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On the Exponentially Bounded C-semigroups

Naoki TANAKA

Waseda University

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Introduction

In this paper we are concerned with exponentially bounded C-semigroups introduced by Davies and Pang [1].

Let X be a Banach space and let $C: X \to X$ be an injective bounded linear operator with dense range. A family $\{S(t): 0 \leqq t < \infty\}$ of bounded linear operators from X into itself is called an *exponentially bounded C-semigroup* if

(0.1) $\quad S(t+s)C = S(t)S(s)$ for $t, s \geqq 0$, and $S(0) = C$,

(0.2) $\quad$ for every $x \in X$, $S(t)x$ is continuous in $t \geqq 0$,

(0.3) $\quad$ there exist $M \geqq 0$ and $a \geqq 0$ such that $\|S(t)\| \leqq Me^{at}$ for $t \geqq 0$.

For every $t \geqq 0$, let $T(t)$ be the closed linear operator defined by $T(t)x = C^{-1}S(t)x$ for $x \in D(T(t)) \equiv \{x \in X: S(t)x \in R(C)\}$. We define the operator G by

$$
\begin{aligned}
&D(G) = \{x \in R(C): \lim_{t \to 0+}(T(t)x - x)/t \text{ exists}\} \quad \text{and} \\
&Gx = \lim_{t \to 0+}(T(t)x - x)/t \qquad \text{for} \quad x \in D(G) .
\end{aligned} \tag{0.4}
$$

For every $\lambda > a$, define the bounded linear operator $L_\lambda: X \to X$ by $L_\lambda x = \int_0^\infty e^{-\lambda t}S(t)xdt$ for $x \in X$. It is known that G is closable with dense domain (see [1]) and by [2, (2.3)]

$$
\begin{aligned}
&(\lambda - \bar{G})L_\lambda x = Cx \qquad \text{for} \quad x \in X \quad \text{and} \quad \lambda > a , \\
&L_\lambda(\lambda - \bar{G})x = Cx \qquad \text{for} \quad x \in D(\bar{G}) \quad \text{and} \quad \lambda > a ,
\end{aligned} \tag{0.5}
$$

where $\bar{G}$ denotes the closure of G. $\bar{G}$ is called the *C-c.i.g.* (*C-complete infinitesimal generator*) of $\{S(t): t \geqq 0\}$.

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§1 is devoted to the representation of exponentially bounded C-semigroups. §2 treats the generation of exponentially bounded C-semigroups, and §3 deals with the abstract Cauchy problem. Finally, §4 concerns the connections with semigroups of growth order $\alpha>0$.

§1. Representation of exponentially bounded C-semigroups.

We start with the following

LEMMA 1.1. *Let T be a closed linear operator satisfying the following conditions*

(1.1) $D(T^n)\supset R(C)$ *for* $n\geqq 1$,

(1.2) *there are* $M>0$ *and* $\alpha\geqq 1$ *such that* $\|T^kC\|\leqq M\alpha^k$ *for* $k\geqq 1$,

(1.3) $TCx=CTx$ *for* $x\in R(C)$.

Then for $x\in X$ *we have*

(i)
$$\left\|e^{-m}\sum_{k=0}^{\infty}\frac{m^k}{k!}T^kC^2x-T^mC^2x\right\|$$
$$\leqq M\alpha^m e^{m(\alpha-1)}\{m^2(\alpha-1)^2+m(\alpha-1)+m\}^{1/2}\|(T-I)Cx\|$$

for $m\geqq 1$ *and*

(ii)
$$\left\|e^{-t/h}\sum_{k=0}^{\infty}\frac{(t/h)^k}{k!}T^kC^2x-T^{[t/h]}C^2x\right\|$$
$$\leqq Me^{t(\alpha-1)/h}\left(\alpha^{t/h}\left\{ht^2\left(\frac{\alpha-1}{h}\right)^2+ht\left(\frac{\alpha-1}{h}\right)+t\right\}^{1/2}+\sqrt{h}\right)\sqrt{h}\,\|A^hCx\|$$

for $t\geqq 0$ *and* $h>0$, *where* $A^h=h^{-1}(T-I)$ *for* $h>0$ *and* [ ] *denotes the Gaussian bracket.*

PROOF. We first note that by (1.2), $\sum_{k=0}^{\infty}(m^k/k!)T^kC$ converges in norm and defines a bounded linear operator.

Let $x\in X$. We have

$$e^{-m}\sum_{k=0}^{\infty}\frac{m^k}{k!}T^kC^2x-T^mC^2x=e^{-m}\sum_{k=0}^{\infty}\frac{m^k(T^kC^2x-T^mC^2x)}{k!}\,.$$

But, by (1.3) for $k>m$,

$$T^kC^2x-T^mC^2x=\sum_{i=m}^{k-1}T^i(TC^2x-C^2x)$$

$$=\sum_{i=m}^{k-1} T^i C(T-I)Cx$$

and hence by (1.2), we have

$$\|T^k C^2 x - T^m C^2 x\| \leqq M\Big(\sum_{i=m}^{k-1} \alpha^i\Big)\|(T-I)Cx\|$$
$$\leqq M\alpha^k (k-m)\|(T-I)Cx\| .$$

On the other hand, for $k<m$,

$$\|T^k C^2 x - T^m C^2 x\| \leqq M\alpha^m (m-k)\|(T-I)Cx\| .$$

Therefore, we obtain that

$$\left\|e^{-m}\sum_{k=0}^{\infty}\frac{m^k}{k!}T^k C^2 x - T^m C^2 x\right\| \leqq M\alpha^m e^{-m}\sum_{k=0}^{\infty}\frac{|k-m|\,\alpha^k m^k}{k!}\|(T-I)Cx\| .$$

By the Schwarz inequality

$$\sum_{k=0}^{\infty}\frac{|k-m|\,\alpha^k m^k}{k!} \leqq e^{m\alpha}\{m^2(\alpha-1)^2+m(\alpha-1)+m\}^{1/2} .$$

So that (i) is proved.

Using (i) with $m=[t/h]$ $(\leqq t/h)$, we have

$$\left\|e^{-[t/h]}\sum_{k=0}^{\infty}\frac{[t/h]^k}{k!}T^k C^2 x - T^{[t/h]}C^2 x\right\| \tag{1.4}$$
$$\leqq M\alpha^{t/h}e^{t(\alpha-1)/h}\left\{t^2\Big(\frac{\alpha-1}{h}\Big)^2+t\Big(\frac{\alpha-1}{h}\Big)+t/h\right\}^{1/2}\|(T-I)Cx\|$$
$$=M\alpha^{t/h}e^{t(\alpha-1)/h}\left\{ht^2\Big(\frac{\alpha-1}{h}\Big)^2+ht\Big(\frac{\alpha-1}{h}\Big)+t\right\}^{1/2}\sqrt{h}\,\|A^h Cx\| .$$

We define the bounded linear operator $S_h(s)$ for every $s\geqq 0$ by

$$S_h(s)=e^{-s/h}\sum_{k=0}^{\infty}\frac{(s/h)^k}{k!}T^k C .$$

It is easy to see that

$$\frac{d}{ds}S_h(s)Cx=S_h(s)A^h Cx .$$

Integrating this from $s=[t/h]h$ to $s=t$, we obtain

$$S_h(t)Cx-S_h([t/h]h)Cx=\int_{[t/h]h}^{t} S_h(s)A^h Cx\,ds .$$

Since $\|S_h(s)\| \leqq e^{-s/h} \sum_{k=0}^{\infty} ((s/h)^k/k!)M\alpha^k = Me^{s(\alpha-1)/h}$, we have

$$\left\| S_h(t)Cx - e^{-[t/h]} \sum_{k=0}^{\infty} \frac{[t/h]^k}{k!} T^k C^2 x \right\| \leqq Mhe^{t(\alpha-1)/h} \|A^h Cx\| .$$

Combining this with (1.4), the desired inequality is proved. Q.E.D.

REMARK. If $C=I$ in Lemma 1.1, the estimates (i) and (ii) are known. (See [3] or [5].)

THEOREM 1.2. *Let $\{S(t): t \geqq 0\}$ be an exponentially bounded C-semigroup. If $\bar{G}$ is the C-c.i.g. of $\{S(t): t \geqq 0\}$, then*

$$S(t)x = \lim_{\lambda \to \infty} S_\lambda(t)x \quad \textit{for} \quad x \in X , \tag{1.5}$$

where $S_\lambda(t)x = e^{-\lambda t} \sum_{n=0}^{\infty} (t^n \lambda^{2n}/n!)(\lambda - \bar{G})^{-n} Cx$ for $x \in X$ and $t \geqq 0$, and the limit is uniform in t on any bounded interval.

PROOF. By (0.3), $\|S(t)\| \leqq Me^{at}$ for $t \geqq 0$ and by [2, Theorem 1], $\|(\lambda - \bar{G})^{-n} C\| \leqq M/(\lambda - a)^n$ so that

$$\|S_\lambda(t)\| \leqq e^{-\lambda t} M e^{t\lambda^2/(\lambda - a)} = Me^{\lambda a t/(\lambda - a)} \leqq Me^{2at} \quad \text{for} \quad \lambda > 2a .$$

It is easily seen that

$$\frac{d}{ds} S_\lambda(s)Cx = S_\lambda(s)\lambda\bar{G}(\lambda - \bar{G})^{-1}Cx \quad \text{for} \quad x \in X ,$$

and by [1, Lemma 8]

$$\frac{d}{ds} S(s)x = S(s)\bar{G}x \quad \text{for} \quad x \in CD(\bar{G}) .$$

By (0.5), $S(s)(\lambda - \bar{G})^{-1}Cx = S(s)L_\lambda x = L_\lambda S(s)x = (\lambda - \bar{G})^{-1}CS(s)x$ for $x \in X$, i.e., $\bar{G}(\lambda - \bar{G})^{-1}C$ $(= \lambda(\lambda - \bar{G})^{-1}C - C)$ commutes with $S(s)$. Now, let $x \in CD(\bar{G})$ and $x = Cy$, $y \in D(\bar{G})$. Then

$$\begin{aligned} \frac{d}{ds}(S_\lambda(t-s)S(s)x) &= -S_\lambda(t-s)\lambda\bar{G}(\lambda - \bar{G})^{-1}CS(s)y \\ &\quad + S_\lambda(t-s)S(s)\bar{G}x \\ &= S_\lambda(t-s)S(s)(\bar{G}x - \lambda\bar{G}(\lambda - \bar{G})^{-1}x) . \end{aligned}$$

Integrating this from $s=0$ to $s=t$, it follows that

$$S(t)Cx - S_\lambda(t)Cx = \int_0^t S_\lambda(t-s)S(s)(\bar{G}x - \lambda\bar{G}(\lambda - \bar{G})^{-1}x)ds .$$

Therefore

$$\|S(t)Cx-S_\lambda(t)Cx\|\leqq\int_0^t\|S_\lambda(t-s)\|\,\|S(s)\|\,ds\,\|\bar{G}x-\lambda\bar{G}(\lambda-\bar{G})^{-1}x\|$$
$$\leqq M^2e^{3at}t\,\|\bar{G}x-\lambda\bar{G}(\lambda-\bar{G})^{-1}x\|$$

and hence we have that for $\lambda>2a$, $T>0$, and $x\in CD(\bar{G})$

$$\sup_{0\leqq t\leqq T}\|S(t)Cx-S_\lambda(t)Cx\|\leqq M^2e^{3aT}T\|\bar{G}x-\lambda\bar{G}(\lambda-\bar{G})^{-1}x\|\ .$$

By [1, Theorem 11 (b)],

$$\lim_{\lambda\to\infty}\lambda\bar{G}(\lambda-\bar{G})^{-1}Cx=\bar{G}Cx\qquad\text{for}\quad x\in D(\bar{G})\ . \tag{1.6}$$

So that

$$\lim_{\lambda\to\infty}\{\sup_{0\leqq t\leqq T}\|S(t)x-S_\lambda(t)x\|\}=0\qquad\text{for}\quad x\in C^2D(\bar{G})\ . \tag{1.7}$$

Since $C^2D(\bar{G})$ is dense in X, $\|S(t)\|\leqq Me^{aT}$ and $\|S_\lambda(t)\|\leqq Me^{2aT}$ for $\lambda>2a$ and $0\leqq t\leqq T$, (1.7) holds for every $x\in X$. Q.E.D.

THEOREM 1.3. *Let $\{S(t): t\geqq 0\}$ be an exponentially bounded C-semigroup. If $\bar{G}$ is the C-c.i.g. of $\{S(t): t\geqq 0\}$, then*

$$S(t)x=\lim_{n\to\infty}\Big(1-\frac{t}{n}\bar{G}\Big)^{-n}Cx\qquad\textit{for}\quad x\in X \tag{1.8}$$

and the limit is uniform in t on any bounded interval.

PROOF. By virtue of [2, Theorem 1], $D((\lambda-\bar{G})^{-n})\supset R(C)$, $\|(\lambda-\bar{G})^{-n}C\|\leqq M(\lambda-a)^{-n}$ for $\lambda>a$ and $n\geqq 1$, and $(\lambda-\bar{G})^{-1}Cx=C(\lambda-\bar{G})^{-1}x$ for $x\in D((\lambda-\bar{G})^{-1})$. Using (ii) in Lemma 1.1 with $T=\lambda(\lambda-\bar{G})^{-1}$, $\alpha=\lambda/(\lambda-a)$ and $h=\lambda^{-1}$, we have

$$\Big\|e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}\lambda^k(\lambda-\bar{G})^{-k}C^2x-(\lambda(\lambda-\bar{G})^{-1})^{[\lambda t]}C^2x\Big\|$$
$$\leqq M\exp\Big(\frac{at}{1-a/\lambda}\Big)\Big((1-a/\lambda)^{-\lambda t}\Big\{\frac{t^2}{\lambda}\Big(\frac{a}{1-a/\lambda}\Big)^2+\frac{t}{\lambda}\Big(\frac{a}{1-a/\lambda}\Big)$$
$$+t\Big\}^{1/2}+1/\sqrt{\lambda}\Big)(1/\sqrt{\lambda})\,\|\lambda\bar{G}(\lambda-\bar{G})^{-1}Cx\|\qquad\text{for}\quad x\in X\ .$$

Here we used $A^hC=\lambda(\lambda(\lambda-\bar{G})^{-1}C-C)=\lambda\bar{G}(\lambda-\bar{G})^{-1}C$. Noting that $(1-a/\lambda)^{-\lambda t}\leqq(1-a/\lambda)^{-\lambda T}\to e^{aT}$ as $\lambda\to\infty$ for $0\leqq t\leqq T$, and by (1.6), $\|\lambda\bar{G}(\lambda-\bar{G})^{-1}Cx\|\to\|\bar{G}Cx\|$ as $\lambda\to\infty$ for $x\in D(\bar{G})$, we have that for $x\in D(\bar{G})$

$$\lim_{\lambda\to\infty}\|S_\lambda(t)Cx-(\lambda(\lambda-\bar{G})^{-1})^{[\lambda t]}C^2x\|=0$$

uniformly in t on any bounded interval. Combining this with (1.5), it follows that

$$S(t)x=\lim_{\lambda\to\infty}(\lambda(\lambda-\bar{G})^{-1})^{[\lambda t]}Cx \quad \text{for} \quad x\in CD(\bar{G})\,. \tag{1.9}$$

Since $CD(\bar{G})$ is dense in X and $\|(\lambda(\lambda-\bar{G})^{-1})^{[\lambda t]}C\|\leqq M(\lambda/(\lambda-a))^{\lambda t}\leqq M(1+(2a/\lambda))^{\lambda t}\leqq Me^{2at}$ for $\lambda>2a$, (1.9) holds for $x\in X$. Setting $\lambda=n/t$ in (1.9), we obtain (1.8). Q.E.D.

REMARK. Let $x\in R(C)$ and $x=Cy$. By (1.5)

$$\begin{aligned} S(t)x&=\lim_{\lambda\to\infty}e^{-\lambda t}\sum_{n=0}^{\infty}\frac{t^n\lambda^{2n}}{n!}(\lambda-\bar{G})^{-n}C\cdot Cy \\ &=C\lim_{\lambda\to\infty}e^{-\lambda t}\sum_{n=0}^{\infty}\frac{t^n\lambda^{2n}}{n!}(\lambda-\bar{G})^{-n}Cy\,. \end{aligned}$$

Therefore we have

$$T(t)x=\lim_{\lambda\to\infty}e^{-\lambda t}\sum_{n=0}^{\infty}\frac{t^n\lambda^{2n}}{n!}(\lambda-\bar{G})^{-n}x \quad \text{for} \quad x\in R(C)\,. \tag{1.10}$$

By the same argument and (1.8), we obtain

$$T(t)x=\lim_{n\to\infty}\left(1-\frac{t}{n}\bar{G}\right)^{-n}x \quad \text{for} \quad x\in R(C)\,. \tag{1.11}$$

§2. Generation of exponentially bounded C-semigroups.

The following is a generalization of generation theorem of (C_0)-semigroups.

THEOREM 2.1. *A closed linear operator T is the C-c.i.g. of an exponentially bounded C-semigroup $\{S(t): t\geqq 0\}$ with $\|S(t)\|\leqq Me^{at}$ if and only if T satisfies the following conditions*

(A1) *$D(T)$ is dense in X,*
(A2) *$\lambda-T$ is injective for $\lambda>a$,*
(A3) *$D((\lambda-T)^{-n})\supset R(C)$ for $n\geqq 1$ and $\lambda>a$,*
(A4) *$\|(\lambda-T)^{-n}C\|\leqq M/(\lambda-a)^n$ for $n\geqq 1$ and $\lambda>a$,*
(A5) *$(\lambda-T)^{-1}Cx=C(\lambda-T)^{-1}x$ for $x\in D((\lambda-T)^{-1})$ and $\lambda>a$,*
(A6) *$CD(T)$ is a core of T.*

PROOF. By virtue of [2, Theorem 1], $\bar{G}$ satisfies (A1)–(A5). To prove that $\bar{G}$ satisfies (A6), i.e., $CD(\bar{G})$ is a core of $\bar{G}$, we first note that

$$\int_0^t S(\tau)z\,d\tau \in D(\bar{G}) \quad \text{and}$$
$$\bar{G}\int_0^t S(\tau)z\,d\tau = S(t)z - Cz \qquad \text{for} \quad z\in X \text{ and } t\geqq 0\,. \tag{2.1}$$

In fact, let $z\in X$ and $t\geqq 0$. Since $D(G)$ is dense in X, we can choose $z_n\in D(G)$ with $\lim_{n\to\infty} z_n = z$. Noting $S(\tau)z_n\in D(G)$ and $(d/d\tau)S(\tau)z_n = GS(\tau)z_n = S(\tau)Gz_n$, we obtain

$$S(t)z_n - Cz_n = \int_0^t S(\tau)Gz_n\,d\tau$$
$$= \int_0^t GS(\tau)z_n\,d\tau = \bar{G}\int_0^t S(\tau)z_n\,d\tau\,.$$

Now, the closedness of $\bar{G}$ implies (2.1) because $\lim_{n\to\infty}\int_0^t S(\tau)z_n\,d\tau = \int_0^t S(\tau)z\,d\tau$ and $\lim_{n\to\infty}\bar{G}\int_0^t S(\tau)z_n\,d\tau = S(t)z - Cz$.

It suffices to show

$$\overline{\bar{G}|_{CD(\bar{G})}} \supset G\,. \tag{2.2}$$

To this end, let $x\in D(G)$ and $\varepsilon>0$. Using (2.1) with $z=C^{-1}x$, we have $t^{-1}\int_0^t S(\tau)C^{-1}x\,d\tau \to x$ and $\bar{G}\left(t^{-1}\int_0^t S(\tau)C^{-1}x\,d\tau\right) = t^{-1}(S(t)C^{-1}x - x) = t^{-1}(T(t)x - x) \to Gx$ as $t\to 0^+$. Hence there is a $t_0>0$ such that $\left\|t_0^{-1}\int_0^{t_0} S(\tau)C^{-1}x\,d\tau - x\right\| + \left\|\bar{G}\left(t_0^{-1}\int_0^{t_0} S(\tau)C^{-1}x\,d\tau\right) - Gx\right\| < \varepsilon/2$. Since $CD(\bar{G})$ is dense in X, we can choose $x_n\in CD(\bar{G})$ such that $x_n\to C^{-1}x$ as $n\to\infty$. By (2.1) again, $t_0^{-1}\int_0^{t_0} S(\tau)x_n\,d\tau \in CD(\bar{G})$ and

$$\bar{G}\left(t_0^{-1}\int_0^{t_0} S(\tau)x_n\,d\tau\right) = (S(t_0)x_n - Cx_n)/t_0$$
$$\to (S(t_0)C^{-1}x - C\cdot C^{-1}x)/t_0 = \bar{G}\left(t_0^{-1}\int_0^{t_0} S(\tau)C^{-1}x\,d\tau\right)$$

as $n\to\infty$. Take an n_0 such that

$$\left\|t_0^{-1}\int_0^{t_0} S(\tau)x_{n_0}\,d\tau - t_0^{-1}\int_0^{t_0} S(\tau)C^{-1}x\,d\tau\right\|$$
$$+\left\|\bar{G}\left(t_0^{-1}\int_0^{t_0} S(\tau)x_{n_0}\,d\tau\right) - \bar{G}\left(t_0^{-1}\int_0^{t_0} S(\tau)C^{-1}x\,d\tau\right)\right\| < \varepsilon/2\,.$$

Then we have that $t_0^{-1}\int_0^{t_0} S(\tau)x_{n_0}\,d\tau \in CD(\bar{G})$ and

$$\left\|t_0^{-1}\int_0^{t_0} S(\tau)x_{n_0}d\tau - x\right\| + \left\|\bar{G}\left(t_0^{-1}\int_0^{t_0} S(\tau)x_{n_0}d\tau\right) - Gx\right\| < \varepsilon \,.$$

So that (2.2) is satisfied.

Conversely, let T satisfies (A1)-(A6). By virtue of [1, Theorem 11], there exists an exponentially bounded C-semigroup $\{S(t): t\geqq 0\}$ satisfying $\|S(t)\|\leqq Me^{at}$ and

$$(2.3) \qquad (\lambda - T)^{-1}Cx = \int_0^\infty e^{-\lambda t}S(t)xdt \qquad \text{for} \quad x\in X \text{ and } \lambda > a \,.$$

By (2.3) and (A5), $L_\lambda(\lambda - T)x = Cx$ for $x\in D(T)$ and $\lambda > a$. Combining this with (0.5), we have $\lambda L_\lambda Tx = \bar{G}(\lambda L_\lambda x)$ for $x\in D(T)$ and $\lambda > a$. Since $\lambda L_\lambda x\to Cx$ and $\bar{G}(\lambda L_\lambda x) = \lambda L_\lambda Tx \to CTx$ as $\lambda\to\infty$, we obtain that $Cx\in D(\bar{G})$ and $\bar{G}Cx = CTx = TCx$ for $x\in D(T)$ and hence $CD(T)\subset D(\bar{G})$ and $T|_{CD(T)} = \bar{G}|_{CD(T)}\subset\bar{G}$. So that (A6) implies $T\subset\bar{G}$. Next, by [2, Lemma], for $x\in D(\bar{G})$ $TCx = C\bar{G}x = \bar{G}Cx$, i.e., $CD(\bar{G})\subset D(T)$ and $\bar{G}|_{CD(\bar{G})} = T|_{CD(\bar{G})}\subset T$. Since $CD(\bar{G})$ is a core of $\bar{G}$, we obtain $\bar{G} = \overline{\bar{G}|_{CD(\bar{G})}}\subset T$. Therefore $T=\bar{G}$, i.e., T is the C-c.i.g. of $\{S(t): t\geqq 0\}$. Q.E.D.

§3. The abstract Cauchy problem.

In this section we consider the following abstract Cauchy problem

$$\text{(ACP)} \quad (d/dt)u(t) = Au(t) \qquad \text{for} \quad t\geqq 0 \text{ and } u(0)=x \,.$$

By a solution $u(t)$ of the (ACP) we mean that $u(t)$ is continuously differentiable in $t\geqq 0$, $u(0)=x$, $u(t)\in D(A)$ and $(d/dt)u(t) = Au(t)$ for every $t\geqq 0$.

The following is a direct consequence of [1, Corollary 13.1].

THEOREM 3.1. *If A is the C-c.i.g. of an exponentially bounded C-semigroup $\{S(t): t\geqq 0\}$ with $\|S(t)\|\leqq Me^{at}$, then the* (ACP) *has a unique solution $u(t)$ satisfying $\|u(t)\|\leqq Me^{at}\|C^{-1}x\|$ for every $x\in CD(A)$.*

Conversely the following theorem holds.

THEOREM 3.2. *Let A be a densely defined closed linear operator which commutes with C. Suppose that the following conditions are satisfied:*

(a) *The* (ACP) *has a unique solution $u(t)$ with $\|u(t)\|\leqq Me^{at}\|C^{-1}x\|$ for $x\in CD(A)$.*

(b) *$CD(A)$ is a core of A.*

Then A is the C-c.i.g. of an exponentially bounded C-semigroup $\{S(t): t\geqq 0\}$ with $\|S(t)\|\leqq Me^{at}$.

PROOF. We define the operator $\tilde{T}(t)\colon CD(A)\to D(A)$ by $\tilde{T}(t)x=u(t)$ for $x\in CD(A)$, and the bounded linear operator $S(t)$ by $S(t)x=\overline{C\tilde{T}(t)}x$ for $x\in X$. Let G be the operator defined by (0.4). Then $\{S(t)\colon t\geqq 0\}$ is an exponentially bounded C-semigroup with $\|S(t)\|\leqq Me^{at}$ and

$$CD(A)\subset D(G) \text{ and } G|_{CD(A)}=A|_{CD(A)}\,, \tag{3.1}$$

so that (b) implies $A\subset\bar{G}$. (See [1, Theorem 14].) To conclude the proof, we will show that $\bar{G}\subset A$. We first note

$$L_\lambda x\in D(A) \text{ and } AL_\lambda x=\lambda L_\lambda x-Cx \qquad \text{for} \quad x\in X\,. \tag{3.2}$$

Indeed, let $x\in X$. Since $CD(A)$ is dense in X, we can choose $x_n\in CD(A)$ with $\lim_{n\to\infty}x_n=x$. Noting that $S(t)x_n=C\tilde{T}(t)x_n\in CD(A)$, we see from (3.1) that $AS(t)x_n=GS(t)x_n=S(t)\bar{G}x_n$. Using the closedness of A,

$$A\left[\int_0^\infty e^{-\lambda t}S(t)x_n dt\right]=\int_0^\infty e^{-\lambda t}AS(t)x_n dt=\int_0^\infty e^{-\lambda t}S(t)\bar{G}x_n dt\,,$$

i.e., $AL_\lambda x_n=L_\lambda\bar{G}x_n$. Combining this with (0.5), $AL_\lambda x_n=\lambda L_\lambda x_n-Cx_n$. Since A is closed, $L_\lambda x_n\to L_\lambda x$ and $AL_\lambda x_n=\lambda L_\lambda x_n-Cx_n\to\lambda L_\lambda x-Cx$ as $n\to\infty$, (3.2) holds. Now, by (3.2) and (0.5), $A(\lambda L_\lambda x)=\lambda L_\lambda\bar{G}x$ for $x\in D(\bar{G})$. Noting that $\lambda L_\lambda x\to Cx$ and $A(\lambda L_\lambda x)=\lambda L_\lambda\bar{G}x\to C\bar{G}x$ as $\lambda\to\infty$, it follows from the closedness of A that

$$Cx\in D(A) \quad \text{and} \quad ACx=C\bar{G}x=\bar{G}Cx \qquad \text{for} \quad x\in D(\bar{G})\,, \quad \text{i.e.,}$$

$$CD(\bar{G})\subset D(A) \quad \text{and} \quad \bar{G}|_{CD(\bar{G})}=A|_{CD(\bar{G})}\subset A\,.$$

Since $CD(\bar{G})$ is a core of $\bar{G}$ we obtain $\bar{G}=\overline{\bar{G}|_{CD(\bar{G})}}\subset A$. Therefore $A=\bar{G}$, i.e., A is the C-c.i.g. of $\{S(t)\colon t\geqq 0\}$. Q.E.D.

§4. Connections with semigroups of growth order $\alpha>0$.

We first recall some results on semigroups of growth order α.

THEOREM 4.1 ([4, Theorem 1.2]). *Let n be the integral part of $\alpha>0$. Then a closed linear operator A in X is the complete infinitesimal generator of a semigroup of growth order α if and only if the following four conditions are satisfied:*

(I) *There is a real number ω such that for each $\xi>\omega$, $R(\xi-A)$ contains $D(A^{n+1})$ and $(\xi-A)^{-1}$ exists.*

(II) *There is a constant $M>0$ such that*

$$\|(\xi-A)^{-m}x\|\leqq\frac{M}{(m-1)!}\,\frac{\Gamma(m-\alpha)}{(\xi-\omega)^{m-\alpha}}\|x\|$$

for $x \in D(A^{n+1})$, $\xi > \omega$ *and* $m = k(n+1)$, $k = 1, 2, \cdots$.

(III) *$D(A^{n+2})$ is a core of A and $D(A)$ is dense in X.*

(IV) *For some $b > \omega$, $(b-A)^{n+1}$ is closable.*

LEMMA 4.2 ([4, Lemma 4.1]). *Let A be a closed linear operator in X satisfying conditions* (I)-(III). *Then for each $\xi > \omega$ there exists a bounded linear operator $V(\xi, A)$ such that*

(a) $AV(\xi, A)x = V(\xi, A)Ax$ *for* $x \in D(A)$,

(b) $V(\xi, A)(\xi - A)^{n+1}x = x$ *for* $x \in D(A^{n+1})$,

(c) *$V(\xi, A)$ is invertible if and only if $(\xi - A)^{n+1}$ is closable.*

Our result of this section is the following

THEOREM 4.3. *Let $\{U(t): t \geqq 0\}$ be a semigroup of growth order $\alpha > 0$. If A is the complete infinitesimal generator of $\{U(t): t \geqq 0\}$ then $T = A$ and $C = V(b, A)$ satisfy* (A1)-(A6), *so that A is the C-c.i.g. of an exponentially bounded C-semigroup $\{S(t): t \geqq 0\}$. Moreover, we have $S(t) = V(b, A)U(t)$ and*

$$U(t)x = \lim_{\lambda \to \infty} e^{-\lambda t} \sum_{m=0}^{\infty} \frac{t^m \lambda^{2m}}{m!} (\lambda - A)^{-m} x \tag{4.1}$$

$$= \lim_{m \to \infty} \left(1 - \frac{t}{m} A\right)^{-m} x \quad \textit{for} \quad x \in D(A^{n+1}) .$$

PROOF. By virtue of [1, Theorem 26], $T = A$ and $C = V(b, A)$ satisfy (A1)-(A5). We will prove that $T = A$ and $C = V(b, A)$ satisfy (A6), i.e., $V(b, A)D(A)$ is a core of A. By (III), it is sufficient to show that

$$V(b, A)D(A) \supset D(A^{n+2}) . \tag{4.2}$$

To this end, let $x \in D(A^{n+2})$ and $y = (b-A)^{n+1}x \in D(A)$. Then it follows from Lemma 4.2 (b) that

$$x = V(b, A)(b-A)^{n+1}x = V(b, A)y \in V(b, A)D(A) .$$

So that (4.2) is satisfied. Finally, it is seen from [1, Remark after Theorem 26], (1.10) and (1.11) that $S(t) = V(b, A)U(t)$ and (4.1) is satisfied. Q.E.D.

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Present Address:
DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCE AND ENGINEERING
WASEDA UNIVERSITY
OKUBO, SHINJUKU-KU, TOKYO 160