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A Note on the Transcendental Continued Fractions

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Introduction

Let $a_1, a_2, a_3, \cdots; b_1, b_2, b_3, \cdots$ be all integers and $a_1 \geqq 0$, $a_2 > 0$, $a_3 > 0, \cdots$; $b_1 \geqq 0$, $b_2 > 0$, $b_3 > 0$, $\cdots$ all along this note.

In [2], G. Nettler proved the following theorem.

NETTLER'S THEOREM. *For*

$$A = a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\cdots \quad \text{and} \quad B = b_1 + \frac{1}{b_2 +}\,\frac{1}{b_3 +}\cdots,$$

if $a_n > b_n > a_{n-1}^{(n-1)^2}$ *for all* n *sufficiently large, then* A, B, $A \pm B$, A/B *and* AB *are all transcendental numbers.*

The aim of this note is to prove the following theorem that is an improvement of Nettler's Theorem.

THEOREM. *For*

$$A = a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\cdots \quad \text{and} \quad B = b_1 + \frac{1}{b_2 +}\,\frac{1}{b_3 +}\cdots,$$

if $a_n > b_n > a_{n-1}^{\gamma(n-1)}$ *for all* n *sufficiently large, then,* A, B, $A \pm B$, A/B *and* AB *are all transcendental numbers, where* γ *is any constant such that* $\gamma > 16$.

We give an elementary proof of this theorem using the method of G. Nettler.

§1. Lemmas.

LEMMA 1. *If*

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$$(1)\qquad A(n)=a_1+\frac{1}{a_2}\,\frac{1}{+\,a_3}+\cdots+\frac{1}{a_n}=\frac{{}^aP_n}{{}^aQ_n}\,,$$

$$(2)\qquad B(n)=b_1+\frac{1}{b_2}\,\frac{1}{+\,b_3}+\cdots+\frac{1}{b_n}=\frac{{}^bP_n}{{}^bQ_n}\,,$$

then, for all $n\geqq 1$, we have

$$(3)\qquad A(n)+B(n)=a_1+b_1+\frac{a_2+b_2}{a_2b_2}\,\frac{a_2b_2F_3}{+\,E_3-F_3}\,\frac{E_3F_4}{+\,E_4-F_4}\,\frac{E_4F_5}{+\,E_5-F_5}+\cdots+\frac{E_{n-1}F_n}{E_n-F_n}\,,$$

$$E_n={}^aQ_n{}^bQ_n({}^aQ_{n-2}{}^aQ_{n-1}+{}^bQ_{n-2}{}^bQ_{n-1})\,,$$

$$F_n={}^aQ_{n-2}{}^bQ_{n-2}({}^aQ_{n-1}{}^aQ_n+{}^bQ_{n-1}{}^bQ_n)\,,$$

$$(4)\qquad B(n)-A(n)=b_1-a_1+\frac{a_2-b_2}{a_2b_2}\,\frac{a_2b_2H_3}{+\,G_3-H_3}\,\frac{G_3H_4}{+\,G_4-H_4}\,\frac{G_4H_5}{+\,G_5-H_5}+\cdots+\frac{G_{n-1}H_n}{G_n-H_n}\,,$$

$$G_n={}^aQ_n{}^bQ_n({}^aQ_{n-2}{}^aQ_{n-1}-{}^bQ_{n-2}{}^bQ_{n-1})\,,$$

$$H_n={}^aQ_{n-2}{}^bQ_{n-2}({}^aQ_{n-1}{}^aQ_n-{}^bQ_{n-1}{}^bQ_n)\,,$$

$$(5)\qquad \frac{A(n)}{B(n)}=\frac{a_1}{b_1}+\frac{b_1b_2-a_1a_2}{a_2b_1(b_1b_2+1)}\,\frac{a_2b_1(b_1b_2+1)J_3}{+\,I_3-J_3}\,\frac{I_3J_4}{+\,I_4-J_4}\,\frac{I_4J_5}{+\,I_5-J_5}+\cdots+\frac{I_{n-1}J_n}{I_n-J_n}\,,$$

$$I_n={}^aQ_n{}^bP_n({}^aQ_{n-2}{}^aP_{n-1}-{}^bQ_{n-2}{}^bP_{n-1})\,,$$

$$J_n={}^aQ_{n-2}{}^bP_{n-2}({}^aQ_n{}^aP_{n-1}-{}^bQ_n{}^bP_{n-1})\,,$$

$$(6)\qquad A(n)B(n)=a_1b_1+\frac{a_1a_2+b_1b_2+1}{a_2b_2}\,\frac{a_2b_2L_3}{+\,K_3-L_3}\,\frac{K_3L_4}{+\,K_4-L_4}\,\frac{K_4L_5}{+\,K_5-L_5}+\cdots+\frac{K_{n-1}L_n}{K_n-L_n}\,,$$

$$K_n={}^aQ_n{}^bQ_n({}^aQ_{n-2}{}^aP_{n-1}+{}^bQ_{n-1}{}^bP_{n-2})\,,$$

$$L_n={}^aQ_{n-2}{}^bQ_{n-2}({}^aQ_n{}^aP_{n-1}+{}^bQ_{n-1}{}^bP_n)\,.$$

PROOF. See Theorem 2.1 of [2].

LEMMA 2. *Let*

$$C=e_1+\frac{d_2}{e_2}+\frac{d_3}{e_3}+\cdots$$

be the continued fraction expansion, given in Lemma 1, *for either* $A\pm B$, A/B *or* AB. *And let*

$$\frac{{}^cP_n}{{}^cQ_n}=e_1+\frac{d_2}{e_2}+\frac{d_3}{e_3}+\cdots+\frac{d_n}{e_n}\,.$$

If $a_n>b_n$ *for sufficiently large* n, *then* ${}^cQ_n<{}^aQ_n^{\alpha n}$, *where* α *is any constant such that* $\alpha>8$.

PROOF. From (3), (4), (5) and (6), we have the following inequalities

$${}^cQ_n=e_n{}^cQ_{n-1}+d_n{}^cQ_{n-2}<{}^cQ_{n-1}(d_n+e_n)<\cdots<\prod_{i=2}^{n}(d_i+e_i)<\prod_{i=2}^{n}{}^aQ_i^{\alpha}<{}^aQ_n^{\alpha n}$$

for all n sufficiently large.

LEMMA 3. *For the continued fraction*

$$A=a_1+\frac{1}{a_2}+\frac{1}{a_3}+\cdots\,,$$

if $a_n>a_{n-1}^{\beta(n-1)}$ *for all* n *sufficiently large, then* ${}^aQ_n<a_n^{\beta}$, *where* β *is any constant such that* $\beta>1$.

PROOF. There is an integer N such that $a_n>a_{n-1}^{\beta(n-1)}$ for all $n\geqq N$. We have the following inequalities

$${}^aQ_n<(a_n+1){}^aQ_{n-1}<\cdots<\prod_{i=N}^{n}(a_i+1){}^aQ_{N-1}={}^aQ_{N-1}\prod_{i=N}^{n}\left(1+\frac{1}{a_i}\right)\prod_{i=N}^{n}a_i\,.$$

There is a positive number M such that

$${}^aQ_{N-1}\prod_{i=N}^{n}\left(1+\frac{1}{a_i}\right)<M\,.$$

Therefore

$$\begin{aligned}{}^aQ_n&<M\prod_{i=N}^{n}a_i<Ma_n^{1+1/(\beta(n-1))+1/(\beta^2(n-1)(n-2))+\cdots+1/(\beta^{n-N}(n-1)(n-2)\cdots N)}\\&<Ma_n^{1+1/(n-1)\cdot\beta/(\beta-1)}<a_n^{\beta}\end{aligned}$$

for all n sufficiently large.

§ 2. Proof of the theorem.

From Lemma 3, we have

$$a_{n+1} > a_n^{\gamma n} > {}^aQ_n^n ,$$

and

$$\left| A - \frac{{}^aP_n}{{}^aQ_n} \right| < \frac{1}{{}^aQ_n {}^aQ_{n+1}} < \frac{1}{a_{n+1} {}^aQ_n^2} < \frac{1}{{}^aQ_n^{n+2}} ,$$

therefore A is a transcendental number. In a similar fashion it can be easily proven that B is also a transcendental number.

Now let

$$A + B = C = e_1 + \frac{d_2}{e_2 +} \frac{d_3}{e_3 +} \cdots$$

and

$$\frac{{}^aP_n}{{}^aQ_n} + \frac{{}^bP_n}{{}^bQ_n} = \frac{{}^cP_n}{{}^cQ_n} \qquad \text{for} \quad n \geqq 1 .$$

Then, for n sufficiently large, we have

$$\left| C - \frac{{}^cP_n}{{}^cQ_n} \right| \leqq \left| A - \frac{{}^aP_n}{{}^aQ_n} \right| + \left| B - \frac{{}^bP_n}{{}^bQ_n} \right|$$

$$< \frac{1}{{}^aQ_n {}^aQ_{n+1}} + \frac{1}{{}^bQ_n {}^bQ_{n+1}} < \frac{2}{{}^bQ_n {}^bQ_{n+1}} < \frac{1}{b_{n+1}} < \frac{1}{a_n^{\gamma n}} .$$

Now we choose β so that

$$\beta > 1 \quad \text{and} \quad \frac{\gamma}{\beta} > 16 .$$

From Lemma 3, we have

$$a_n^{\gamma n} = a_n^{\beta \cdot (\gamma/\beta) n} > {}^aQ_n^{(\gamma/\beta) n} ,$$

therefore

$$\left| C - \frac{{}^cP_n}{{}^cQ_n} \right| < \frac{1}{{}^aQ_n^{(\gamma/\beta) n}} .$$

Now we choose α so that

$$8<\alpha<\frac{\gamma}{2\beta}\,.$$

From Lemma 2, we have the following inequalities

$$\left|C-\frac{{}^cP_n}{{}^cQ_n}\right|<\frac{1}{({}^aQ_n^{\alpha n})^{\gamma/(\alpha\beta)}}<\frac{1}{{}^cQ_n^{\gamma/(\alpha\beta)}}\,.$$

Therefore $A+B$ is transcendental by Roth's theorem. Similarly, it can easily be proven that $A-B$ is also a transcendental number.

To prove that A/B is transcendental, let

$$\frac{A}{B}=C=e_1+\frac{d_2}{e_2\,+}\,\frac{d_3}{e_3\,+\cdots}\quad\text{and}\quad\frac{{}^aP_n}{{}^aQ_n}\Big/\frac{{}^bP_n}{{}^bQ_n}=\frac{{}^cP_n}{{}^cQ_n}$$

for $n\geqq 1$. As

$$\frac{{}^aP_n}{{}^aQ_n}\leqq 2a_1\quad\text{and}\quad\frac{1}{2b_2}\leqq\frac{{}^bP_n}{{}^bQ_n}$$

for $n\geqq 3$, we obtain

$$\begin{aligned}\left|C-\frac{{}^cP_n}{{}^cQ_n}\right|&=\left|\frac{A}{B}-\frac{{}^aP_n/{}^aQ_n}{{}^bP_n/{}^bQ_n}\right|\\&=\frac{\left|\frac{{}^bP_n}{{}^bQ_n}\left(A-\frac{{}^aP_n}{{}^aQ_n}\right)+\frac{{}^aP_n}{{}^aQ_n}\left(\frac{{}^bP_n}{{}^bQ_n}-B\right)\right|}{B({}^bP_n/{}^bQ_n)}\\&\leqq\frac{\left|A-\frac{{}^aP_n}{{}^aQ_n}\right|}{B}+\frac{2a_1\left|B-\frac{{}^bP_n}{{}^bQ_n}\right|}{B/2b_2}<\frac{4a_1b_2+1}{B}\cdot\frac{1}{{}^bQ_n{}^bQ_{n+1}}<\frac{1}{b_{n+1}}<\frac{1}{{}^cQ_n^{\gamma/(\alpha\beta)}}\end{aligned}$$

for all n sufficiently large which proves that A/B is a transcendental number.

Finally, to prove that AB is transcendental, let

$$AB=C=e_1+\frac{d_2}{e_2\,+}\,\frac{d_3}{e_3\,+\cdots}\quad\text{and}\quad\frac{{}^aP_n}{{}^aQ_n}\cdot\frac{{}^bP_n}{{}^bQ_n}=\frac{{}^cP_n}{{}^cQ_n}$$

for $n\geqq 1$. We obtain

$$\begin{aligned}\left|C-\frac{{}^cP_n}{{}^cQ_n}\right|&=\left|AB-\frac{{}^aP_n}{{}^aQ_n}\cdot\frac{{}^bP_n}{{}^bQ_n}\right|\leqq\left|A-\frac{{}^aP_n}{{}^aQ_n}\right|B+\frac{{}^aP_n}{{}^aQ_n}\left|B-\frac{{}^bP_n}{{}^bQ_n}\right|\\&\leqq\frac{B}{{}^aQ_n{}^aQ_{n+1}}+\frac{{}^aP_n}{{}^aQ_n}\cdot\frac{1}{{}^bQ_n{}^bQ_{n+1}}\leqq(B+2a_1)\frac{1}{{}^bQ_n{}^bQ_{n+1}}<\frac{1}{b_{n+1}}\end{aligned}$$

for all n sufficiently large. Therefore AB is a transcendental number.

§ 3. Example.

If

$$A=2^{2!}+\frac{1}{2^{4!}}+\frac{1}{2^{6!}}+\cdots+\frac{1}{2^{(2n)!}}+\cdots$$

and

$$B=2^{9\cdot 1!}+\frac{1}{2^{9\cdot 3!}}+\frac{1}{2^{9\cdot 5!}}+\cdots+\frac{1}{2^{9(2n-1)!}}+\cdots,$$

then A, B, $A\pm B$, A/B and AB are all transcendental numbers.

PROOF. Now we put $a_n=2^{(2n)!}$, $b_n=2^{9(2n-1)!}$. First we can see easily that $a_n>b_n$ for $n\geqq 5$. And we have

$$\frac{\log b_n}{(n-1)\log a_{n-1}}=\frac{9(2n-1)!\log 2}{(n-1)(2n-2)!\log 2}=\frac{9(2n-1)}{n-1}>18$$

for $n\geqq 2$. Therefore $b_n>a_{n-1}^{18(n-1)}$ for $n\geqq 2$. From the theorem, A, B, $A\pm B$, A/B and AB are all transcendental numbers.

References

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