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Reducibility of Flow-Spines

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The notion of flow-spines was introduced in [2]. A flow-spine is a standard spine of a closed 3-manifold M and is generated by a normal pair which is a pair of a non-singular flow on M and its compact local section. In this paper, we consider methods for constructing a simpler flow-spine than given one. In general, a spine P_1 (not necessarily a flow-spine) is thought to be simpler than P_2 when P_1 has less third singularities than P_2. And, for example in [1], several methods for obtaining a spine with less third singularities are discovered by Ikeda, Yamashita and Yokoyama. However a spine obtained by applying those methods to a flow-spine is not always a flow-spine. Hence, in order to leave our discussion within an extent of flow-spines, we must consider other "reducibility" of flow-spines.

In § 4 we will give one of reasonable definitions of the reducibility of flow-spines. In § 3 a "simply reduced flow-spine" is defined, and our reducibility will be considered within this sub-class of simply reduced flow-spines. And in §§ 5–6 we will give some conditions for a flow-spine to be reducible in our sense. §§ 1–2 are devoted to preparations. Especially in § 2, we will precisely formulate the concept of a "singularity-data" introduced in [2], and give a necessary condition for a singularity-data to be realized by a normal pair.

§ 1. Preliminaries.

Let M be a smooth closed 3-manifold, and ψ_t be a smooth non-singular flow on M. A pair of ψ_t and its compact local section Σ is said to be a *normal pair* (see [2] for the precise definition), if (ψ_t, Σ) satisfies that

(i) Σ is homeomorphic to a compact 2-disk,
(ii) $|T_{\pm}(\psi_t, \Sigma)(x)| < \infty$ for any $x \in M$,
(iii) $\partial\Sigma$ is ψ_t-transversal at $(x, T_+(\psi_t, \Sigma)(x))$ for any $x \in \partial\Sigma$, and
(iv) if $x \in \partial\Sigma$ and $x_1 = \hat{T}_+(\psi_t, \Sigma)(x) \in \partial\Sigma$, then $\hat{T}_+(\psi_t, \Sigma)(x_1)$ is contained

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where $T_\pm(\psi_t, \Sigma): M \to \boldsymbol{R}$ and $\hat{T}_\pm(\psi_t, \Sigma): M \to \Sigma$ are defined by

$$T_+(\psi_t, \Sigma)(x) = \inf\{t>0 \mid \psi_t(x) \in \Sigma\}$$
$$T_-(\psi_t, \Sigma)(x) = \sup\{t<0 \mid \psi_t(x) \in \Sigma\}$$
$$\hat{T}_\pm(\psi_t, \Sigma)(x) = \psi_\sigma(x) \quad (\sigma = T_\pm(\psi_t, \Sigma)(x)) .$$

Then *flow-spines* $P_-(\psi_t, \Sigma)$ and $P_+(\psi_t, \Sigma)$ generated by a normal pair (ψ_t, Σ) are given by

$$P_-(\psi_t, \Sigma) = \Sigma \cup \{\psi_t(x) \mid x \in \partial\Sigma,\ T_-(\psi_t, \Sigma)(x) \leqq t \leqq 0\}$$
$$P_+(\psi_t, \Sigma) = \Sigma \cup \{\psi_t(x) \mid x \in \partial\Sigma,\ 0 \leqq t \leqq T_+(\psi_t, \Sigma)(x)\} .$$

It was shown in [2] that every closed 3-manifold admits a normal pair, and that each of $P_-(\psi_t, \Sigma)$ and $P_+(\psi_t, \Sigma)$ forms a standard spine of the phase manifold.

When there is no fear of confusion, we simply write $T_\pm$, $\hat{T}_\pm$ and $P_\pm$ for $T_\pm(\psi_t, \Sigma)$, $\hat{T}_\pm(\psi_t, \Sigma)$ and $P_\pm(\psi_t, \Sigma)$ respectively. For a given normal pair (ψ_t, Σ), the following notation are used throughout this paper, which are the same as in [2].

NOTATION.

(1) For a closed fake surface P, $\mathfrak{S}_j(P)$ denotes the set of the j-th singularities of P (see [1], [2]).

(2) ν denotes the number of the elements of $\mathfrak{S}_3(P_-)$ ($P_- = P_-(\psi_t, \Sigma)$).

(3) By $a_1, a_2, \cdots, a_\nu$ we denote the elements of $\mathfrak{S}_3(P_-)$; i.e., $\mathfrak{S}_3(P_-) = \{a_1, \cdots, a_\nu\} = \{x \in \text{Int}\,\Sigma \mid \hat{T}_+(x)$ and $\hat{T}_+^2(x)$ are both on $\partial\Sigma\}$.

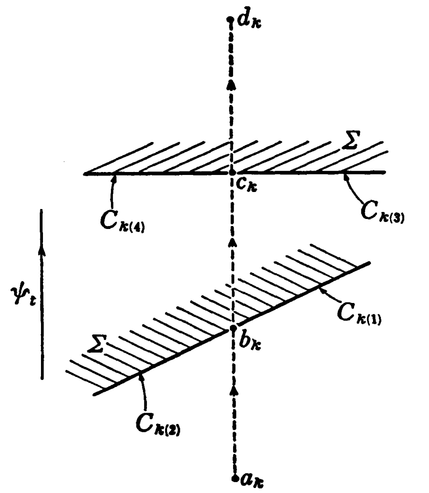

FIGURE 1

(4) $b_k=\hat{T}_+(a_k)$, $c_k=\hat{T}_+^2(a_k)$ and $d_k=\hat{T}_+^3(a_k)$ $(k=1, \cdots, \nu)$. Notice that $b_k, c_k \in \partial\Sigma$, and that $\{d_1, \cdots, d_\nu\}=\mathfrak{S}_3(P_+)\subset \mathrm{Int}\,\Sigma$.

(5) $C_1, C_2, \cdots, C_{2\nu}$ denote the connected components of $\partial\Sigma-\{b_1, \cdots, b_\nu, c_1, \cdots, c_\nu\}$.

We always assume that the assignments of numbers to a_k's and C_m's are fixed once for all.

For each $k=1, \cdots, \nu$, we define four integers $k(j)$ $(j=1, \cdots, 4, 1\leqq k(j)\leqq 2\nu)$ so that the components $C_{k(j)}$ are like as in Figure 1 (see [2] for the precise).

§2. Singularity-data.

In [2] the notion of the sigularity-data was introduced. We give its precise formulation in this section.

Let (ψ_t, Σ) be a normal pair on some manifold M. Fixing an orientation on $\partial\Sigma$, we denote by $\widehat{xy}$ $(x, y\in\partial\Sigma)$ the subarc of $\partial\Sigma$ going from x to y in the positive direction. For each $m=1, \cdots, 2\nu$, take a point w_m on the component C_m of $\partial\Sigma-\{b_1, \cdots, b_\nu, c_1, \cdots, c_\nu\}$. Then each $a_k\in\mathfrak{S}_3(P_-)$ satisfies one of the following four conditions:

$$(+)\quad b_k\in\widehat{w_{k(1)}w_{k(2)}} \quad\text{and}\quad c_k\in\widehat{w_{k(3)}w_{k(4)}}$$
$$(-)\quad b_k\in\widehat{w_{k(2)}w_{k(1)}} \quad\text{and}\quad c_k\in\widehat{w_{k(4)}w_{k(3)}}$$
$$(+^*)\quad b_k\in\widehat{w_{k(1)}w_{k(2)}} \quad\text{and}\quad c_k\in\widehat{w_{k(4)}w_{k(3)}}$$
$$(-^*)\quad b_k\in\widehat{w_{k(2)}w_{k(1)}} \quad\text{and}\quad c_k\in\widehat{w_{k(3)}w_{k(4)}}\,.$$

As is shown in [2], any a_k satisfies the condition $(+)$ or $(-)$ if M is orientable. In [2], the following two informations (a) and (b) about the third singularities of $P_\pm$ are called a *singularity-data*.

(a) The arrangement of b_k's and c_k's on $\partial\Sigma$.

(b) The condition $(+)$ or $(-)$ or $(+^*)$ or $(-^*)$ which is satisfied by each of a_k's.

How a singularity-data determines a flow-spine is stated in [2].

Now we shall give a more precise formulation of a singularity-data. Let B^+, B^-, C^+ and C^- be mutually disjoint finite subsets of the circle S^1 such that $\#(B^+\cup B^-)=\#(C^+\cup C^-)$. Let θ be a one-to-one correspondence between $B^+\cup B^-$ and $C^+\cup C^-$, and σ be an orientation on S^1. Then we call the six-tuple $(\sigma; B^+, B^-; C^+, C^-; \theta)$ a *singularity-data* of a flow-spine. Namely, putting $\{b_1, \cdots, b_\nu\}=B^+\cup B^-$ and $c_k=\theta(b_k)$, we determine the condition $(\pm)$ or $(\pm^*)$ with respect to the given orientation σ on $S^1=\partial\Sigma$

which is satisfied by $a_k \in \mathfrak{S}_3(P_-)$ corresponding to b_k in the following way:

(i) a_k satisfies $(+)$ iff $b_k \in B^+$ and $c_k \in C^+$,

(ii) a_k satisfies $(-)$ iff $b_k \in B^-$ and $c_k \in C^-$,

(iii) a_k satisfies $(+*)$ iff $b_k \in B^+$ and $c_k \in C^-$,

(iv) a_k satisfies $(-*)$ iff $b_k \in B^-$ and $c_k \in C^+$.

Let $\Delta=(\sigma; B^+, B^-; C^+, C^-; \theta)$ be a singularity-data, and Γ_l $(l=1, \cdots, \nu)$ be the connected components of $S^1-(B^+\cup B^-)$, and w_l be a point on Γ_l. For each $k=1, \cdots, \nu$, we define three integers $k\{j\}$ $(j=1, 2, 3, 1\leqq k\{j\}\leqq \nu)$ so that $\Gamma_{k\{j\}}$ satisfy the following conditions (i)–(iii).

(i) $\Gamma_{k\{1\}}$ and $\Gamma_{k\{2\}}$ are components having b_k as their end point,

(ii) $b_k \in \widehat{w_{k\{1\}}w_{k\{2\}}}$ iff $b_k \in B^+$, and $b_k \in \widehat{w_{k\{2\}}w_{k\{1\}}}$ iff $b_k \in B^-$,

(iii) $c_k \in \Gamma_{k\{3\}}$.

And define a group $\Pi(\Delta)$ by

$$\Pi(\Delta)\equiv\langle g_1, \cdots, g_\nu; r_1, \cdots, r_\nu\rangle , \quad r_k=g_{k\{1\}}g_{k\{3\}}g_{k\{2\}}^{-1} .$$

The following theorem was shown in [2].

THEOREM 2.1. *If a singularity-data Δ is realized by a normal pair on M, then $\pi_1(M)=\Pi(\Delta)$.*

For a singularity-data $\Delta=(\sigma; B^+, B^-; C^+, C^-; \theta)$, we define the *reversed singularity-data* Δ^r by $\Delta^r=(-\sigma; C^+, C^-; B^+, B^-; \theta^{-1})$. If Δ is realized by a normal pair (ψ_t, Σ) on M, then Δ^r is realized by $(\bar{\psi}_t, \Sigma)$ where $\bar{\psi}_t$ is the time-reversed flow given by $\bar{\psi}_t=\psi_{-t}$. Hence, by the above theorem, we must have $\Pi(\Delta)=\Pi(\Delta^r)=\pi_1(M)$, namely we get the following necessary condition for the realizability of a singularity-data.

PROPOSITION 2.2. *If a singularity-data Δ is realized by some normal pair, then $\Pi(\Delta)=\Pi(\Delta^r)$.*

§3. Simple third singularities, simply reduced normal pairs.

Let (ψ_t, Σ) be a normal pair. A third singularity a_k of $P_-=P_-(\psi_t, \Sigma)$ (or d_k of P_+) is said to be *simple*, if $C_{k(2)}=C_{k(3)}$. If $a_k\in\mathfrak{S}_3(P_-)$ is simple, then each of $\{a_k\}\cup\hat{T}_-(C_{k(2)})$ and $\{d_k\}\cup\hat{T}_+(C_{k(3)})$ forms a simple closed curve in Σ (cf. Figure 2).

DEFINITION 3.1. A normal pair (ψ_t, Σ) is said to be *simply reduced,* if any simple $a_k\in\mathfrak{S}_3(P_-)$ satisfies that

$$\partial C_{k(1)}\subset\hat{T}_+(\mathfrak{S}_3(P_-))=\{b_1, \cdots, b_\nu\} , \quad \text{and}$$
$$\partial C_{k(4)}\subset\hat{T}_-(\mathfrak{S}_3(P_+))=\{c_1, \cdots, c_\nu\} .$$

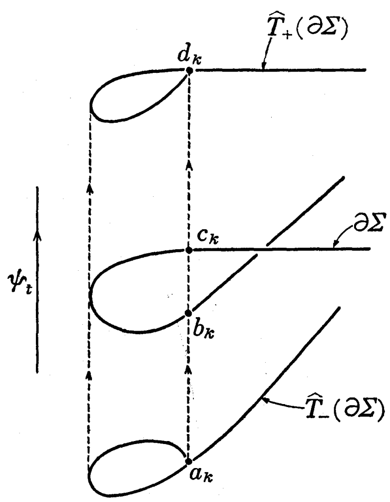

FIGURE 2

In what follows, we shall give a method for obtaining a simply reduced normal pair from given one.

Suppose that $\#\mathfrak{S}_3(P_-)\geqq 2$, and let $a_k\in\mathfrak{S}_3(P_-)$ be a simple third singularity such that $\partial C_{k(1)}\cap\hat{T}_-(S_3(P_+))\neq\varnothing$. Then $C_{k(1)}=C_{k'(3)}$ or $C_{k(1)}=C_{k'(4)}$ for some $k'\neq k$. First we shall consider the case $C_{k(1)}=C_{k'(3)}$. Assume that b_k, c_k, $b_{k'}$ and $c_{k'}$ are arranged as in Figure 3 (a). Then $\widehat{c_{k'}c_k}$ and $C_{k'(1)}\cup C_{k'(2)}$ are mapped by $\hat{T}_-$ into the figure like as in Figure 3 (b).

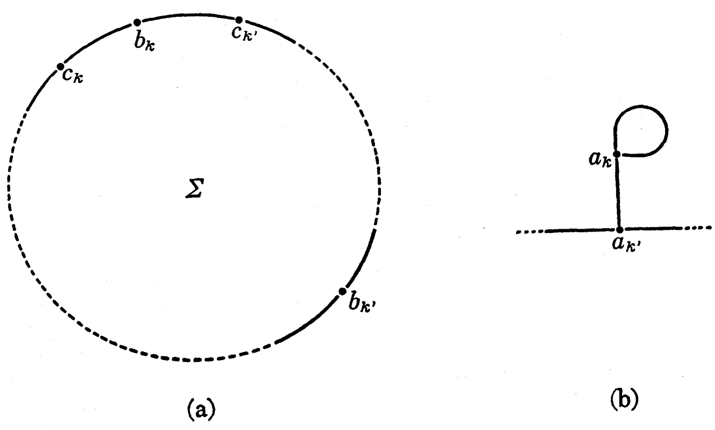

FIGURE 3

Take a compact 2-disk Y in Σ so that $(\operatorname{Int} Y)\cap\hat{T}_-(\partial\Sigma)=\hat{T}_-(\widehat{c_{k'}c_k})$ and $\gamma\equiv\partial Y\cap\hat{T}_-(\partial\Sigma)=\hat{T}_-(\gamma')$ for some small subarc γ' of $\partial\Sigma$ containing $b_{k'}$. And choose a continuous function $f\colon Y\to\boldsymbol{R}$ so that $f(x)=\hat{T}_+(x)$ for $x\in\gamma$ and $0<f(x)<T_+(x)$ for $x\in Y-\gamma$. A new compact local section Σ' is defined by $\Sigma'=\Sigma\cup\{\psi_t(x)\,|\,x\in Y,\ t=f(x)\}$. Then (ψ_t, Σ') is also a normal pair and

has less third singularity than (ψ_t, Σ) (see Figure 4). If $\Delta=(\sigma; B^+, B^-; C^+, C^-; \theta)$ is the singularity-data for (ψ_t, Σ), then the singularity-data Δ' of (ψ_t, Σ') is given by $\Delta'=(\sigma; B_1^+, B_1^-; C_1^+, C_1^-; \theta_1)$ where $B_1^{\pm}=B^{\pm}-\{b_{k'}\}$, $C_1^{\pm}=C^{\pm}-\{c_{k'}\}$ and $\theta_1=\theta|_{B_1^+\cup B_1^-}$.

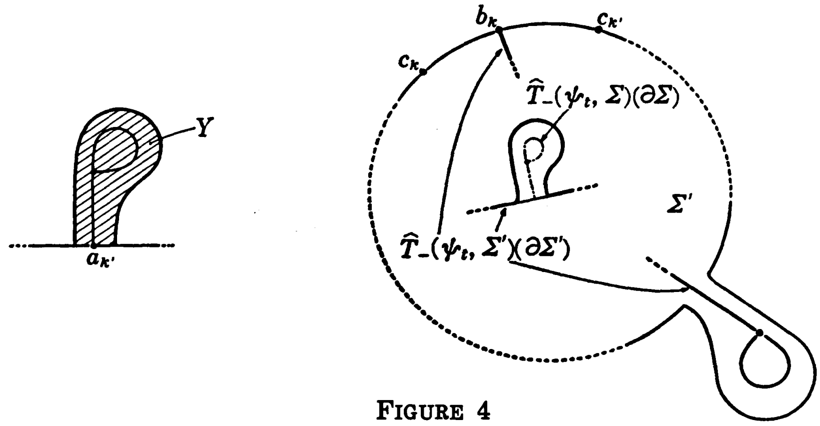

FIGURE 4

Now we shall consider the case $C_{k(1)}=C_{k'(4)}$. In this case, $\hat{T}_-(\widehat{c_{k'}c_k})$ is like as in Figure 5. First we shall show that, deforming ψ_t if necessary, we may assume that $\hat{T}_-(\widehat{c_{k'}c_k})$ is disjoint from $\hat{T}_+(\partial\Sigma)$.

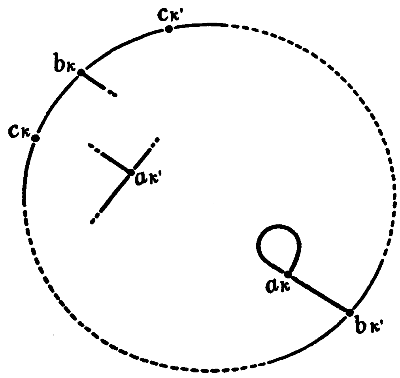

FIGURE 5

Let X be the vector field generating ψ_t, and define U to be $U=\{\psi_t(x) \mid x\in \operatorname{Int}\Sigma, -\delta<t<0\}$, where $\delta>0$ is a collar-size for (ψ_t, Σ) (see [2] for the definition of a collar-size). Let (x, y) be a smooth coordinate on Σ. Then, by the mapping $(x, y, t)\mapsto\psi_t(x, y)$, (x, y, t) becomes a coordinate on U. Consider a vector field $\tilde{X}$ on M such that $\tilde{X}\equiv 0$ on the outside of U and $\tilde{X}(x, y, t)=a(x, y, t)\partial/\partial x+b(x, y, t)\partial/\partial y$ on U. And let ψ'_t be a flow generated by $X+\tilde{X}$. Then obviously (ψ'_t, Σ) is a normal pair and has the same singularity-data as (ψ_t, Σ). Moreover it is easy to see that, for an adequate choice of $\tilde{X}$, $\hat{T}_-(\psi'_t, \Sigma)(\widehat{c_{k'}c_k})$ does not intersect with $\hat{T}_+(\psi'_t, \Sigma)(\partial\Sigma)$

(cf. Figure 6).

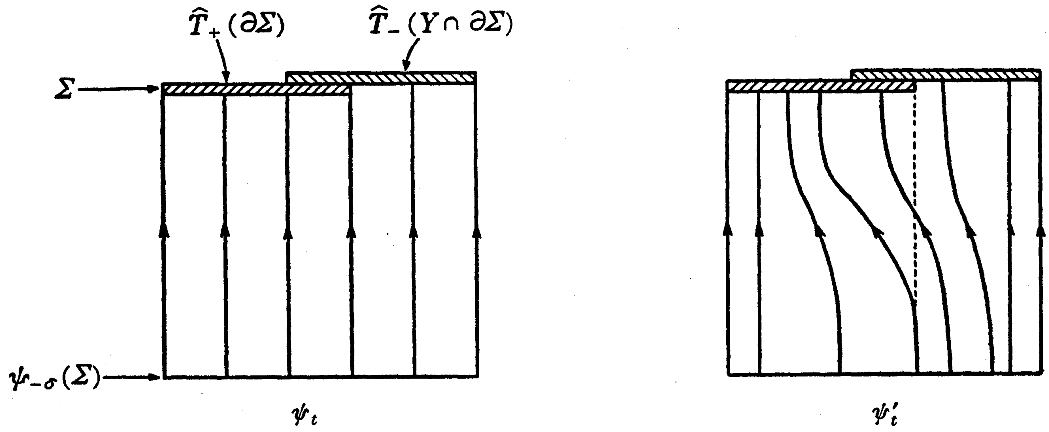

FIGURE 6

Hence we may assume that the original (ψ_t, Σ) has this property. Then we can take a compact 2-disk $Y\subset\Sigma$ so that $(\mathrm{Int}\,\Sigma)\cap\hat{T}_-(\partial\Sigma)=\hat{T}_-(\widehat{c_{k'}c_k})$ and $Y\cap\hat{T}_+(\partial\Sigma)=\varnothing$ (see Figure 7(a)). Then, for a compact local section $\Sigma'=\mathrm{Cl}(\Sigma-Y)$, (ψ_t, Σ') is a normal pair and $P_-(\psi_t, \Sigma')$ has less third singularity than $P_-(\psi_t, \Sigma)$. Also in this case, the singularity-data of (ψ_t, Σ') is obtained by omitting $b_{k'}$ and $c_{k'}$ from the one of (ψ_t, Σ).

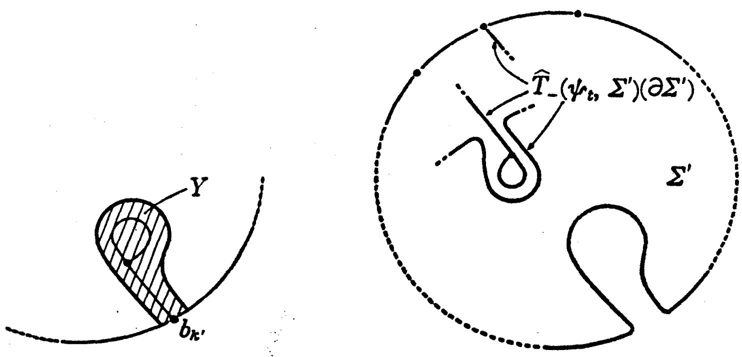

FIGURE 7

If $a_k\in\mathfrak{S}_3(P_-)$ is simple and $b_{k'}\in\partial C_{k(4)}$, then, considering the graph $\hat{T}_+(\partial\Sigma)$ instead of $\hat{T}_-(\partial\Sigma)$, we can see that the third singularity $d_{k'}\in P_+$ can be removed in the same way as above. Repeating this procedure, we get a simply reduced normal pair or a normal pair with only one third singularity. If M admits a normal pair with one third singularity, then M is the 3-sphere S^3 (see [2]). Hence we have that

THEOREM 3.1. *If $M \neq S^3$, then by the above procedure we get a simply reduced normal pair. And in the case of $M = S^3$ we obtain a simply reduced normal pair or a normal pair with only one third singularity.*

§4. Reducibility.

Let (ψ_t, Σ) be a normal pair on M, and $A \equiv \{a_{k_1}, \cdots, a_{k_r}\}$ be the set of simple third singularities of $P_-(\psi_t, \Sigma)$. Then $\gamma_j \equiv \{a_{k_j}\} \cup \hat{T}_-(C_{kj(2)})$ is a simple closed curve in Σ for each $a_{k_j} \in A$. We denote by $D_j \subset \Sigma$ the domain bounded by γ_j, and define V to be

$$V = P_- \cup \{\psi_t(x) \mid x \in D_1 \cup D_2 \cup \cdots \cup D_r,\ 0 < t \leqq \delta\}$$

where $\delta > 0$ is a collar-size for (ψ_t, Σ). Evidently V collapses to P_-, and has free faces

$$F_j = \{\psi_t(x) \mid x \in \gamma_j - \{a_{k_j}\},\ 0 < t < \delta\} .$$

Collapsing V from these free faces, we obtain

$$V' = (P_- \cup \psi_\delta(D_1) \cup \cdots \cup \psi_\delta(D_r)) - (\tilde{\gamma}_1 \cup \cdots \cup \tilde{\gamma}_r)$$
$$\tilde{\gamma}_j = \{\psi_t(x) \mid x \in \gamma_j - \{a_{k_j}\},\ 0 < t < \delta\} ,$$

(see Figure 8). This V' still has free faces $L_j = \{\psi_t(a_{k_j}) \mid 0 < t < \delta\}$.

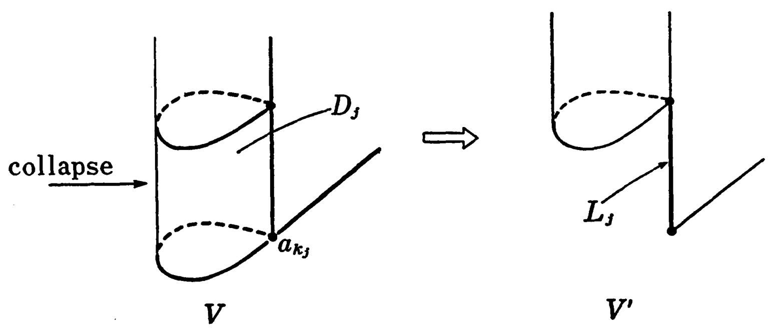

FIGURE 8

Hence, continuing the collapsing process, we get a spine $\tilde{P}$ of M. Maybe $\tilde{P}$ depends on the collapsing process. And, in general, $\tilde{P}$ is not a flow-spine. However it is known that

THEOREM 4.1 ([1]). *If we get a $\tilde{P}$ which is not a closed fake surface, then $H_1(M; Z)$ is not trivial or $M = S^3$.*

And the next proposition can be easily seen by the way in which we collapse V to $\tilde{P}$.

PROPOSITION 4.2. *If $\tilde{P}$ is a closed fake surface, then $\mathfrak{S}_3(\tilde{P})$ is included in $\mathfrak{S}_3(P_-)-A$. And moreover if $b_k \in \partial C_{k_j(1)}$ for some $a_{k_j} \in A$, then $a_k \notin \mathfrak{S}_3(\tilde{P})$.*

This proposition implies that a simply reduced flow-spine having many simple third singularities results in a spine with few third singularities. Taking account of this, we define the reducibility of a flow-spine in what follows.

DEFINITION 4.1. Two simple third singularities a_{k_1} and a_{k_2} are said to be *twin*, if $C_{k_1(1)}$ and $C_{k_2(1)}$ has the same boundary point.

DEFINITION 4.2. (1) $\kappa_0=\kappa_0(\psi_t, \Sigma)$ denotes the number of the simple third singularities of $P_-(\psi_t, \Sigma)$.

(2) $\kappa_1=\kappa_1(\psi_t, \Sigma)$ denotes the number of pairs of twin simple third singularities of $P_-(\psi_t, \Sigma)$.

(3) $\kappa=\kappa(\psi_t, \Sigma)$ is defined by $\kappa=\nu-2\kappa_0+\kappa_1$ $(\nu=\#\mathfrak{S}_3(P_-))$.

We define the reducibility as follows.

DEFINITION 4.3. A simply reduced normal pair (ψ_t, Σ) (or its flow-spine $P_-(\psi_t, \Sigma)$) on M is said to be *reducible*, if there is a simply reduced normal pair (ψ'_t, Σ') on M satisfying either of the following (i) or (ii).

(i) $\kappa(\psi'_t, \Sigma')<\kappa(\psi_t, \Sigma)$.

(ii) $\kappa(\psi'_t, \Sigma')=\kappa(\psi_t, \Sigma)$ and $\kappa_0(\psi'_t, \Sigma')-2\kappa_1(\psi'_t, \Sigma')<\kappa_0(\psi_t, \Sigma)-2\kappa_1(\psi_t, \Sigma)$.

The next theorem will give a reasonability of this definition of the reducibility.

THEOREM 4.3. *If M admits a simply reduced normal pair (ψ_t, Σ) such that $\kappa(\psi_t, \Sigma)\leqq 0$, then either $H_1(M; \mathbf{Z})\neq\{0\}$ or $M=S^3$.*

First we shall prove that

LEMMA 4.4. *Let a_{k_1} and a_{k_2} be twin simple third singularities of $P_-(\psi_t, \Sigma)$. If (ψ_t, Σ) is simply reduced and $H_1(M; \mathbf{Z})=\{0\}$, then $\partial C_{k_1(1)} \cap \partial C_{k_2(1)}=\{b_{k_3}\}$ for some $k_3\neq k_1, k_2$.*

PROOF. Since (ψ_t, Σ) is simply reduced, $C_{k_1(1)}=C_{k_2(1)}$ if the conclusion of the lemma does not hold. In this case, setting $L=\mathrm{Cl}(C_{k_1(2)} \cup C_{k_1(1)} \cup C_{k_2(2)})$, we can see that $\hat{T}_-(L)$ forms a component of $\partial\Sigma \cup \hat{T}_-(\partial\Sigma)$, that is, $\partial\Sigma \cup \hat{T}_-(\partial\Sigma)$ is not connected. As is shown in Theorem 4.3 of [2], $H_1(M; \mathbf{Z})$ is not trivial if $\partial\Sigma \cup \hat{T}_-(\partial\Sigma)$ is not connected. This completes the proof.

Proof of Theorem 4.3. Suppose that $H_1(M; \mathbf{Z})=\{0\}$ and $M\neq S^3$, and consider the spine $\tilde{P}$ constructed in the beginning of this section. Because of Theorem 4.1, $\tilde{P}$ is a closed fake surface.

Let A be the set of simple third singularities of $P_-(\psi_t, \Sigma)$, and A_0 be a set of third singularities $a_{k'}$ such that $b_{k'}\in\partial C_{k(1)}$ for some $a_k\in A$ $(A\subset A_0)$. Then, since (ψ_t, Σ) is simply reduced, we have $\# A_0=2(\kappa_0-\kappa_1)+\kappa_1$ by Lemma 4.4, and hence $\#\mathfrak{S}_3(\tilde{P})\leqq(\nu-\kappa_1)-2(\kappa_0-\kappa_1)$ by Proposition 4.2. On the other hand, M has no standard spine without third singularities if $H_1(M; \mathbf{Z})=\{0\}$ (see [1]). Therefore we must have $\kappa(\psi_t, \Sigma)\geqq\#\mathfrak{S}_3(P)>0$. This proves the theorem.

According to Theorem 4.3, an affirmative answer to the following problem implies the Poincaré conjecture.

Problem. Let M be a homotopy sphere and (ψ_t, Σ) be a simply reduced normal pair on M. Is (ψ_t, Σ) reducible whenever $\kappa(\psi_t, \Sigma)>0$?

§5. Examples of reducing methods.

In this section, we explain by examples how we can see the reducibility of a flow-spine. As an example, we consider the singularity-data $(\sigma; B^+, B^-; C^+, C^-; \theta)$ given in Figure 9, where $B^+=\{b_1, b_3\}$, $B^-=\{b_2, b_4\}$, $C^+=\{c_1, c_3\}$, $C^-=\{c_2, c_4\}$ and $c_k=\theta(b_k)$. It can be shown that this singularity-data is realized by a normal pair on S^3, and $\hat{T}_-(\partial\Sigma)$ and $\hat{T}_+(\partial\Sigma)$ are like as in Figure 10.

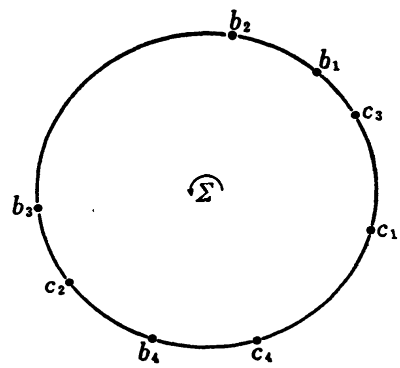

Figure 9

We shall show the reducibility of this normal pair (ψ_t, Σ) in three different ways.

The First Method. Take a compact 2-disk $Y\subset\Sigma$ like as in Figure

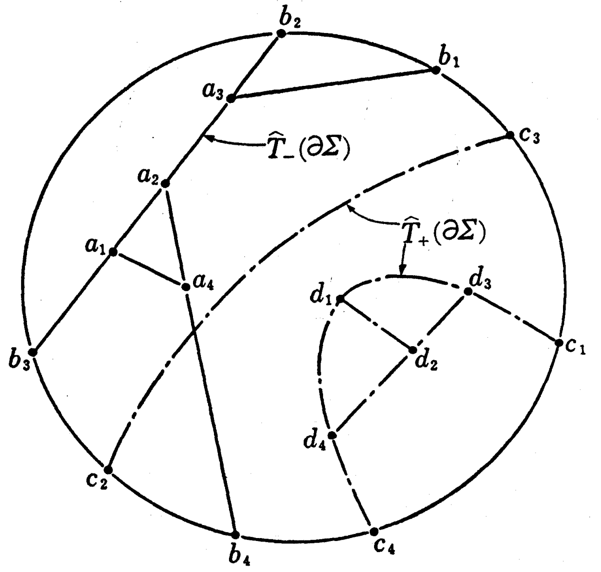

FIGURE 10

11 (a). Next choose a continuous function f: $Y\to \boldsymbol{R}$ such that $f(x)=T_+(x)$ for $x\in Y\cap \hat{T}_-(\partial\Sigma)$ and $0<f(x)<T_+(x)$ otherwise. Then, setting $\Sigma'=\Sigma\cup\{\psi_t(x)\,|\,x\in Y,\ t=f(x)\}$, we get a new normal pair (ψ_t, Σ'). For this (ψ_t, Σ'), $\hat{T}_-(\partial\Sigma')$ is like as in Figure 11 (b). Evidently $\kappa(\psi_t, \Sigma')=\kappa(\psi_t, \Sigma)-1$.

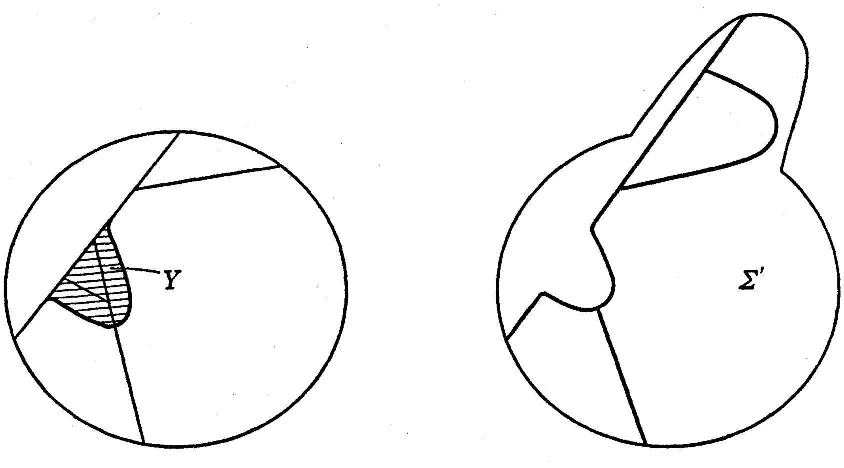

FIGURE 11

The Second Method. In this case, we take a compact 2-disk $Y\subset\Sigma$ like as in Figure 12. Then, applying the method used in § 3, we may assume that $Y\cap\hat{T}_-(\partial\Sigma)=\varnothing$. Take another 2-disk U like as in Figure 12.

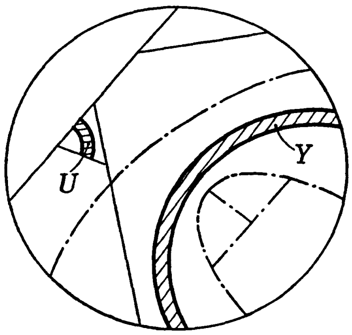

FIGURE 12

Now choose a continuous function $f: U \to \boldsymbol{R}$ such that $f(x) = T_+(x)$ for $x \in U \cap \hat{T}_-(\partial \Sigma)$ and $0 < f(x) < T_+(x)$ otherwise. Then, setting $\Sigma' = (\mathrm{Cl}(\Sigma - Y)) \cup \{\psi_t(x) \mid x \in U, t = f(x)\}$, we obtain a normal pair (ψ_t, Σ'). The singularity-data of (ψ_t, Σ') is given by Figure 13, and this normal pair has a simple third singularity a_1. Hence, applying the procedure in § 3, we get a simply reduced normal pair (ψ'_t, Σ'') such that $\kappa(\psi'_t, \Sigma'') < \kappa(\psi_t, \Sigma)$.

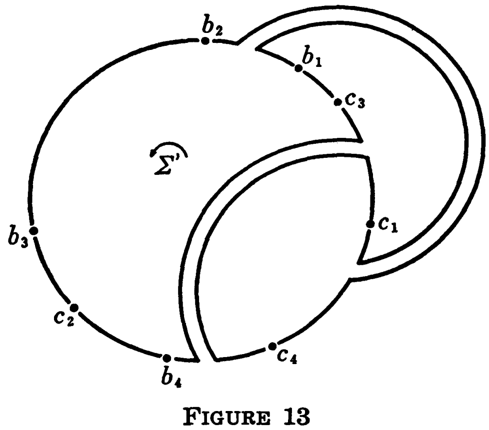

FIGURE 13

The Third Method. In this case, we take three 2-disks Y_j $(j = 1, 2, 3)$ like as in Figure 14 (a), (b). And let γ_l $(l = 1, \cdots, 6)$ be subarcs of ∂Y_j indicated in the figure. We can choose continuous functions $f_j: Y_j \to \boldsymbol{R}$ such that

(i) $0 < f_j(x) < T_+(x)$ for any j and $x \in Y_j$,

(ii) $f_1(x) \equiv \delta$ (δ is a collar-size),

(iii) $f_2(x) = T_+(x) + \delta$ for $x \in \gamma_4$,

(iv) $f_3(x) = T_+(x) + \delta$ for $x \in \gamma_6$, and

(v) $f_3(x) = T_+(x) + f_2(\hat{T}_+(x))$ for $x \in \gamma_5$ $(\hat{T}_+(x) \in \gamma_3)$.

Then $D = \{\psi_t(x) \mid x \in Y_j, t = f_j(x), j = 1, 2, 3\}$ is a compact local section and homeomorphic to a 2-disk. Now take another compact 2-disk U like as in Figure 14 (b), and choose a continuous function $f: U \to \boldsymbol{R}$ such that $f(x) = T_+(x)$ for $x \in U \cap \hat{T}_-(\partial \Sigma)$, $f(x) = \delta$ for $x \in U \cap Y_1$ and $0 < f(x) < T_+(x)$ otherwise.

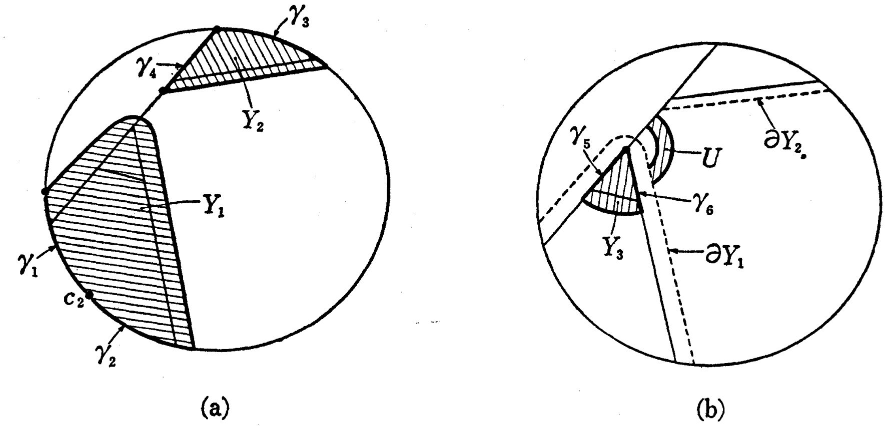

FIGURE 14

Then, defining Σ' by $\Sigma' = \Sigma \cup D \cup \{\psi_t(x) \mid x \in U,\ t = f(x)\}$, we get a normal pair (ψ_t, Σ). We can easily see that, applying the procedure used in § 3 to this (ψ_t, Σ'), we obtain a normal pair (ψ'_t, Σ'') with $\#\mathfrak{S}_3(P_-(\psi'_t, \Sigma'')) = 1$.

In the next section, we shall give a generalization of the third method. The first and the second methods will be discussed in the forthcoming paper.

§ 6. A condition for the reducibility of flow-spines.

In order to give a condition for the reducibility which is a generalization of the third method of the preceding section, we first prepare a definition.

DEFINITION 6.1. A simple closed curve β in M is said to be *nice* (with respect to a normal pair (ψ_t, Σ)), if it satisfies that

(i) $\beta \cap (\Sigma \cup \mathfrak{S}_2(P_-) \cup \mathfrak{S}_2(P_+)) = \varnothing$,

(ii) $\psi_t(x) \notin \beta$ for any $x \in \beta$ and $0 < t < T_+(x)$,

(iii) β is nowhere tangential to ψ_t, and transversal to P_- and P_+,

(iv) $\beta \cap P_- = \{x_\beta\}$ is a singleton and $x_\beta \notin C_{k(2)} = C_{k(3)}$ for any simple third singularity a_k of P_-,

(v) there is an embedded 2-disk $D_\beta \subset M - \Sigma$ such that $\partial D_\beta = \beta$ and D_β is a compact local section of ψ_t, and

(vi) $D_\beta \cap P_- \cap P_+ \neq \varnothing$ or $\hat{T}_+(x_\beta) \notin C_{k(1)} \cup C_{k(4)}$ for any simple third singularity a_k of P_-.

Then we can show that

THEOREM 6.1. *A simply reduced normal pair* (ψ_t, Σ) *is reducible, if it admits a nice closed curve* β *such that* $\tilde{x}_\beta \equiv \{\psi_t(x_\beta) \mid 0<t<T_+(x_\beta)\}$ *does not intersect with* D_β.

Moreover in the case where $H_1(M; \boldsymbol{Z})$ is trivial, we have that

THEOREM 6.2. *A simply reduced normal pair* (ψ_t, Σ) *on* M *is reducible, if* $H_1(M; \boldsymbol{Z})$ *is trivial and* (ψ_t, Σ) *admits a nice closed curve.*

PROOF OF THEOREM 6.1. Let β be a nice closed curve with respect to (ψ_t, Σ), and B_0 be a subset of $\hat{T}_+(\mathfrak{S}_3(P_-))$ consisting of the points b such that $\psi_t(b) \notin D_\beta$ for any $0<t<T_+(b)$. First we shall prove that

LEMMA 6.3. $b_k = \hat{T}_+(a_k)$ *is contained in* B_0, *if* a_k *is a simple third singularity of* $P_-(\psi_t, \Sigma)$.

PROOF. Let the third singularity a_k be simple, and $V \subset \Sigma$ be the domain bounded by $\{a_k\} \cup \hat{T}_-(C_{k(2)})$. And define $\tilde{V}$ to be $\tilde{V} = \{\psi_t(x) \mid x \in \mathrm{Cl}(V), 0 \leqq t \leqq T_+(x)\}$. Then, according to the conditions (iv) and (v) in Definition 6.1, each component of $D_\beta \cap \partial\tilde{V}$ is a closed curve in $\partial\tilde{V} - (V \cup \hat{T}_+(V) \cup C_{k(2)})$, and nowhere tangential to ψ_t. Therefore $D_\beta \cap \partial\tilde{V}$ cannot intersect with the orbit segment from b_k to $\hat{T}_+(b_k)$. This completes the proof of the lemma.

Now suppose that $\tilde{x}_\beta \cap D_\beta = \emptyset$, and denote by C_β the component of $\partial\Sigma - (\hat{T}_+(\mathfrak{S}_3(P_-)) \cup \hat{T}_+^2(\mathfrak{S}_3(P_-)))$ which contains $\hat{T}_+(x_\beta)$. Then, since $\tilde{x}_\beta \cap D_\beta = \emptyset$, we can take a compact 2-disk $U \subset \Sigma$ like as in Figure 15 and a continuous function $f: U \to \boldsymbol{R}$ which satisfy that

(i) $U \cap \hat{T}_-(\partial\Sigma) \subset \hat{T}_-(C_\beta)$,
(ii) $f(x) = T_+(x)$ for $x \in U \cap \hat{T}_-(C_\beta)$,
(iii) $f(\hat{T}_-(x)) = -T_-(x)$ for $x \in \beta$ $(\hat{T}_-(x) \in U \cap \hat{T}_-(\beta))$, and
(iv) $\psi_{f(x)}(x) \notin \Sigma \cap D_\beta$ for $x \in U - \hat{T}_-(\partial\Sigma \cup \beta)$.

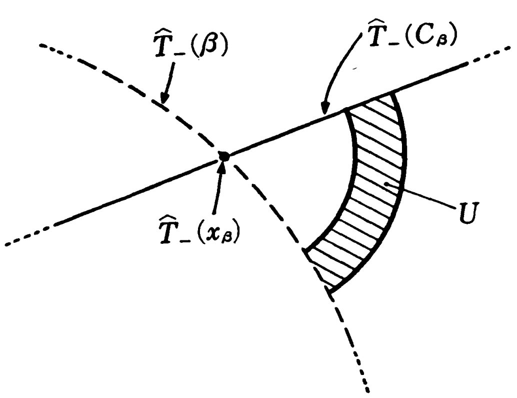

FIGURE 15

Define Σ' to be $\Sigma'=\Sigma\cup D_\beta\cup\{\psi_t(x)\mid x\in U,\ t=f(x)\}$. Then (ψ_t, Σ') is a normal pair, and $a_0'=\hat{T}_-(\psi_t, \Sigma')(x_\beta)$ is a simple third singularity of $P_-(\psi_t, \Sigma')$. In the remainder of the proof, we denote $T_\pm(\psi_t, \Sigma')$ and $\hat{T}_\pm(\psi_t, \Sigma')$ by $T'_\pm$ and $\hat{T}'_\pm$ respectively, and $T_\pm(\psi_t, \Sigma)$ and $\hat{T}_\pm(\psi_t, \Sigma)$ by $T_\pm$ and $\hat{T}_\pm$ respectively. Let B^* be the set of points $c\in\beta\cap P_+(\psi_t, \Sigma)$ such that $\psi_t(c)\notin D_\beta$ for any $T_-(c)<t<0$, and define B_1 by $B_1=\hat{T}'_-(B^*)$. Then it is evident that $\mathfrak{S}_3(P_-(\psi_t, \Sigma'))=\{a_0'\}\cup\hat{T}'_-(B_0)\cup\hat{T}'_-(B_1)$. Let Δ be the singularity-data of (ψ_t, Σ'), and Δ' be the one obtained by removing B_1 and $\hat{T}'_+(B_1)$ from Δ. Then, noticing that a_0' is simple, we can easily see that Δ' can be realized by some normal pair (ψ_t', Σ'') on M. We shall consider a simply reduced normal pair (ψ_t^*, Σ^*) which is obtained by applying the procedure in §3 to this (ψ_t', Σ''), and show that $\kappa(\psi_t^*, \Sigma^*)<\kappa(\psi_t, \Sigma)$ or $\kappa_0(\psi_t^*, \Sigma^*)-2\kappa_1(\psi_t^*, \Sigma^*)<\kappa_0(\psi_t, \Sigma)-2\kappa_1(\psi_t, \Sigma)$.

First we shall consider the case where $D_\beta\cap P_+\cap P_-\neq\emptyset$ $(P_\pm=P_\pm(\psi_t, \Sigma))$. In this case $\#B_0<\#\mathfrak{S}_3(P_-)$, and hence $\#\mathfrak{S}_3(P_-^*)\leqq\#B_0+1\leqq\#\mathfrak{S}_3(P_-)$ $(P_-^*=P_-(\psi_t^*, \Sigma^*))$. It follows from Lemma 6.3 and the procedure for getting (ψ_t^*, Σ^*) that $\kappa_j^*=\kappa_j+1$ or κ_j $(j=0, 1,\ \kappa_j^*=\kappa_j(\psi_t^*, \Sigma^*)$ and $\kappa_j=\kappa_j(\psi_t, \Sigma))$, and that $\kappa_0^*=\kappa_0+1$ if $\#\mathfrak{S}_3(P_-^*)=\#\mathfrak{S}_3(P_-)$. Therefore we have $\kappa(\psi_t^*, \Sigma^*)>\kappa(\psi_t, \Sigma)$ except the case where $\kappa_0^*=\kappa_0$ and $\#\mathfrak{S}_3(P_-^*)=\#\mathfrak{S}_3(P_-)-1$. This case can occur only when $\#(D_\beta\cap P_+\cap P_-)=1$ and $\hat{T}_+(x_\beta)$ is contained in $C_{k(1)}$ or $C_{k(4)}$ for some simple third singularity a_k of P_-. And in this case we can see that $\kappa_1^*=\kappa_1$, and hence $\kappa(\psi_t^*, \Sigma^*)<\kappa(\psi_t, \Sigma)$ also in this case.

Next we shall consider the case where $D_\beta\cap P_+\cap P_-=\emptyset$ and $\hat{T}_+(x_\beta)\notin C_{k(1)}\cup C_{k(4)}$ for any simple third singularity a_k of P_-. Let y be the end point of C_β which is not included in $\hat{T}_+(D_\beta)$. If $y\in\hat{T}_+(\mathfrak{S}_3(P_-))$, then using the condition that $\hat{T}_+(x_\beta)\notin C_{k(1)}$ for any simple a_k, we can see that $\#\mathfrak{S}_3(P_-^*)=\#\mathfrak{S}_3(P_-)+1$, $\kappa_0^*=\kappa_0+1$ and $\kappa_1^*=\kappa_1$ or κ_1+1. And in the case where y is contained in $\hat{T}_+^2(\mathfrak{S}_3(P_-))$, by the condition $\hat{T}_+(x_\beta)\notin C_{k(4)}$ for simple a_k, we have that $\#\mathfrak{S}_3(P_-^*)\leqq\#\mathfrak{S}_3(P_-)$, $\kappa_0^*=\kappa_0+1$ or κ_0 and $\kappa_1^*=\kappa_1$ or κ_1+1, and moreover that $\#\mathfrak{S}_3(P_-^*)<\#\mathfrak{S}_3(P_-)$ if $\kappa_0^*=\kappa_0$. Hence in any cases, we get $\kappa(\psi_t^*, \Sigma^*)<\kappa(\psi_t, \Sigma)$ or $\kappa_0^*-2\kappa_1^*<\kappa_0-2\kappa_1$. This completes the proof.

PROOF OF THEOREM 6.2. According to Theorem 6.1, it is sufficient for the proof of Theorem 6.2 to show that $\tilde{x}_\beta\cap D_\beta=\emptyset$ for any nice closed curve β if $H_1(M; \boldsymbol{Z})$ is trivial.

Assume that $\tilde{x}_\beta\cap D_\beta\neq\emptyset$, and define $F\colon \beta\to\boldsymbol{R}$ by

$$F(x)=\inf\{t>0\mid \psi_t(x)\in D_\beta\}\ .$$

According to the conditions (ii), (iv) and (v) of Definition 6.1, F is continuous on β and $F(x)<T_+(x)$ for any $x\in\beta$. Hence the 2-dimensional polyhedron $D_\beta\cup\tilde{\beta}$ defines a 2-cycle, where $\tilde{\beta}=\{\psi_t(x)\mid x\in\beta,\ 0\leqq t\leqq F(x)\}$. Therefore if $H_1(M; \boldsymbol{Z})$ is trivial, then $D_\beta\cup\tilde{\beta}$ divides M into two domains V_1 and V_2. Let x_0 be a point D_β which is not contained in the domain bounded by $\{\psi_t(x)\mid x\in\beta,\ t=F(x)\}$. We can choose x_0 so that the orbit through x_0 does not intersect with β. Without loss of generality, we assume that $\psi_\delta(x_0)\in V_1$ for small $\delta>0$ and $\psi_{-\delta}(x_0)\in V_2$. Because $(D_\beta\cup\tilde{\beta})\cap\Sigma=\emptyset$, Σ is completely included in either of these two domains.

Let $\Sigma\subset V_1$. Then there must exist a t_0 $(T_-(x_0)<t_0<0)$ such that $\psi_t(x_0)\in V_2$ for $t_0<t<0$ and $\psi_{t_0}(x_0)\in D_\beta$. However this is obviously impossible. Also in the case of $\Sigma\subset V_2$, we have a contradiction that $\psi_t(x_0)\in D_\beta-U$ for some $0<t<T_+(x_0)$ where $U\subset D_\beta$ is the domain bounded by $\{\psi_t(x)\mid x\in\beta,\ t=F(x)\}$. This completes the proof.

REMARK 1. The assumption of Theorem 6.2 seems to be somewhat weakened, that is, we can show the following (a) and (b).

(a) If $H_1(M; \boldsymbol{Z})=\{0\}$ and β is a simple closed curve satisfying that
(1) β satisfies (i)–(iv) in Definition 6.1,
(2) $\mathrm{LK}(\beta, \psi_\delta(\beta))=0$ for sufficiently small $\delta>0$ where $\mathrm{LK}(\cdot,\cdot)$ is the linking number,
(3) there is an "immersed" 2-disk $D'_\beta\subset M-\Sigma$ such that $\partial D'_\beta=\beta$ and D'_β is nowhere tangential to ψ_t,
then we can take an embedded 2-disk $D''_\beta\subset M-\Sigma$ with $\partial D''_\beta=\beta$.

(b) Any simple closed curve satisfying (i)–(iv) in Definition 6.1 has the above property (2) if $H_1(M; \boldsymbol{Z})$ is trivial.

However, in (a), it is not yet known whether we can take D''_β so that it is transversal to ψ_t.

REMARK 2. Recently Ikeda and Inoue ([3], [4]) introduced the concept of DS-diagrams and DS-diagrams with E-cycle. As is pointed out in [4], a flow-spine defines a DS-diagram with E-cycle. The converse can be also proved, namely, we can construct a normal pair which generates a given DS-diagram with E-cycle. Especilly we can say that if a singularity-data is realizable in the sense of [2], then it is really generated by some normal pair. This fact will be discussed in the forthcoming paper.

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