we can define a holomorphic immersion of U into $C^{N'}$ by $h=(h^1, \cdots, h^{N'})$. The new diastasis D_h of the induced metric g_h is given by

$$D_h(p, q)=\sum_{r=1}^{N'}|h^r(q)|^2 \quad (q\in U) .$$

Then the holomorphic mapping $\psi=(\tilde{\psi}^1, \tilde{\psi}^2, \tilde{\psi}^3, \cdots)$ of the Kaehler manifold (U, g_h) into l^2 is isometric. Hence $\tilde{\psi}(U)$ lies in some complex N'-plane of l^2 by Proposition 1.7. But we can easily take a subsequence of $\{\tilde{\psi}^l\}_{l=1,2,3,\cdots}$ which are linearly independent. This makes a contradiction.

From Propositions 2.1, 2.2 and 2.3, the theorem stated in the introduction is directly obtained.

Appendix

Here we introduce "diastasis", following Calabi [1], and prove that it is equivalent to ours.

LEMMA. *Let f be a real analytic function on a complex n-manifold M. Then there exists a unique holomorphic function F on an open neighborhood of the diagonal set in $M\times\bar{M}$ such that*

$$F(p, p)=f(p) \quad (p \in M) ,$$

where $\bar{M}$ is the conjugate manifold of M.

Proof. For a sufficiently small coordinate neighborhood $(U; z^1, \cdots, z^n)$ of M, f has a power series expansion:

$$f(p)=\sum_{I,J} b_{IJ}(z(p))^I\overline{(z(p))}^J \quad (p \in U) ,$$

where by $\sum_{I,J}$ we mean the infinite sum for multi-indices I, J ranging over all n-tuples of non-negative integers. Then F should be expressed as follows:

$$F(p, q)=\sum_{I,J} b_{IJ}(z(p))^I\overline{(z(q))}^J \quad (p, q \in U) .$$

Such a function F is called the *complexification* of f. Let M be an analytic Kaehler manifold with the metric g, and f a primitive function of g defined on some open subset U of M. Then the functional element of *diastasis* is defined by

$$D_M(p, q)=F(p, p)+F(q, q)-F(p, q)-F(q, p) \quad (p, q \in U) ,$$

where F is the complexification of f. Since the primitive functions are differ by the real part of some holomorphic functions, D_M is uniquely determined. Obviously $D_M(p, q)$ is symmetric in p and q. On the other hand, since f is real analytic, we have $\overline{F(p, q)}=F(p, q)$ $(p, q \in U)$. Hence D_M is real valued. We prove the following.

Proposition. *Let M be an analytic Kaehler n-manifold and $p \in M$ any fixed point. Then there exists a unique primitive function $\tilde{f}$ defined on some neighborhood U of p such that for each multi-index $I=\{i_1, \cdots, i_n\}$ $(i_1, \cdots, i_n \geqq 0)$*

$$(1) \qquad \partial^{|I|}\tilde{f}/\partial z^I=0 \quad \text{and} \quad \partial^{|I|}\tilde{f}/\partial \bar{z}^I=0 \quad \text{at} \quad p ,$$

where $(z^1, \cdots, z^n)$ is a local complex coordinate. Moreover $\tilde{f}$ satisfies the following:

$$\tilde{f}(q)=D_M(p, q) \quad (q \in U) .$$

Proof. Let $(z^1, \cdots, z^n)$ be a local complex coordinate with the origin $p \in M$ and f a primitive function of the metric g, which has a following power series expansion:

$$f(q)=\sum_{I,J} b_{IJ}(z(q))^I\overline{(z(q))}^J \quad (q \in U) .$$

Then the diastasis $D_M(p, q)$ is expressed by

$$D_M(p, q)=\sum_{I,J\neq 0} b_{IJ}(z(q))^I(\overline{z(q)})^J \quad (q\in U)\ ,$$

where $0=\{0, \cdots, 0\}$. Hence $\tilde{f}(q)=D_M(p, q)$ satisfies (1) obviously. On the other hand, since primitive functions are differ by the real part of some holomorphic function, uniqueness of $\tilde{f}$ is also obvious.

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