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Number Theoretical Transformations with Finite Range Structure and Their Ergodic Properties

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Introduction

Various number theoretical transformations, such as transformations for simple continued fraction [1], nearest integers continued fractions [10], complex continued fractions [2] [7], Jacobi-Perron's Algorithm [14], etc., have the following structure: Let $X \subset \boldsymbol{R}^n$ and T be a map of X onto itself. Then there exist a partition $\{X_a : a \in I\}$ of X, and a finite number of range sets $U_0, \cdots, U_N$ such that $T^n(X_{a_1} \cap T^{-1} X_{a_2} \cap \cdots \cap T^{-(n-1)} X_{a_n}) \in \{U_i : 0 \leqq i \leqq N\}$. The system $(X, T, \{U_0, \cdots, U_N\}, \{X_a ; a \in I\})$ with such a structure will be called a *number theoretical transformation with finite range structure* (see the definition in §1).

In this paper, we first summarize ergodic properties of number theoretical transformations with finite range structure which have already been obtained in [10] and [8]. Namely, Theorem 1 in §1 states that the number theoretical transformation satisfying a transitivity condition and Renyi's condition is ergodic, exact, and admits a finite invariant measure whose density is bounded. Moreover, according to a result of Schweiger [15], Theorem 2 gives a sufficient condition in order that such a transformation possesses a σ-finite invariant measure. The maps defined by

$$T_1(\theta, \varphi) = \left(-\left[-\frac{1}{\theta}\right] - \frac{1}{\theta}, -\left[-\frac{\varphi}{\theta}\right] - \frac{\varphi}{\theta}\right)$$

and

$$T_2(\theta, \varphi) = \left(-\left[-\frac{1}{\theta}\right] - \frac{1}{\theta}, \frac{\varphi}{\theta} - \left[\frac{\varphi}{\theta}\right]\right)$$

are interesting examples in view of number theoretical applications [18] (cf. (3)).

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In §3 and §5, we verify that these maps are number theoretical transformations with finite range structure and satisfy conditions of Theorem 2, and therefore, they possess σ-finite invariant measures.

In §4 and §6, we give the density functions of the invariant measures μ_1 and μ_2 for T_1 and T_2 explicitly as follows:

$$\frac{d\mu_1}{d\lambda}=\begin{cases}\dfrac{2-\theta}{2(1-\theta)^2} & \text{if } \theta<\varphi\\[2mm] \dfrac{1}{2(1-\theta)} & \text{if } \theta>\varphi\end{cases}$$

$$\frac{d\mu_2}{d\lambda}=\begin{cases}\dfrac{2-\theta}{2(1-\theta)^2} & \text{if } \theta+\varphi<1\\[2mm] \dfrac{1}{2(1-\theta)} & \text{if } \theta+\varphi>1\,.\end{cases}$$

Our method of determining the density is to construct the dual algorithm or natural extension [10], [5], [9]. This method is very useful for determining explicitly the density of an invariant measure in the case of maps defined on higher dimensional spaces. In the papers [3], [19] the densities of invariant measures for some number theoretical transformations on a 2-dimensional space were computed by using this method.

§1. A class of number theoretical transformations.

In this section, we propose a class of transformations, whose element will be called a *number theoretical transformation with finite range structure*.

Let Y be a bounded measurable subset with piecewise smooth boundary of $\boldsymbol{R}^n$ and $\lambda(\cdot)$ be the normalized Lebesgue measure on Y. Let I be a countable set. In this section, let us consider a map $S: Y\to Y$ satisfying the following conditions:

(0) There exists a countable partition $\xi=\{Y_a : a\in I\}$ with an index set I elements Y_a of which are measurable and connected subsets of Y with piecewise smooth boundary such that $S|_{Y_a}$ is injective, of class C^1 and $\det(DS|_{Y_a})\neq 0$.

We introduce some notations and definitions: *A cylinder of rank n with respect to S* is defined by

$$Y_{a_1\cdots a_n}=Y_{a_1}\cap S^{-1}Y_{a_2}\cap\cdots\cap S^{-(n-1)}Y_{a_n}\quad\text{if}$$
$$(Y_{a_1})^\circ\cap(S^{-1}Y_{a_2})^\circ\cap\cdots\cap(S^{-(n-1)}Y_{a_n})^\circ\neq\varnothing\,.$$

$\mathscr{L}^{(n)}$ denotes the family of all cylinders $Y_{a_1\cdots a_n}$ of rank n, and $\mathscr{L}$ denotes

$\bigcup_{n=1}^{\infty} \mathscr{L}^{(n)}$. If $Y_{a_1\cdots a_n} \in \mathscr{L}^{(n)}$, we call the sequence $(a_1, \cdots, a_n)$ *S-admissible*. Denote the set of all S-admissible sequences of length n by $A(n)$.

Using the notation as above, we give the definition of number theoretical transformation with finite range structure as follows.

For a map S satisfying (0), if there exists a finite number of subsets $V_0, \cdots, V_N$ of Y with positive measure such that for each n and for all $(a_1 \cdots a_n) \in A(n)$

$$S^n Y_{a_1\cdots a_n} \in \{V_0, \cdots, V_N\} ,$$

then we call S a *number theoretical transformation with finite range structure*. Hereafter, we denote such a system by $(Y, S, \{V_0, \cdots, V_N\}, \{Y_a : a \in I\})$ throughout this paper. In case there exists $k \geqq 0$ such that all range sets are $\mathscr{L}^{(k)}$-measurable, that is,

$$V_i = \bigcup_{Y_{a_1\cdots a_k} \subset V_i} Y_{a_1\cdots a_k} \qquad \text{for all } i \text{ with } \quad 0 \leqq i \leqq N ,$$

we call S a *number theoretical transformation with Markov structure* and call such a system $(Y, S, \{V_0, \cdots, V_N\}, \{Y_a ; a \in I\})$ a *number theoretical Markov system* (abbrev. N. M. S.). In particular, in case $k=0$, that is, $S^n Y_{a_1\cdots a_n} = Y$ for all n and $(a_1 \cdots a_n) \in A(n)$, we call such an S a *number theoretical transformation with Bernoulli structure*.

We write Ψ_a for $(S|_{Y_a})^{-1}$ and define inductively

$$\Psi_{a_1\cdots a_n} = \Psi_{a_1\cdots a_{n-1}} \circ \Psi_{a_n} .$$

Thus $\Psi_{a_1\cdots a_n}$ is a map of $S^n Y_{a_1\cdots a_n}$ onto $Y_{a_1\cdots a_n}$ for all $(a_1 \cdots a_n) \in A(n)$, and the domain of $\Psi_{a_1\cdots a_n}$ is contained in $\{V_0, \cdots, V_N\}$.

Given a constant $C \geqq 1$, we call a cylinder $Y_{a_1\cdots a_n}$ an "*R.C-cylinder*" if it satisfies "Renyi's condition", i.e.

$$\sup_{x \in S^n Y_{a_1\cdots a_n}} |\det D\Psi_{a_1\cdots a_n}(x)| \leqq C \inf_{x \in S^n Y_{a_1\cdots a_n}} |\det D\Psi_{a_1\cdots a_n}(x)| .$$

$R(C.S)$ denotes the set of all $R.C$-cylinders.

With the above notation, we give the next theorem which is an improvement of Nakada's theorem [8] due to Waterman [21].

THEOREM 1. *Suppose that* $(S, Y, \{V_0, \cdots, V_N\}, \{Y_a ; a \in I\})$ *is a number theoretical transformation with finite range structure and satisfies the following conditions*:

(C.1) (*transitivity condition*):

$V_0 = Y$ and for each j with $0 \leqq j \leqq N$, there exists a $Y_{a_1\cdots a_{s_j}} \subset V_j$ and $S^{s_j} Y_{a_1\cdots a_{s_j}} = Y$.

(C.2) (*generator condition*):
$\bigvee_{m=1}^{\infty} S^{-m}\xi = \varepsilon$, *i.e. the partition into points.*
(C.3) (*Renyi's condition*):
There exists a constant $C \geqq 1$ *such that* $R(C.S) = \mathscr{L}$.
Then S *is ergodic, exact with respect to* λ *and admits a finite invariant measure* $\nu \sim \lambda$ *such that its density is bounded above and below.*

Before proving Theorem 1, we give a remark and several lemmas.

REMARK 1. The following formula will be used often throughout this paper.

$$Y_{a_1 \cdots a_k a_{k+1} \cdots a_n} = \Psi_{a_1 \cdots a_k}(S^k Y_{a_1 \cdots a_k} \cap Y_{a_{k+1} \cdots a_n})$$

if

$$(S^k Y_{a_1 \cdots a_k})^\circ \cap (Y_{a_{k+1} \cdots a_n})^\circ \neq \varnothing .$$

Thus we find the following symbolical property:
$(a_1 \cdots a_n) \in A(n)$ if and only if $(a_1 \cdots a_k) \in A(k)$, $(a_{k+1}, \cdots, a_n) \in A(n-k)$, and $(S^k Y_{a_1 \cdots a_k})^\circ \cap (Y_{a_{k+1} \cdots a_n})^\circ \neq \varnothing$.

LEMMA 1.1. *Put* $s = \text{L.C.M.}_{1 \leq j \leq N}\{s_j\}$. *Then for each* j *there exists a* $Y_{a_1 \cdots a_s}$ *such that*

$$Y_{a_1 \cdots a_s} \subset V_j \quad \textit{and} \quad S^s Y_{a_1 \cdots a_s} = Y .$$

PROOF. By using Remark 1, we can easily verify that

$$Y_{\underbrace{a_1 \cdots a_{s_j}} \cdots \underbrace{a_1 \cdots a_{s_j}}} \text{ is well defined, and } S^s Y_{\underbrace{\underbrace{a_1 \cdots a_{s_j}} \cdots \underbrace{a_1 \cdots a_{s_j}}}_{s}} = Y . \qquad \square$$

LEMMA 1.2. *Put* $A^0(n) = \{(a_1 \cdots a_n) \in A(n) \colon S^n Y_{a_1 \cdots a_n} = Y\}$. *Then,*

for any $Y_{b_1 \cdots b_k} \in \mathscr{L}^{(k)}$ *there exists*
$Y_{b_1 \cdots b_k a_1 \cdots a_s}$ *such that* $Y_{b_1 \cdots b_k a_1 \cdots a_s} \in A^0(k+s)$.

PROOF. Let $S^k Y_{b_1 \cdots b_k} = V_j$. Lemma 1.1 implies that there exists $Y_{a_1 \cdots a_s}$ which is contained in V_j and so the result follows from Remark 1 immediately. $\square$

Thus, for all $n \geqq s$, we have $A^0(n) \neq \varnothing$. Next, for the proof of Theorem 1, we need the following lemma, which corresponds to the condition (q) in [21].

LEMMA 1.3. *There is a constant* D *such that for any* $Y_{a_1 \cdots a_k}$ *and all*

$m \geqq s$,

$$\sum_{\substack{(b_1\cdots b_m);\\ (a_1\cdots a_k b_1\cdots b_m)\in A^0(k+m)}} \lambda(Y_{a_1\cdots a_k b_1\cdots b_m}) \geqq D\lambda(Y_{a_1\cdots a_k}) .$$

In particular, $\sum_{(b_1\cdots b_m)\in A^0(m)} \lambda(Y_{b_1\cdots b_m}) \geqq D$.

PROOF OF LEMMA 1.3. For convenience, we denote $\boldsymbol{a}(n)=(a_1\cdots a_n)$, $\boldsymbol{b}(m)=(b_1\cdots b_m)$, and $\boldsymbol{a}(n)\boldsymbol{b}(m)=(a_1\cdots a_n b_1\cdots b_m)\in A(n+m)$. First, let us suppose $m=s$. Using the fact from Remark 1 that

$$(1)\qquad \begin{aligned}&\{\boldsymbol{b}(s)\colon Y_{\boldsymbol{b}(s)}\in \mathscr{L}^{(s)},\ \boldsymbol{a}(k)\boldsymbol{b}(s)\in A^0(k+s)\}\\ &\quad\supset\{\boldsymbol{b}(s)\colon \boldsymbol{b}(s)\in A^0(s),\ Y_{\boldsymbol{b}(s)}\subset S^k Y_{\boldsymbol{a}(k)}\} ,\end{aligned}$$

we obtain from (C.3)

$$\begin{aligned}\sum_{\substack{\boldsymbol{b}(s);\\ \boldsymbol{a}(k)\boldsymbol{b}(s)\in A^0(k+s)}} \lambda(Y_{\boldsymbol{a}(k)\boldsymbol{b}(s)}) &\geqq \sum_{\substack{\boldsymbol{b}(s);\\ Y_{\boldsymbol{b}(s)}\subset S^k Y_{\boldsymbol{a}(k)}\\ \boldsymbol{b}(s)\in A^0(s)}} \lambda(\Psi_{\boldsymbol{a}(k)} Y_{\boldsymbol{b}(s)})\\ &\geqq \sum_{\substack{\boldsymbol{b}(s);\\ Y_{\boldsymbol{b}(s)}\subset S^k Y_{\boldsymbol{a}(k)}\\ \boldsymbol{b}(s)\in A^0(s)}} \int_{Y_{\boldsymbol{b}(s)}} |\det D\Psi_{\boldsymbol{a}(k)}(x)|d\lambda(x)\\ &\geqq \sum_{\substack{\boldsymbol{b}(s);\\ Y_{\boldsymbol{b}(s)}\subset S^k Y_{\boldsymbol{a}(k)}\\ \boldsymbol{b}(s)\in A^0(s)}} \lambda(Y_{\boldsymbol{b}(s)})\cdot\frac{\lambda(Y_{\boldsymbol{a}(k)})}{C} .\end{aligned}$$

Put $V_j=S^k Y_{\boldsymbol{a}(k)}$. Then by Lemma 1.1 there exists

$$\tilde{\boldsymbol{b}}_j(s)\in A^0(s) \quad \text{such that} \quad Y_{\tilde{\boldsymbol{b}}_j(s)}\subset V_j ,$$

and hence

$$\sum_{\substack{\boldsymbol{b}(s);\\ \boldsymbol{a}(k)\boldsymbol{b}(s)\in A^0(k+s)}} \lambda(Y_{\boldsymbol{a}(k)\boldsymbol{b}(s)}) \geqq \frac{\lambda(Y_{\tilde{\boldsymbol{b}}_j(s)})}{C}\cdot\lambda(Y_{\boldsymbol{a}(k)}) .$$

Put $D=C^{-1}\min_{1\leqq j\leqq N}\{\lambda(Y_{\tilde{\boldsymbol{b}}_j(s)})\}$. Then we have

$$(2)\qquad \sum_{\substack{\boldsymbol{b}(s);\\ \boldsymbol{a}(k)\boldsymbol{b}(s)\in A^0(k+s)}} \lambda(Y_{\boldsymbol{a}(k)\boldsymbol{b}(s)}) \geqq D\cdot\lambda(Y_{\boldsymbol{a}(k)}) .$$

Next we prove the lemma in case $m>s$. Let $(b_1\cdots b_{m-s})$ be S-admissible such that $\boldsymbol{a}(k)b_1\cdots b_{m-s}\in A(k+m-s)$. Then by Lemma 1.1 there exists $(b_{m-s+1}\cdots b_m)$ such that $(b_{m-s+1}\cdots b_m)\in A^0(s)$ and $S^{k+m-s}Y_{\boldsymbol{a}(k)b_1\cdots b_{m-s}}\supset Y_{b_{m-s+1}\cdots b_m}$. This implies

$$\boldsymbol{a}(k)b_1\cdots b_{m-s}b_{m-s+1}\cdots b_m\in A^0(k+m) ,$$

so we obtain the following:

$$\sum_{\substack{b(m);\\ a(k)b(m)\in A^0(k+m)}} \lambda(Y_{a(k)b(m)})$$
$$\geqq \sum_{\substack{b_1\cdots b_{m-s};\\ a(k)b_1\cdots b_{m-s}\in A(k+m-s)}} \Bigg(\sum_{\substack{b_{m-s+1}\cdots b_m\in A^0(s)\\ S^{k+m-s}Y_{a(k)b_1\cdots b_{m-s}}\supset Y_{b_{m-s+1}\cdots b_m}}} \lambda(Y_{a(k)b_1\cdots b_{m-s}b_{m-s+1}\cdots b_m})\Bigg) .$$

From this and (2), we have

$$\sum_{\substack{b(m);\\ a(k)b(m)\in A^0(k+m)}} \lambda(Y_{a(k)b(m)}) \geqq \sum_{\substack{(b_1\cdots b_{m-s});\\ a(k)b_1\cdots b_{m-s}\in A(k+m-s)}} (D\cdot\lambda(Y_{a(k)b_1\cdots b_{m-s}}))$$
$$= D\cdot\lambda(Y_{a(k)}) . \qquad \square$$

PROOF OF THEOREM 1. First we prove the ergodicity. Suppose that $S^{-1}E=E$ and $\lambda(E)>0$. For all $m\geqq 0$ and $a(m)\in A(m)$,

$$\lambda(E\cap Y_{a(m)}) \geqq \sum_{\substack{b(s);\\ a(m)b(s)\in A^0(m+s)}} \int_{Y_{a(m)b(s)}} I_E(x)d\lambda(x)$$

where I_E is the indicator function of E.

Since $S^{-1}E=E$, we obtain

$$\lambda(E\cap Y_{a(m)}) \geqq \sum_{\substack{b(s);\\ a(m)b(s)\in A^0(m+s)}} \int_{Y_{a(m)b(s)}} I_E(S^{m+s}x)d\lambda(x)$$
$$= \sum_{\substack{b(s);\\ a(m)b(s)\in A^0(m+s)}} \int_Y |\det D\Psi_{a(m)b(s)}(x)|\, I_E(x)d\lambda(x) .$$

By (C.3) we have

$$\int_Y |\det D\Psi_{a(m)b(s)}(x)|\, I_E(x)d\lambda(x) \geqq \frac{1}{C}\lambda(Y_{a(m)b(s)})\cdot\lambda(E) ,$$

and so by Lemma 1.3

$$\lambda(E\cap Y_{a(m)}) \geqq C^{-1}\cdot D\cdot\lambda(Y_{a(m)})\cdot\lambda(E) .$$

It follows from this that S is ergodic with respect to λ.

Next we show the existence of an invariant measure. By Lemma 1.3 we obtain for all $k\geqq s$

$$\lambda(S^{-k}E) = \sum_{a(k)\in A(k)} \lambda(S^{-k}E\cap Y_{a(k)})$$
$$= \sum_{a(k)\in A(k)} \int_{E\cap S^kY_{a(k)}} |\det D\Psi_{a(k)}(x)|\, d\lambda(x)$$
$$\geqq \sum_{a(k)\in A^0(k)} \int_E |\det D\Psi_{a(k)}(x)|\, d\lambda(x)$$

$$\geqq \sum_{a(k)\in A^0(k)} \lambda(Y_{a(k)})\frac{\lambda(E)}{C}$$
$$\geqq C^{-1}\cdot D\cdot \lambda(E)\ .$$

On the other hand, since for any $a(k)\in A(k)$

$$\inf_{x\in Y_{a(k)}} |\det D\Psi_{a(k)}(x)|\cdot \lambda(S^k Y_{a(k)}) \leqq \lambda(Y_{a(k)})$$

holds, by (C.3) we see

$$\lambda(S^{-k}E) \leqq \sum_{a(k)\in A(k)} \sup|\det D\Psi_{a(k)}(x)|\cdot \lambda(E)$$
$$\leqq \sum_{a(k)\in A(k)} C\cdot \inf|\det D\Psi_{a(k)}(x)|\cdot \lambda(E)$$
$$\leqq \sum_{a(k)\in A(k)} \frac{\lambda(Y_{a(k)})}{\lambda(S^k Y_{a(k)})}\cdot C\cdot \lambda(E)$$
$$\leqq (\min_{1\leqq j\leqq N} \lambda(V_j))^{-1}\cdot C\cdot \lambda(E)\ .$$

Put $G=\max\{C(\min_j \lambda(V_j))^{-1},\ CD^{-1}\}$. Then we have

$$G^{-1}\cdot \lambda(E) \leqq \lambda(S^{-k}E) \leqq G\cdot \lambda(E) \qquad \text{for all} \quad k\geqq s\ .$$

This inequality implies that there exists an invariant measure μ and the measure μ is equivalent to λ.

The proof of exactness is similar to that of Theorem 5.3 in [21] and is, therefore, omitted. □

At the end of this section, we make a remark on the assumptions made in Theorem 1, and give examples of number theoretical transformations with finite range structure.

For various number theoretical transformations, the assumptions of Theorem 1 are satisfied. On the other hand, we also know some examples not satisfying them, that is, examples for which $S^n Y_{a_1\cdots a_n} \neq Y$ for any $Y_{a_1\cdots a_n} \in \mathscr{L}$. They sometimes appear in the dual algorithm. In view of these examples, we weaken the assumption (C.1) of Theorem 1, and obtain an analogue of Theorem 1 as follows.

THEOREM 1*. *Suppose that* $(Y, S, \{V_0, \cdots, V_N\}, \{Y_a; a\in I\})$ *is a number theoretical transformation with finite range structure and the following conditions are satisfied:*

(C.1)* *There exist* $V_{n_1}, \cdots, V_{n_k}$ *and* $Y=\cup_{i=1}^k V_{n_i}$. *And for each pair* V_j *and* V_{n_i}, *there exists a* $Y_{a(j,i)(s_j)} \in \mathscr{L}$ *such that* $Y_{a(j,i)(s_j)} \subset V_j$ *and* $S^{s_j} Y_{a(j,i)(s_j)} = V_{n_i}$.

(C.2)* $\vee_{m=1}^{\infty} S^{-m}\xi=\varepsilon$.

(C.3)* *There exists a constant* $C\geqq 1$ *such that* $R(C.S)=\mathscr{L}$.

Then S is ergodic, exact with respect to λ *and admits a finite invariant measure* $\nu\sim\lambda$ *such that its density is bounded.*

The proof of this theorem is quite similar to that of Theorem 1, and hence it is omitted.

A typical and important example of number theoretical transformation with Bernoulli structure satisfying the conditions (C.1) and (C.2) is a simple continued fraction transformation [12]. The transformation which induces n-adic expansion or decimal expansions [6] are included in this class.

As examples of N.M.S. satisfying the conditions (C.1), (C.2) and (C.3), we mention the nearest integer continued fraction transformation [10], Jacobi-Perron algorithm [14] and a skew product transformation T_1 of $(0, 1)^2$ defined by

$$T_1(x, y)=\left(\frac{1}{x}-\left[\frac{1}{x}\right], \frac{y}{x}-\left[\frac{y}{x}\right]\right) \qquad ([3]) .$$

We remark that a nonsingular continued fraction expansion [10], which gives a dual algorithm to the nearest integer continued fraction transformation, is a simple example of N.M.S. satisfying (C.1)*, (C.2), and (C.3).

As examples of number theoretical transformations with finite range structure satisfying the conditions (C.1), (C.2) and (C.3), we know complex continued fraction expansions on $Z(i)$ or $Z((1+\sqrt{3i})/2)$ ([2], [8], [17]) and a skew product transformation T_2 of $(0, 1)^2$ defined by

$$T_2(x, y)=\left(\frac{1}{x}-\left[\frac{1}{x}\right], -\left[-\frac{y}{x}\right]-\frac{y}{x}\right) \qquad ([3]) .$$

§2. Number theoretical transformations with σ-finite invariant measure.

In §1, we have treated the class of maps which satisfy "Renyi's condition", (C.3). On the other hand, we can find many interesting examples not satisfying (C.3). For this reason, we direct our attention now to the class of maps which do not satisfy (C.3).

Now let $(X, T, \{U_0, \cdots, U_N\}, \{X_a: a\in I\})$ be a number theoretical transformation with finite range structure and assume that the conditions (C.1), (C.2) are satisfied.

Given a constant $C \geqq 1$. We define

$$\mathscr{D}_n = \{X_{a_1 \cdots a_n} \in \mathscr{L}^{(n)}\colon X_{a_1 \cdots a_j} \in \mathscr{L} \backslash R(C.T) \text{ for } 1 \leqq j \leqq n\} ,$$
$$D_n = \bigcup_{X_{a_1 \cdots a_n} \in \mathscr{D}_n} X_{a_1 \cdots a_n} ,$$
$$\beta_n = \{X_{a_1 \cdots a_n} \in \mathscr{L}^{(n)}\colon X_{a_1 \cdots a_{n-1}} \in \mathscr{D}_{n-1}, X_{a_1 \cdots a_n} \in R(C.T)\} ,$$
$$B_n = \bigcup_{X_{a_1 \cdots a_n} \in \beta_n} X_{a_1 \cdots a_n} .$$

It is easy to see that for each n $\bigcup_{j=1}^{n} B_j \cup D_n$ is a disjoint covering of X. Using the above notations, we prove the following theorem.

THEOREM 2. *Suppose that* $(X, T, \{U_0, \cdots, U_N\}, \{X_a\colon a \in I\})$ *is a number theoretical transformation with finite range structure and the conditions* (C.1), (C.2) *are satisfied. Assume that there is a constant* $C \geqq 1$ *such that*

(C.3)$_a$ $\lim_{n \to \infty} \lambda(D_n) = 0$,

(C.3)$_b$ *if* $X_{a_1 \cdots a_n} \in R(C.T)$, *then* $X_{b_1 \cdots b_k a_1 \cdots a_n} \in R(C.T)$ *for any* $(b_1 \cdots b_k a_1 \cdots a_n) \in A(k+n)$,

(C.3)$_c$ *there is an element of the partition,* X_a *such that*

$$TX_a = X \quad \textit{and} \quad X_a \in R(C.T) .$$

Then T *is ergodic with respect to* λ *and admits a* σ*-finite invariant measure* $\mu \sim \lambda$.

REMARK 2. If we replace (C.3)$_a$ by

$$(\text{C.3})_a^* \qquad \sum_{n=1}^{\infty} \lambda(D_n) < \infty ,$$

then μ is finite.

In case T has a Bernoulli structure, that is,

$$T^k X_{a_1 \cdots a_k} = X \ (= U_0) \quad \text{for all } k \text{ and } X_{a_1 \cdots a_k} ,$$

F. Shweiger obtained the same result as this theorem in ([15]). Therefore this theorem is a generalization of Schweiger's. But the process of the proof is essentially that of Schweiger's ([15], [16]).

LEMMA 2.1. *Any cylinder is within a set of* λ*-measure zero a disjoint union of R.C-cylinders.*

PROOF. Since $X = \bigcup_{j=1}^{m} B_j \cup D_m$, if $X_{a(n)} \in \mathscr{L} \backslash R(C.T)$ and $T^n X_{a(n)} = U_i$ then we see

$$\begin{aligned} X_{a(n)} &= \Psi_{a(n)}(X \cap U_i) \\ &= \bigcup_{j=1}^{m} \Psi_{a(n)}(B_j \cap U_i) \cup \Psi_{a(n)}(D_m \cap U_i) \\ &= \bigcup_{j=1}^{m} \bigcup_{X_{b(j)} \in \beta_j} \Psi_{a(n)}(X_{b(j)} \cap U_i) \cup \Psi_{a(n)}(D_m \cap U_i) \\ &= \bigcup_{j=1}^{m} \bigcup_{X_{b(j)} \in \beta_j} X_{a(n)b(j)} \cup \Psi_{a(n)}(D_m \cap U_i) . \end{aligned}$$

$(C.3)_a$ and $(C.3)_b$ imply $X_{a(n)b(j)} \in R(C.T)$ and $\lim_{m\to\infty} \lambda(\Psi_{a(n)}(D_m \cap U_i))=0$. Thus Lemma 2.1 is proved. □

REMARK 3. $$X=\bigcup_{j=1}^{\infty} B_j \quad (\text{mod } 0) .$$

Now let us consider a new map T_R defined by

$$T_R x = T^j x \quad \text{for} \quad x \in B_j . \tag{3}$$

Then, T_R can be considered as a map of $\bigcup_{j=1}^{\infty} B_j$ with a partition indexed by

$$J=\bigcup_{n=1}^{\infty} \{(a_1\cdots a_n) \in A(n) \colon X_{a_1\cdots a_n} \in \beta_n\} .$$

Hereafter for $\alpha=(a_1\cdots a_n) \in J$ we denote

$$X_\alpha = X_{a_1\cdots a_n} .$$

It is easy to see that a cylinder $X_{\alpha_1\cdots\alpha_n}$ of rank n with respect to T_R is given by

$$X_{\alpha_1\cdots\alpha_n} = X_{a_1^1\cdots a_{k(1)}^1\cdots a_1^n\cdots a_{k(n)}^n} , \tag{4}$$

where $\alpha_1=(a_1^1\cdots a_{k(1)}^1)$, $\cdots$, $\alpha_n=(a_1^n\cdots a_{k(n)}^n)$, and

$$T_R^n(X_{\alpha_1\cdots\alpha_n}) = T^{k(1)+\cdots+k(n)}(X_{a_1^1\cdots a_{k(n)}^n}) .$$

Therefore

$$T_R^n(X_{\alpha_1\cdots\alpha_n}) \in \{U_0, U_1, \cdots, U_N\} \quad \text{for all } n\geqq 1 \text{ and } X_{\alpha_1\cdots\alpha_n} .$$

Put $\{U_0', \cdots, U_M'\}=\{T_R^n(X_{\alpha_1\cdots\alpha_n})\colon \text{all } n\geqq 1 \text{ and all } X_{\alpha_1\cdots\alpha_n}\}$. Then the system $(\bigcup_{j=1}^{\infty} B_j, T_R, \{U_0', \cdots, U_M'\}, \{X_\alpha; \alpha \in J\})$ is a number theoretical transformation with finite range structure.

LEMMA 2.2 (Main lemma). $(\bigcup_{j=1}^{\infty} B_j, T_R, \{U_0', \cdots, U_M'\}, \{X_\alpha\colon \alpha \in J\})$ *satisfies the assumptions of Theorem* 1.

To prove (C.2) we prepare the next sublemma. (See [15]).

SUBLEMMA. *Let $\mathscr{F}^{(n)}$ denote the σ-algebra generated by all the cylinders of rank n with respect to T and let $\mathscr{F}=\bigvee_{n=1}^{\infty}\mathscr{F}^{(n)}$ denote their join. And define $\mathscr{F}_R=\bigvee_{n=1}^{\infty}\mathscr{F}_R^{(n)}$ similarly with respect to T_R. Then $\mathscr{F}=\mathscr{F}_R$.*

PROOF. It is clear that $\mathscr{F}_R\subset\mathscr{F}$. Hence to prove Sublemma it suffices to show that every cylinder with respect to T is a disjoint union of cylinders with respect to T_R. The equality $D_n=B_{n+1}\cup D_{n+1}$ shows $D_n=\bigcup_{k=1}^{\infty}B_{n+k}$ mod 0. If $X_{a_1\cdots a_n}\in\mathscr{D}_n$, then

$$X_{a_1\cdots a_n}=\bigcup_{k=1}^{\infty}\Big(\bigcup_{X_{b(n+k)}\in\beta_{n+k}}X_{b(n+k)}\cap X_{a_1\cdots a_n}\Big)$$
$$=\bigcup_{k=1}^{\infty}\Big(\bigcup_{\substack{b_{n+1}\cdots b_{n+k};\\ X_{a_1\cdots a_n b_{n+1}\cdots b_{n+k}}\in\beta_{n+k}}}X_{a_1\cdots a_n b_{n+1}\cdots b_{n+k}}\Big)\quad(\text{mod }0)\,.$$

Let $X_{a_1\cdots a_n}\notin\mathscr{D}_n$. Then there is a maximal $k_0\in[1,n]$ such that $X_{a_1\cdots a_{k_0}}\in R(C.T)$. The condition $(C.3)_b$ implies that $X_{a_{k_0+1}\cdots a_n}\in\mathscr{D}_{n-k_0}$. In fact if $X_{a_{k_0+1}\cdots a_n}\notin\mathscr{D}_{n-k_0}$, then there is a number $l<n$ such that $X_{a_1\cdots a_{k_0}a_{k_0+1}\cdots a_l}\in R(C.T)$ by $(C.3)_b$. This contradicts the choice of k_0. Thus

$$X_{a_{k_0+1}\cdots a_n}=\bigcup_{k=1}^{\infty}B_{n-k_0+k}\cap X_{a_{k_0+1}\cdots a_n}\quad(\text{mod }0)\,,$$

and hence

$$X_{a_1\cdots a_n}=X_{a_1\cdots a_{k_0}}\cap T^{-k_0}\Big(\bigcup_{k=1}^{\infty}B_{n-k_0+k}\cap X_{a_{k_0+1}\cdots a_n}\Big)$$

mod 0. This gives a disjoint union representation by cylinders with respect to T_R. □

PROOF OF LEMMA 2.2. Using Sublemma, we can verify that the system $(\bigcup_{j=1}^{\infty}B_j, T_R, \{U_0', \cdots, U_M'\}, \{X_{\boldsymbol{\alpha}}; \boldsymbol{\alpha}\in J\})$ satisfies (C.2) immediately. From the definition of T_R, (4) and $(C.3)_b$, we can also verify that the system satisfies (C.3). For each $1\leq j\leq M$, from the condition (C.1) of $(X, T, \{U_0, \cdots, U_N\}, \{X_a; a\in I\})$ we find a cylinder $X_{a_1\cdots a_{s_j}}$ such that

$$X_{a_1\cdots a_{s_j}}\subset U_j' \quad\text{and}\quad X_{a_1\cdots a_{s_j}}\in A^0(s_j)\,.$$

Let X_a satisfy $(C.3)_c$. Then by Remark 1 we see $X_{a_1\cdots a_{s_j}a}\neq\emptyset$ and $T^{s_j+1}X_{a_1\cdots a_{s_j}a}=X$. Further we can see that there are $\boldsymbol{\alpha}_1, \cdots, \boldsymbol{\alpha}_l\in J$ such that $X_{a_1\cdots a_{s_j}a}=X_{\alpha_1\cdots\alpha_l}$. Indeed if $X_{a_1\cdots a_{s_j}}=X_{b(k)d(s_j-k)}$, where $\boldsymbol{b}(k)=(\boldsymbol{\alpha}_1\cdots\boldsymbol{\alpha}_m)\in J^m$ and $X_{d(s_j-k)}\in\mathscr{D}_{s_j-k}$, then by $(C.3)_c$ $X_{d(s_j-k)a}\neq\emptyset$ and

$X_{d(sj-k)a} \in \beta_{sj-k+1}$. Thus for all U'_j there exists a $X_{a_1 \cdots a_{sj}a} \subset U'_j$ such that

$$T^l_R X_{a_1 \cdots a_l} = X \quad \text{and} \quad X_{a_1 \cdots a_l} \in R(C.T_R) .$$

This completes the proof of Lemma 2.2. □

PROOF OF THEOREM 2. The proof is similar to that of theorem 4 and 5 in [15], and so we give only a sketch of the proof. It follows from Lemma 2.2 that T_R is ergodic with respect to λ and admits a finite invariant measure $\nu \sim \lambda$.

Let $T^{-1}E = E$, then the relation

$$T_R^{-1}E = \bigcup_{n=1}^{\infty} (B_n \cap T^{-n}E)$$

implies $T_R^{-1}E = E$. Therefore T is ergodic too. We put $D_0 = \bigcup_{n=1}^{\infty} B_n$ and define

$$\mu(A) = \sum_{n=0}^{\infty} \nu(D_n \cap T^{-n}A) .$$

Then we can see $\mu(T^{-1}A) = \mu(A)$, since $D_0 = X$ and ν is an invariant measure for T_R.

For every $X_{a(m)} \in R(C.T)$ from $(C.3)_b$, we have

$$\bigcup_{X_{k(n)} \in \mathscr{D}_n} (T^{-n}X_{a(m)} \cap X_{k(n)}) = \bigcup_{X_{k(n)} \in \mathscr{D}_n} X_{k(n)a(m)} \subseteqq D_n \backslash D_{n+m} .$$

Therefore

$$\mu(X_{a(m)}) \leqq \sum_{n=0}^{\infty} \nu(D_n \backslash D_{n+m}) = \sum_{n=1}^{m-1} \nu(D_n) < \infty .$$

Thus for $X_{a(m)} \in \beta_m$ the measure $\mu(X_{a(m)})$ is finite. Since $\bigcup_{j=0}^{\infty} B_j = X \bmod 0$, this implies that μ is a σ-finite invariant measure. This completes the proof of Theorem 2. □

If $\sum_{n=1}^{\infty} \lambda(D_n) < \infty$, the following inequality shows that μ is finite:

$$\mu(A) \leqq \nu(A) + \sum_{n=1}^{\infty} \nu(D_n)$$

for any measurable set A.

We have the next result which corresponds to Theorem 1*.

THEOREM 2*. *Suppose that* $(X, T, \{U_0, \cdots, U_N\}, \{X_a: a \in I\})$ *is the number theoretical transformation with finite range structure and satisfies* (C.2), $(C.3)_a$, $(C.3)_b$ *and the following conditions*:

(C.1)** *There exists* $U_{n_1}, \cdots, U_{n_k}$ *and* $X = \cup_{i=1}^{k} U_{n_i}$.
And for each j *with* $0 \leqq j \leqq N$, *there exists a* $X_{a_1 \cdots a_{sj}}$ *such that* $X_{a_1 \cdots a_{sj}} \subset U_j$ *and* $T^{s_j} X_{a_1 \cdots a_{sj}} = X$.

$(C.3)^*_c$ *There are* $X_{c_1} \cdots X_{c_k}$ *such that* $X_{c_l} \in R(C.T)$ *and* $TX_{c_l} = U_{n_l}$ $(l = 1 \cdots K)$.

Then T *is ergodic with respect to* λ *and admits a* σ*-finite invariant measure* $\mu \sim \lambda$.

The proof of this theorem is quite similar to that of Theorem 2. Therefore it is omitted.

A typical and important example of number theoretical transformations with Bernoulli structure satisfying the conditions (C.1), (C.2) and (C.3) is a transformation T of $(0, 1)$ defined by $Tx = -[-(1/x)] - (1/x)$. The density of the invariant measure is known:

$$h(x) = \frac{1}{1-x} .$$

This transformation is conjugate to a transformation T defined by $Tx = (x/(1-x)) \pmod 1$, whose invariant measure has the density given by $\Psi(x) = 1 - x$ ([11]).

As an example of number theoretical transformations with Bernoulli structure satisfying (C.1), (C.2) and $(C.3)^*_a$, we mention the transformation T of D defined by

$$T(x, y) = \left(\frac{1}{x} - \left[\frac{1-y}{x}\right] - \left[-\frac{y}{x}\right], -\left[-\frac{y}{x}\right] - \frac{y}{x}\right),$$

where

$$D = \{(x, y);\, 0 \leqq y \leqq 1 , \quad -y \leqq x < -y + 1\} ,$$

which induces a inhomogeneous linear approximation. The density of the invariant measure is known:

$$h(x, y) = \frac{1}{2 \log 2(1-x^2)} \quad ([4]) .$$

As an example of N.M.S which satisfies the conditions (C.1) (C.2) and $(C.3)^*_a$ we mention the complex continued fraction transformation of $Z(1+i)$ ([19]). More detailed investigation will be found in [20].

In the rest of this paper, we will give two examples of number theoretical transformations with finite range structure satisfying (C.1), (C.2), $(C.3)_a$, $(C.3)_b$ and $(C.3)_c$. We remark that one of the examples is N.M.S.

§3. Definition of a map T_1 and its ergodic properties.

In this section we give the first example which satisfies conditions of Theorem 2.

Let $X=\{(\theta, \varphi): 0 \leqq \theta, \varphi<1\}$ and functions $a(\theta)$ on $[0, 1]$ and $b(\theta, \varphi)$ on X be defined by

$$(5) \qquad a(\theta)=-\left[-\frac{1}{\theta}\right] \quad \text{and} \quad b(\theta, \varphi)=-\left[-\frac{\varphi}{\theta}\right],$$

where $[x]=\max\{n; n \leqq x, n \in \boldsymbol{Z}\}$. Let us define a map T_1 of X onto itself by

$$(6) \qquad T_1(\theta, \varphi)=\left(-\frac{1}{\theta}+a(\theta), -\frac{\varphi}{\theta}+b(\theta, \varphi)\right). \quad \text{(See Figure 1)}.$$

It is well known that the first coordinate map $(\theta \mapsto -(1/\theta)+a(\theta))$ has a σ-finite invariant measure whose density is $(1/(1-\theta))$. We define for $(\theta, \varphi) \in X$ the sequences $a_n=a_n(\theta)$ and $b_n=b_n(\theta, \varphi)$ by $a_n(\theta)=a(r_{n-1}(\theta))$ and $b_n(\theta, \varphi)=b(r_{n-1}(\theta), s_{n-1}(\theta, \varphi))$, where $r_n(\theta)$ and $s_n(\theta, \varphi)$ are components of $T_1^n(\theta, \varphi)$, that is,

$$(r_n(\theta), s_n(\theta, \varphi))=T_1^n(\theta, \varphi) \quad (n \geqq 0).$$

Note that $T_1^n(\theta, \varphi) \notin X$ may occur for some n. To avoid these difficulties we must consider the Algorithm on the restricted set

$$(7) \qquad X_{T_1}=\{(\theta, \varphi) \in X: T_1^n(\theta, \varphi) \in X \text{ for } n=1, 2, \cdots\}.$$

Since the Lebesgue measure of this set X_{T_1} is equal to 1, we denote for convenience X_{T_1} by X throughout this paper. From the definition of $a_i(\theta)$, $b_i(\theta, \varphi)$, $r_i(\theta)$ and $s_i(\theta, \varphi)$ it is easy to see that $a_i(\theta) \geqq 2$ and for each $(\theta, \varphi) \in X$

$$(8) \qquad \theta=\cfrac{1}{a_1(\theta)-\cfrac{1}{a_2(\theta)-\cfrac{\ddots}{-\cfrac{1}{a_n(\theta)-r_n(\theta)}}}};$$

$$\varphi=\sum_{k=1}^{n}(-1)^{k-1}\theta\theta_1\cdots\theta_{k-1}b_k(\theta,\varphi)$$
$$+(-1)^n\theta\theta_1\cdots\theta_{n-1}s_n(\theta,\varphi)\ ,$$

where

$$\theta_{k-1}=\cfrac{1}{a_k(\theta)-\cfrac{1}{a_{k+1}(\theta)-\ddots-\cfrac{1}{a_n(\theta)-r_n(\theta)}}}\ .$$

We denote

(9) $$\frac{p_n(\theta)}{q_n(\theta)}=\cfrac{1}{a_1(\theta)-\ddots-\cfrac{1}{a_n(\theta)}}\qquad(n\geqq 1)\ .$$

(10) $$\alpha_n(\theta)=q_n(\theta)\cdot\theta-p_n(\theta)\qquad(n\geqq 1)\ .$$

Then we have

(11) $$\begin{cases}q_n(\theta)=a_n(\theta)q_{n-1}(\theta)-q_{n-2}(\theta)\\ p_n(\theta)=a_n(\theta)p_{n-1}(\theta)-p_{n-2}(\theta)\ ,\end{cases}$$

where $(q_0(\theta),\ q_{-1}(\theta))=(1,\ 0)$ and $(p_0(\theta),\ p_{-1}(\theta))=(0,\ -1)$, and

(12) $$\theta\cdot\theta_1\cdot\theta_2\cdots\theta_n=\alpha_n(\theta)\ .$$

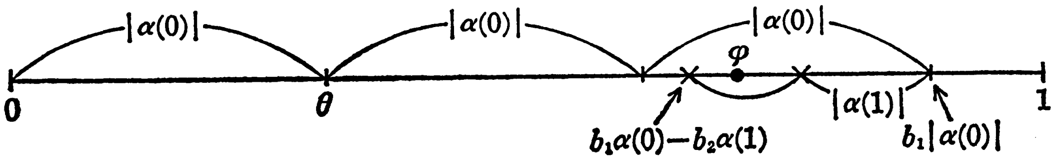
In the sequel for simplicity we sometimes write a_n, b_n, p_n, q_n, r_n, s_n, and α_n for $a_n(\theta)$, $b_n(\theta,\varphi)$, $p_n(\theta)$, $q_n(\theta)$, $r_n(\theta)$, $s_n(\theta,\varphi)$ and $\alpha_n(\theta)$, respectively. By induction we can easily see that

(13) $$q_{n+1}=a_{n+1}q_n-q_{n-1}>q_n\ .$$

Thus the sequence $\{q_n\}_{n\geqq 1}$ is monotone increasing. From the identity (9) and the relation (11) and (12), it follows that the expansion (8) of $(\theta,\varphi)\in X$ can be written in the form

(14) $$\theta=\frac{p_n-r_np_{n-1}}{q_n-r_nq_{n-1}}\ ,$$
$$\varphi=\sum_{k=1}^{n}(-1)^{k-1}\alpha_{k-1}b_k+\frac{(-1)^ns_n}{q_n-r_nq_{n-1}}\ .$$

We call p_n/q_n and $\sum_{k=1}^{n}(-1)^{k-1}\alpha_{k-1}b_k$ the n-th approximants of θ and φ respectively, with respect to the algorithm T_1. The second expression of (14) represents the approximation of the real number φ corresponding to the following geometric picture:



It is easy to see that

$$A^1(n)=\left\{\begin{pmatrix} a_1(\theta)\cdots a_n(\theta) \\ b_1(\theta,\varphi)\cdots b_n(\theta,\varphi)\end{pmatrix}: (\theta,\varphi)\in X\right\},$$

and for an n-tuple of pairs $\begin{pmatrix} a_1\cdots a_n \\ b_1\cdots b_n\end{pmatrix}$

$$X_{\binom{a_1\cdots a_n}{b_1\cdots b_n}}=\left\{(\theta,\varphi): \begin{pmatrix} a_i(\theta) \\ b_i(\theta,\varphi)\end{pmatrix}=\begin{pmatrix} a_i \\ b_i\end{pmatrix},\quad 1\leqq i\leqq n\right\}$$

is a cylinder of rank n with respect to T_1 in the sense of §2. Moreover we can easily verify that each T_1-admissible sequence $\begin{pmatrix} a_1\cdots a_n \\ b_1\cdots b_n\end{pmatrix}\in A^1(n)$ satisfies the following:

(A) a_i, b_i are positive integers such that

$$a_i\geqq 2\,,\quad a_i\geqq b_i\geqq 1\qquad (1\leqq i\leqq n)\,,$$

(B) if $a_i=b_i$, then $b_{i+1}\neq 1$.

Let U_1 be the set $\{(\theta,\varphi): 0<\theta<1 \text{ and } \theta<\varphi\leqq 1\}$ and $U_0=X$. Then

$$T_1X_{\binom{a}{b}}=\begin{cases} U_0 & \text{if } a\neq b \\ U_1 & \text{if } a=b\,,\end{cases}$$

and

(15) $$U_1=\bigcup_{\substack{\binom{a_1\cdots a_n}{b_1\cdots b_n}\in A^1(n)\\ b_1\neq 1}} X_{\binom{a_1\cdots a_n}{b_1\cdots b_n}}\,.$$

Moreover we have

Lemma 3.1. *For* $\begin{pmatrix} a_1\cdots a_n \\ b_1\cdots b_n\end{pmatrix}\in A^1(n)$, *we have*

$$T_1^n X_{\binom{a_1\cdots a_n}{b_1\cdots b_n}} = \begin{cases} U_0 & \text{if } a_n \neq b_n \\ U_1 & \text{if } a_n = b_n . \end{cases} \tag{3.1}$$

that is, the transformation T_1 is a number theoretical transformation with Markov structure.

We denote sometimes $\binom{a_1\cdots a_n}{b_1\cdots b_n} \in A^1(n)$ by $\binom{a(n)}{b(n)}$ for the sake of simplicity.

Now for each $\binom{a(n)}{b(n)} \in A^1(n)$ we define a map $\psi_{\binom{a(n)}{b(n)}}$ by

$$\psi_{\binom{a(n)}{b(n)}}(\theta, \varphi) = \left(\frac{p_n - p_{n-1}\theta}{q_n - q_{n-1}\theta}, \sum_{k=1}^{n} (-1)^{k-1}\theta_0(\theta)\theta_1(\theta)\cdots\theta_{k-1}(\theta)b_k + (-1)^n\theta_0(\theta)\cdots\theta_{n-1}(\theta)\varphi \right),$$

where

$$\theta_{k-1}(\theta) = \cfrac{1}{a_k - \cfrac{\ddots}{\cfrac{1}{a_n - \theta}}} \quad (1 \leqq k \leqq n) .$$

From (14) and Lemma 3.1 we see that the map $\varphi_{\binom{a(n)}{b(n)}}$ is the inverse map of T_1^n on $T_1^n X_{\binom{a(n)}{b(n)}} = U_0$ or U_1,

$$\psi_{\binom{a(n)}{b(n)}}(T_1^n X_{\binom{a(n)}{b(n)}}) = X_{\binom{a(n)}{b(n)}} ,$$

and on each $U_i (i=0, 1)$ its Jacobian $J(\psi_{\binom{a(n)}{b(n)}})$ is

$$J(\psi_{\binom{a(n)}{b(n)}})(\theta, \varphi) = \frac{1}{(q_n - q_{n-1}\theta)^3} . \tag{16}$$

Lemma 3.2. *Let $C = 2^3$. Then for each $\binom{a(n)}{b(n)} \in A^1(n)$ such that $a_n \neq 2$ we have*

$$\sup_{(\theta,\varphi)\in T_1^n X_{\binom{a(n)}{b(n)}}} J(\psi_{\binom{a(n)}{b(n)}})(\theta, \varphi) < C \cdot \inf_{(\theta,\varphi)\in T_1^n X_{\binom{a(n)}{b(n)}}} J(\psi_{\binom{a(n)}{b(n)}})(\theta, \varphi) ,$$

that is, $X_{\binom{a_1\cdots a_n}{b_1\cdots b_n}}$ for $a_n \neq 2$ belongs to $R(C, T_1)$.

Proof. The result follows from the following inequalities:

$$\frac{1}{q_n^3}<\frac{1}{q_n^3\left(1-\frac{q_{n-1}}{q_n}\theta\right)^3}=\frac{1}{q_n^3\left(1-\frac{1}{a_n-\frac{q_{n-2}}{q_{n-1}}}\theta\right)^3}<\frac{1}{q_n^3}\cdot 2^3 . \qquad \square$$

Lemma 3.3. *For $C=2^3$, $\mathscr{D}_n$ defined in §1 satisfies*

$$\sum_{X_{a(n)}\in\mathscr{D}_n}\lambda(X_{a(n)})\longrightarrow 0 \quad as \quad n\longrightarrow\infty ,$$

where λ is the Lebesgue measure on X.

Proof. Let $\mathscr{D}_n'$ be

$$\mathscr{D}_n'=\{X_{\binom{2\ 2\ \cdots 2}{b_1\ b_2\cdots b_n}};\ (b_1\cdots b_n)=(\underbrace{1\cdots 1}_{k},\ \underbrace{2\cdots 2}_{n-k}),\ 0\leq k\leq n\} .$$

Then from Lemma 3.2, (A) and (B) we have $\mathscr{D}_n\subset\mathscr{D}_n'$. Therefore to prove Lemma 3.3 it suffices to prove

$$\sum_{X_{\binom{a(n)}{b(n)}}\in\mathscr{D}_n'}\lambda(X_{\binom{a(n)}{b(n)}})\longrightarrow 0 \quad \text{as} \quad n\longrightarrow\infty .$$

By Lemma 3.1 and (16) we obtain

$$\sum_{X_{\binom{a(n)}{b(n)}}\in\mathscr{D}_n'}\lambda(X_{\binom{a(n)}{b(n)}})=\sum_{\substack{b_n=2\\ X_{\binom{a(n)}{b(n)}}\in\mathscr{D}_n'}}\int_{\psi_{\binom{a(n)}{b(n)}}(U_1)}d\lambda+\int_{\psi_{\binom{2\ 2\cdots 2}{1\ 1\cdots 1}}(U_0)}d\lambda$$

$$<n\cdot\iint_{U_1}\frac{d\theta d\varphi}{(q_n-q_{n-1}\theta)^3}+\iint_{U_0}\frac{d\theta d\varphi}{(q_n-q_{n-1}\theta)^3} .$$

Note that $q_n=n+1$ if $a_1=a_2=\cdots=a_n=2$. Therefore we have

$$\sum_{X_{\binom{a(n)}{b(n)}}\in\mathscr{D}_n'}\lambda(X_{\binom{a(n)}{b(n)}})\leq n\iint_{U_1}\frac{d\theta d\varphi}{(n+1-n\theta)^3}+\iint_{U_0}\frac{d\theta d\varphi}{(n+1-n\theta)^3}$$

$$=\frac{1}{8n}+\frac{1}{16n^2} .$$

This completes the proof of Lemma 3.3. $\square$

Corollary 3. *The partition $\left\{X_{\binom{m}{n}};\ \binom{m}{n}\in A^1(1)\right\}$ is a generator with respect to the map T_1.*

Proof. Let Y be the set:

$$Y=\{(\theta,\varphi)\in X;\ \text{there exists } n=n(\theta,\varphi) \text{ such that } a_m(\theta)=2 \text{ for all } m\geq n\} .$$

Then from the proof of Lemma 3.3 we see that $\lambda(Y)=0$. Since for $(\theta, \varphi)\in X\setminus Y$ there exist infinitely many n_k such that $a_{n_k}(\theta)\geqq 3$, we have

$$\begin{aligned} q_{n_k}-q_{n_k-1} &= (a_{n_k}-1)q_{n_k-1}-q_{n_k-2} \\ &\geqq 2q_{n_k-1}-q_{n_k-2}>q_{n_k-1} . \end{aligned}$$

Thus

$$\lim_{n\to\infty}(q_n-q_{n-1})=\infty . \tag{17}$$

(14) and (17) imply that if $(\theta, \varphi)\in X\setminus Y$, then

$$\lim_{n\to\infty}\left(\frac{p_n}{q_n}, \sum_{k=1}^{n}(-1)^{k-1}\alpha_{k-1}b_k\right)=(\theta, \varphi) .$$

This gives Corollary 3. □

With these facts in hand, we can now verify the assumptions of Theorem 2 immediately, and obtain the following.

THEOREM 3. *The map T_1 on X is ergodic with respect to λ and admits a σ-finite invariant measure $\mu_1\sim\lambda$.*

§4. Dual algorithm and natural extension of a map T_1.

In this section, we construct a natural extension $\bar{T}_1$ of the map T_1 by means of a dual algorithm and derive the invariant density function for T_1 explicitly. We define

$$D=\{(\xi, \eta)\colon 0\leqq\xi<1,\ \xi<\eta<\xi+1\} , \quad E=\{(\xi, \eta)\colon 0\leqq\xi<1,\ \xi<\eta<1\} ,$$

and the sets D_{mn}, which will be called the basic neighbourhood of the integer vector (m, n), as follows:

$$D_{mn}=\begin{cases} D-(m, n) & \text{for } m\geqq n \text{ and } n\neq 1 \\ E-(m, n) & \text{for } n=1 \end{cases}$$

where

$$D-(m, n)=\{(\xi, \eta)-(m, n)\colon (\xi, \eta)\in D\}$$

and

$$E-(m, n)=\{(\xi, \eta)-(m, n)\colon (\xi, \eta)\in E\} .$$

These neighbourhoods $\{D_{mn}\colon m\geqq n\geqq 1,\ m, n\in N\}$ give a disjoint covering of the domain $R=\{(\xi, \eta)\colon \xi<-1,\ \xi<\eta<0\}$; that is, $D_{mn}\cap D_{m'n'}=\varnothing$ for $(m, n)\neq$

(m', n') and $\bigcup_{(m,n)} D_{mn}=R$.

Now we define a map $T_1^*: D\to D$ which will be called a "dual algorithm with respect to T_1" as follows:

$$T_1^*(\xi, \eta)=\left(m-\frac{1}{\xi}, n+\frac{\xi-\eta}{\xi}\right) \quad \text{if} \quad \left(-\frac{1}{\xi}, \frac{\xi-\eta}{\xi}\right)\in D_{mn} .$$

Note that this map T_1^* is constructed from a map S: $(\xi, \eta)\mapsto(-1/\xi, (\xi-\eta)/\xi)$ and the map S satisfies $S(D)=R$. (See Figure 2).

Let functions $c(\xi)$, $d(\xi, \eta)$ on D be defined by

$$\begin{cases} c(\xi)=m \\ d(\xi, \eta)=n \end{cases} \quad \text{if} \quad \left(-\frac{1}{\xi}, \frac{\xi-\eta}{\xi}\right)\in D_{mn}$$

and define for $(\xi, \eta)\in D$ the sequences $c_n(\xi)$ and $d_n(\xi, \eta)$ by $c_n(\xi)=c(t_{n-1}(\xi))$ and $d_n(\xi, \eta)=d(t_{n-1}(\xi), u_{n-1}(\xi, \eta))$ for $n\geqq 1$ where $t_n(\xi)$ and $u_n(\xi, \eta)$ are the components of $(T_1^*)^n(\xi, \eta)$, that is, $(t_n(\xi), u_n(\xi, \eta))=(T_1^*)^n(\xi, \eta)$ $(n\geqq 0)$. For each $(\xi, \eta)\in D$ we denote $\begin{pmatrix} c_1(\xi), & \cdots\cdots c_n(\xi) \\ d_1(\xi, \eta), & \cdots d_n(\xi, \eta)\end{pmatrix}$ by $\begin{pmatrix} c_1\cdots c_n \\ d_1\cdots d_n\end{pmatrix}$ for convenience. We can easily see that the sequence of integer vectors $\begin{pmatrix} c_1\cdots c_n \\ d_1\cdots d_n\end{pmatrix}$ satisfies the following properties:

(A)′ $c_i, d_i\in N$ and $c_i\geqq d_i\geqq 1$,

(B)′ if $d_i=1$, then $d_{i+1}\neq c_{i+1}$.

This implies that $\begin{pmatrix} c_n, & c_{n-1}, & \cdots c_1 \\ d_n, & d_{n-1}, & \cdots d_1\end{pmatrix}\in A^1(n)$, that is, the sequence of integer vector $\begin{pmatrix} c_1\cdots c_n \\ d_1\cdots d_n\end{pmatrix}$ is a word dual to the one in $A^1(n)$ (or the word in $A^1(n)$ read backwards); in other words, the map T_1^* is the dual algorithm with respect to T_1. Now we consider for each (m, n) the map $\varphi_{\binom{m}{n}}$ and its range $Y_{\binom{m}{n}}$, which will be the inverse map of T_1^* and the domain of T_1^*, respectively, i.e., we define by

$$\varphi_{\binom{m}{n}}(\xi, \eta)=\left(\frac{1}{m-\xi}, \frac{(n+1)-\eta}{m-\xi}\right) \quad \text{on } D \text{ for } m\geqq n\geqq 2$$

and define similarly on E for $n=1$. We denote $\varphi_{\binom{m}{n}}(D)$ and $\varphi_{\binom{m}{n}}(E)$ by $Y_{\binom{m}{n}}$. By the definition of T_1^* we see that the family $\{Y_{\binom{m}{n}}: m\geqq n\geqq 1\}$ has the following properties:

(a) $\{Y_{\binom{m}{n}}: m\geqq n\geqq 1\}$ is a partition of D, that is, $Y_{\binom{m}{n}}\cap Y_{\binom{m'}{n'}}=\emptyset$ if $(m, n)\neq(m', n')$ and $\bigcup Y_{\binom{m}{n}}=D$,

(b) $\{Y_{\binom{m}{n}}: m\neq n\}$ is a partition of E, that is, $\bigcup_{m\neq n} Y_{\binom{m}{n}}=E$,

(c) $T_1^*(Y_{\binom{m}{n}})=\begin{cases} D & \text{if} \quad m\geqq n\geqq 2 \\ E & \text{if} \quad n=1. \end{cases}$

We put $V_{\binom{m}{n}}=T_1^*(Y_{\binom{m}{n}})$ and $U_{\binom{m}{n}}=T_1(X_{\binom{m}{n}})$, and let the set M be defined by

$$M=\left\{(\xi, \eta, \theta, \varphi)\in Y\times X;\ \begin{pmatrix} c_1(\xi) & a_1(\theta) \\ d_1(\xi, \eta) & b_1(\theta, \varphi)\end{pmatrix}\in A^1(2)\right\}.$$

Then, the set M is seen to have the following two partitions:

$$\begin{aligned} M&=\bigcup_{\binom{m}{n}\in A^1(1)} V_{\binom{m}{n}}\times X_{\binom{m}{n}} \\ &=\bigcup_{\binom{m}{n}\in A^1(1)} Y_{\binom{m}{n}}\times U_{\binom{m}{n}}.\end{aligned}$$

Now we define the map $\bar{T}_1\colon M\to M$ as follows:

$$\begin{aligned} \bar{T}_1(\xi, \eta, \theta, \varphi)&=(\varphi_{\binom{m}{n}}(\xi, \eta),\ T_1(\theta, \varphi)) \\ &=\left(\frac{1}{m-\xi}, \frac{(n+1)-\eta}{m-\xi}, m-\frac{1}{\theta}, n-\frac{\varphi}{\theta}\right)\end{aligned}$$

for $(\xi, \eta, \theta, \varphi)\in V_{\binom{m}{n}}\times X_{\binom{m}{n}}$.

THEOREM 4. *The map $\bar{T}_1$ defined above is a natural extension of T_1, and it has an invariant measure $\bar{\mu}$ such that*

$$\frac{d\bar{\mu}}{d\bar{\lambda}}=\frac{C}{(1-\xi\theta)^3},$$

where $\bar{\lambda}$ is the Lebesgue measure on M, furthermore, $\bar{T}_1$ is ergodic.

PROOF. By the definition of $\bar{T}_1$ and from the properties (A)′ and (B)′ we can show that for each $V_{\binom{m}{n}}\times X_{\binom{m}{n}}$ $\bar{T}_1$ is a one to one map of $V_{\binom{m}{n}}\times X_{\binom{m}{n}}$ onto $Y_{\binom{m}{n}}\times U_{\binom{m}{n}}$. Therefore the map $\bar{T}_1$ is a natural extension of T_1. Let the kernel function $K(\xi, \eta, \theta, \varphi)$ on M be defined by

$$K(\xi, \eta, \theta, \varphi)=\frac{1}{(1-\xi\theta)^3}.$$

Then, since the Jacobian $J\bar{T}_1(\xi, \eta, \theta, \varphi)=1/((m-\xi)^3\theta^3)$ we have the following:

$$\begin{aligned} &K(\bar{T}_1(\xi, \eta, \theta, \varphi))\cdot|J\bar{T}_1(\xi, \eta, \theta, \varphi)| \\ &=\frac{1}{\left(1-\frac{1}{m-\xi}\left(m-\frac{1}{\theta}\right)\right)^3}\cdot\frac{1}{(m-\xi)^3\theta^3} \\ &=K(\xi, \eta, \theta, \varphi)\end{aligned}$$

for each $(\xi, \eta, \theta, \varphi) \in V_{\binom{m}{n}} \times X_{\binom{m}{n}}$.

This means that $K(\xi, \eta, \theta, \varphi)$ is an invariant density function for $\bar{T}_1$. Ergodicity of $\bar{T}_1$ is due to Theorem in [13]. □

COROLLARY 4. *The map T_1 of X has the following invariant density function*:

$$\frac{d\mu_1}{d\lambda} = \begin{cases} \dfrac{2-\theta}{2(1-\theta)^2} & \text{if } \theta < \varphi \\ \dfrac{1}{2(1-\theta)} & \text{if } \theta > \varphi . \end{cases}$$

PROOF. The result is obtained by calculating

$$\iint_D \frac{d\xi d\eta}{(1-\xi\theta)^3} \quad \text{and} \quad \iint_E \frac{d\xi d\eta}{(1-\xi\theta)^3} .$$ □

§5. Definition of the map T_2 and its ergodic properties.

In this section we give another number theoretical algorithm T_2 which is similar to the algorithm T_1. This algorithm gives us an example of number theoretical transformation with finite range structure (not Markov structure).

Let $X = \{(\theta, \varphi); 0 \leqq \theta, \varphi < 1\}$ and functions $a(\theta)$ on $[0, 1]$ and $b(\theta, \varphi)$ on X be defined by

$$a(\theta) = -\left[-\frac{1}{\theta}\right] \quad \text{and} \quad b(\theta, \varphi) = \left[\frac{\varphi}{\theta}\right].$$

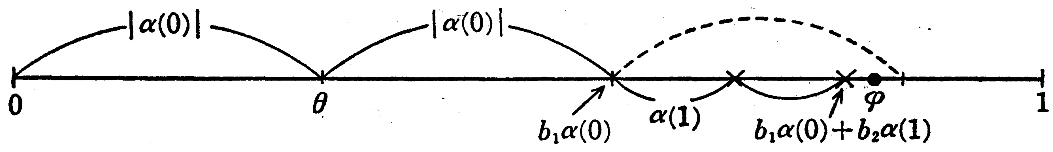
Let us define a second map T_2 of X onto itself by

$$T_2(\theta, \varphi) = \left(a(\theta) - \frac{1}{\theta}, \frac{\varphi}{\theta} - b(\theta, \varphi)\right). \quad \text{(See Figure 3)} .$$

Let $r_n(\theta)$ and $s_n(\theta, \varphi)$ be given by $(r_n(\theta), s_n(\theta, \varphi)) = T_2^n(\theta, \varphi) (n \geqq 0)$ and let $a_n(\theta) = a(r_{n-1}(\theta))$, $b_n(\theta, \varphi) = b(r_{n-1}(\theta), s_{n-1}(\theta, \varphi))$ $(n \geqq 1)$. Then we have the following expansions similar to (12) in §3:

$$\begin{cases} \theta = \dfrac{p_n - r_n(\theta) q_{n-1}}{q_n - r_n(\theta) q_{n-1}} \\ \varphi = \displaystyle\sum_{k=1}^{n} \alpha_{k-1}(\theta) b_k(\theta, \varphi) + \frac{s_n(\theta, \varphi)}{q_n - r_n(\theta) q_{n-1}} , \end{cases}$$

where p_n, q_n are as in §3. The second identity above, which gives the approximation of the real number φ by means of the algorithm T_2, corresponds to the following geometric picture:



Now we put

$$A^2(n)=\left\{\begin{pmatrix} a_1(\theta) & \cdots a_n(\theta) \\ b_1(\theta,\varphi) & \cdots b_n(\theta,\varphi) \end{pmatrix} : (\theta,\varphi)\in X\right\} .$$

We can easily see that each T_2-admissible sequence, $\begin{pmatrix} a_1\cdots a_n \\ b_1\cdots b_n \end{pmatrix}\in A^2(n)$, satisfies the following properties:

(A)'' $a_i\in \boldsymbol{N}$, $b_i\in \boldsymbol{N}\cup\{0\}$, and $a_i>b_i$,

(B)'' if there exists a k $(1\leqq k\leqq n-1)$ such that $a_k-b_k=1$, then $a_{k+1}-b_{k+1}\geqq 2$, and if there exists a j $(1\leqq j\leqq n-k-1)$ such that $a_{k+i}-b_{k+i}=2$ for $1\leqq i\leqq j$, then $a_{k+j+1}-b_{k+j+1}\geqq 2$.

The set of T_2-admissible sequences $A^2(n)$ can be decomposed as follows: $A^2(n)=B_0(n)\cup B_1(n)$, where

$$B_0(n)=\left\{\begin{pmatrix} a_1\cdots a_n \\ b_1\cdots b_n \end{pmatrix}\in A^2(n)\text{: there exists a } k\ (1\leqq k\leqq n) \text{ such that } a_k-b_k>2 \text{ and } a_j-b_j=2 \text{ for } k<j\leqq n\right\}$$

and

$$B_1(n)=\left\{\begin{pmatrix} a_1\cdots a_n \\ b_1\cdots b_n \end{pmatrix}\in A^2(n)\text{; there exists a } k\ (1\leqq k\leqq n) \text{ such that } a_k-b_k=1 \text{ and } a_j-b_j=2 \text{ for } k<j\leqq n\right\} .$$

In particular $B_0(1)=\left\{\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}; a_1-b_1\geqq 2\right\}$ and $B_1(1)=\left\{\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}; a_1-b_1=1\right\}$. With the above definitions and multi-Markov properties, we have

LEMMA 5.1.

(5.1.1) *If* $\begin{pmatrix} a_1\cdots a_{n+1} \\ b_1\cdots b_{n+1} \end{pmatrix}\in B_0(n+1)$ *and* $a_{n+1}-b_{n+1}=2$, *then*

$$\begin{pmatrix} a_1\cdots a_n \\ b_1\cdots b_n \end{pmatrix}\in B_0(n) .$$

(5.1.2) *If* $\begin{pmatrix} a_1 \cdots a_{n+1} \\ b_1 \cdots b_{n+1} \end{pmatrix} \in B_1(n+1)$, *then*

$$\begin{cases} \begin{pmatrix} a_1 \cdots a_n \\ b_1 \cdots b_n \end{pmatrix} \in B_0(n) \text{ when } a_{n+1}-b_{n+1}=1 \\ \begin{pmatrix} a_1 \cdots a_n \\ b_1 \cdots b_n \end{pmatrix} \in B_1(n) \text{ when } a_{n+1}-b_{n+1}=2 . \end{cases}$$

We denote sometimes $\begin{pmatrix} a_1 \cdots a_n \\ b_1 \cdots b_n \end{pmatrix} \in A^2(n)$ by $\begin{pmatrix} a(n) \\ b(n) \end{pmatrix}$ for the sake of simplicity. Let us define for $\begin{pmatrix} a_1 \cdots a_n \\ b_1 \cdots b_n \end{pmatrix} \in A^2(n)$

$$X_{\binom{a(n)}{b(n)}} = X_{\binom{a_1 \cdots a_n}{b_1 \cdots b_n}} = \left\{ (\theta, \varphi) : \begin{pmatrix} a_i(\theta) \\ b_i(\theta, \varphi) \end{pmatrix} = \begin{pmatrix} a_i \\ b_i \end{pmatrix} , \ 1 \leqq i \leqq n \right\} .$$

Then $\left\{ X_{\binom{a(n)}{b(n)}} : \begin{pmatrix} a(n) \\ b(n) \end{pmatrix} \in A^2(n) \right\}$ forms a partition of X. Let $U_0 = X$ and $U_1 = \{(\theta, \varphi) \in X; \theta + \varphi \leqq 1\}$, then we have

LEMMA 5.2.

$$(5.2) \qquad T_2^n X_{\binom{a(n)}{b(n)}} = U_i \quad if \quad \begin{pmatrix} a(n) \\ b(n) \end{pmatrix} \in B_i(n) \quad (i=0, 1) .$$

PROOF. By induction on n. If $n=1$, then by the definition of T_2,

$$T_2 X_{\binom{a_1}{b_1}} = \begin{cases} U_0 & \text{if } a_1 - b_1 \geqq 2 \\ U_1 & \text{if } a_1 - b_1 = 1 . \end{cases}$$

Suppose that (5.2) holds for all $1 \leqq k \leqq n$. Then, if $\begin{pmatrix} a(n+1) \\ b(n+1) \end{pmatrix} \in B_0(n+1)$ it follows from (5.1.1) that

$$\begin{aligned} T_2^{n+1} X_{\binom{a(n+1)}{b(n+1)}} &= T_2 (T_2^n X_{\binom{a(n)}{b(n)}} \cap X_{\binom{a_{n+1}}{b_{n+1}}}) \\ &= \begin{cases} T_2(U_0 \cap X_{\binom{a_{n+1}}{b_{n+1}}}) \text{ or } T_2(U_1 \cap X_{\binom{a_{n+1}}{b_{n+1}}}) & \text{when } a_{n+1} - b_{n+1} > 2 \\ T_2(U_0 \cap X_{\binom{a_{n+1}}{b_{n+1}}}) & \text{when } a_{n+1} - b_{n+1} = 2 \end{cases} \\ &= U_0 . \end{aligned}$$

If $\begin{pmatrix} a(n+1) \\ b(n+1) \end{pmatrix} \in B_1(n+1)$, it follows from (5.1.2) that

$$T_2^{n+1} X_{\binom{a(n+1)}{b(n+1)}} = \begin{cases} T_2(U_0 \cap X_{\binom{a_{n+1}}{b_{n+1}}}) & \text{when } a_{n+1} - b_{n+1} = 1 \\ T_2(U_1 \cap X_{\binom{a_{n+1}}{b_{n+1}}}) & \text{when } a_{n+1} - b_{n+1} = 2 \end{cases}$$

$$= U_1 . \qquad \square$$

For each $\binom{a(n)}{b(n)} \in A^2(n)$, we have a map $\psi_{\binom{a(n)}{b(n)}}$ by

$$\psi_{\binom{a(n)}{b(n)}}(\theta, \varphi) = \left(\frac{p_n - p_{n-1}\theta}{q_n - q_{n-1}\theta}, \sum_{k=1}^{n} b_k \theta \cdot \theta_1 \cdots \theta_{k-1}(\theta) + \frac{\varphi}{q_n - q_{n-1}\theta} \right) .$$

Then we see easily that $\psi_{\binom{a(n)}{b(n)}}$ is the inverse map of T_2^n on $X_{\binom{a(n)}{b(n)}}$. Its Jacobian $J(\psi_{\binom{a(n)}{b(n)}})$ is given by (14) and we can see that the result of Lemma 3.2 holds (the corresponding arguments are analogous to the proof of Lemma 3.2).

Let $\mathscr{D}'_n$ be

$$\left\{ X_{\binom{2\cdots 2}{b_1\cdots b_n}} ; \begin{pmatrix} 2 \cdots 2 \\ b_1 \cdots b_n \end{pmatrix} \in A^2(n) \right\} .$$

Then by (B)″ we see that

$$\mathscr{D}'_n = \{ X_{\binom{2\cdots 2}{b_1\cdots b_n}} ; (b_1 \cdots b_n) = (0 \cdots 010 \cdots 0) \text{ or } (00 \cdots 00) \}$$

and

$$\mathscr{D}_n \subset \mathscr{D}'_n .$$

Therefore we obtain

$$\sum_{X_{\binom{a(n)}{b(n)}} \in \mathscr{D}'_n} \lambda(X_{\binom{a(n)}{b(n)}}) = \sum_{\substack{b_i = 1 \\ \binom{a(n)}{b(n)} \in \mathscr{D}'_n}} \int_{\psi_{\binom{a(n)}{b(n)}}(U_1)} d\lambda + \int_{\psi_{\binom{2\cdots 2}{0\cdots 0}}(U_0)} d\lambda .$$

LEMMA 5.3.

(5.3.1) $$\sum_{X_{\binom{a(n)}{b(n)}} \in \mathscr{D}_n} \lambda(X_{\binom{a(n)}{b(n)}}) \longrightarrow 0 \quad \textit{as} \quad n \longrightarrow \infty .$$

(5.3.2) *The partition* $\left\{ X_{\binom{a}{b}} ; \binom{a}{b} \in A^2(1) \right\}$ *is a generator with respect to the map* T_2.

The proof of Lemma 5.3 is similar to that of Corollary 3.

THEOREM 5. *The map T_2 on X is ergodic with respect to λ and admits a σ-finite invariant measure $\mu_2 \sim \lambda$.*

§6. Natural extension of the map T_2 and its invariant density function.

In this section, we construct a natural extension $\bar{T}_2$ of the map T_2 and derive the invariant density function for T_2 explicitly. Since the properties of T_2-admissibility are more complicated than those of T_1, we need some lemmas about T_2-admissibility.

Let $B_{i,j}$ $(i, j=1, 2)$ be subsets of $\bigcup_{n=1}^{\infty} A^2(n)$ defined as follows:

$$B_{11}=\left\{\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \in A^2(1):\ a_1-b_1>2\right\} ,$$

$$B_{12}=\bigcup_{k=1}^{\infty}\left\{\begin{pmatrix} a_1\cdots a_{k+1} \\ b_1\cdots b_{k+1} \end{pmatrix} \in A^2(k+1): a_j-b_j=2,\ 1\leqq j\leqq k \text{ and } a_{k+1}-b_{k+1}>2\right\} ,$$

$$B_{21}=\left\{\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \in A^2(1): a_1-b_1=1\right\} ,$$

$$B_{22}=\bigcup_{k=1}^{\infty}\left\{\begin{pmatrix} a_1\cdots a_{k+1} \\ b_1\cdots b_{k+1} \end{pmatrix} \in A^2(k+1): a_j-b_j=2,\ 1\leqq j\leqq k \text{ and } a_{k+1}-b_{k+1}=1\right\} .$$

We put $B_1=B_{11}\cup B_{12}$ and $B_2=B_{21}\cup B_{22}$.

Lemma 6.1. *Under the above notations, we have the following:*

(6.1.1) *if* $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \in B_{11}$, *then* $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}\cdot \boldsymbol{c}$ *is admissible for all* $\boldsymbol{c}\in B_1\cup B_2$,

(6.1.2) *if* $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \in B_{21}$, *then* $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}\cdot \boldsymbol{c}$ *is admissible for all* $\boldsymbol{c}\in B_1$,

(6.1.3) *if* $\begin{pmatrix} a_1\cdots a_{k+1} \\ b_1\cdots b_{k+1} \end{pmatrix} \in B_{12}$, *then* $\begin{pmatrix} a_2\cdots a_{k+1} \\ b_2\cdots b_{k+1} \end{pmatrix} \in \begin{cases} B_{11} & \text{if } k=1 \\ B_{12} & \text{if } k\geqq 2 , \end{cases}$

(6.1.4) *if* $\begin{pmatrix} a_1\cdots a_{k+1} \\ b_1\cdots b_{k+1} \end{pmatrix} \in B_{22}$, *then* $\begin{pmatrix} a_2\cdots a_{k+1} \\ b_2\cdots b_{k+1} \end{pmatrix} \in \begin{cases} B_{21} & \text{if } k=1 \\ B_{22} & \text{if } k\geqq 2 . \end{cases}$

Let $D=\{(\xi, \eta): 0\leqq \xi, \eta<1\}$, $E=\{(\xi, \eta): 0\leqq \xi, \eta<1 \text{ and } \xi+\eta<1\}$ and define a map S on D by

$$S(\xi, \eta)=\left(-\frac{1}{\xi}, \frac{\eta}{\xi}\right) .$$

Then $S(D)=\{(\xi, \eta): \xi<-1, \eta\geqq 0 \text{ and } \xi+\eta<0\}$, $S(E)=\{(\xi, \eta): \xi<-1, \eta\geqq 0$ and $\xi+\eta<-1\}$. For both $S(D)$ and $S(E)$, let us consider the partitions consisting of the basic neighbourhoods of integer vectors (m, n). Namely, let the partition $\{D_{mn}: m-1\geqq n\geqq 0\}$ of $S(D)$ be defined by

$$D_{mn}=\begin{cases} X+(-m, n) & \text{if } m-1>n\geqq 0 \\ U_1+(-m, n) & \text{if } m-1=n \end{cases},$$

and the partition $\{E_{mn}: m-2\geqq n\geqq 0\}$ of $S(E)$ be defined by

$$E_{mn}=\begin{cases} X+(-m, n) & \text{if } m-2>n\geqq 0 \\ U_1+(-m, n) & \text{if } m-2=n \end{cases}.$$

Next we define maps $\varphi^{(i)}_{\binom{m}{n}}$ $(i=1, 2)$ by $\varphi^{(i)}_{\binom{m}{n}}(\xi, \eta)=(1/(m-\xi), (n+\eta)/(m-\xi))$ for each (m, n) where the domains and ranges of $\varphi^{(i)}_{\binom{m}{n}}$ are as follows:

$$\begin{cases} \varphi^{(1)}_{\binom{m}{n}}: D\longrightarrow D & \text{if } m-n>1 , \\ \varphi^{(1)}_{\binom{m}{n}}: E\longrightarrow D & \text{if } m-n=1 , \end{cases}$$

$$\begin{cases} \varphi^{(2)}_{\binom{m}{n}}: D\longrightarrow E & \text{if } m-n>2 , \\ \varphi^{(2)}_{\binom{m}{n}}: E\longrightarrow E & \text{if } m-n=2 . \end{cases} \qquad \text{(See Figure 4)} .$$

We denote $\varphi^{(1)}_{\binom{m}{n}}(D)$ and $\varphi^{(1)}_{\binom{m}{n}}(E)$ by $Y_{\binom{m}{n}}$ and $\varphi^{(2)}_{\binom{m}{n}}(D)$ and $\varphi^{(2)}_{\binom{m}{n}}(E)$ by $Z_{\binom{m}{n}}$. Then $Y_{\binom{m}{n}}$ and $Z_{\binom{m}{n}}$ satisfy the following:

$$S(Y_{\binom{m}{n}})=D_{mn} , \qquad S(Z_{\binom{m}{n}})=E_{mn} .$$

That is, $\{Y_{\binom{m}{n}}: m-1\geqq n\geqq 0\}$ and $\{Z_{\binom{m}{n}}: m-2\geqq n\geqq 0\}$ give the partitions of D and E, respectively.

Now we consider a set M which will be the domain of the natural extension $\bar{T}_2$: $M=D\times U_1\cup E\times U_2$, where $U_2=X\setminus U_1$.

Let

$$\mathscr{U}^{(1)}_1=\{X_{a\cdot b}: a\in B_{11} \text{ and } \boldsymbol{b}\in B_1\cup B_2\}\cup\{X_a: \boldsymbol{a}\in B_{12}\} ,$$

$$\mathscr{U}^{(1)}_2=\{X_a: \boldsymbol{a}\in B_1\} ,$$

$$\mathscr{U}^{(2)}_1=\{X_{a\cdot b}: a\in B_{21} \text{ and } \boldsymbol{b}\in B_1\}\cup\{X_a: \boldsymbol{a}\in B_{22}\} ,$$

and

$$\mathscr{U}^{(2)}_2=\{X_a: \boldsymbol{a}\in B_2\} .$$

It is easy to see that $\mathscr{U}^{(j)}_i$ $(i=1, 2)$ are partitions of U_j and $\mathscr{U}^{(j)}_1$ is a refinement of $\mathscr{U}^{(j)}_2$. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be the partitions of $D\times U_1$ and $E\times U_2$, respectively, given as follows:

$$\mathscr{P}_1=\{D\times v;\ v\in\mathscr{U}_1^{(1)}\}\ ,\qquad \mathscr{P}_2=\{E\times v;\ v\in\mathscr{U}_1^{(2)}\}\ .$$

Define a partition $\mathscr{P}$ on M given by

$$\mathscr{P}=\mathscr{P}_1\cup\mathscr{P}_2\ .$$

And let $\mathscr{Q}_1$ and $\mathscr{Q}_2$ be another partition of $D\times U_1$ and $E\times U_2$, respectively, given as follows:

$$\mathscr{Q}_1=\{Y_{\binom{m}{n}}\times v;\ v\in\mathscr{U}_2^{(1)},\ m-1\geqq n\geqq 0\}\ ,$$

$$\mathscr{Q}_2=\{Z_{\binom{m}{n}}\times v;\ v\in\mathscr{U}_2^{(2)},\ m-2\geqq n\geqq 0\}\ .$$

Let $\mathscr{Q}=\mathscr{Q}_1\cup\mathscr{Q}_2$. We denote elements of $\mathscr{P}$, $D\times X_{\binom{a_1\cdots a_l}{b_1\cdots b_l}}$ or $E\times X_{\binom{a_1\cdots a_l}{b_1\cdots b_l}}$ by $\Delta_{\binom{a_1\cdots a_l}{b_1\cdots b_l}}$ and elements of $\mathscr{Q}$, $Y_{\binom{a_1}{b_1}}\times X_{\binom{a_2\cdots a_l}{b_2\cdots b_l}}$ or $Z_{\binom{a_l}{b_l}}\times X_{\binom{a_2\cdots a_l}{b_2\cdots b_l}}$ by $\Gamma_{\binom{a_1\cdots a_l}{b_1\cdots b_l}}$. Now define the map $\bar{T}_2$ on M by

$$\begin{aligned}\bar{T}_2(\xi,\eta,\theta,\varphi)&=(\varphi^{(4)}_{\binom{a_1}{b_1}}(\xi,\eta),\ T_2(\theta,\varphi))\\&=\left(\frac{1}{a_1-\xi},\ \frac{b_1+\eta}{a_1-\xi},\ a_1-\frac{1}{\theta},\ \frac{\varphi}{\theta}-b_1\right)\end{aligned}$$

for $(\xi,\eta,\theta,\varphi)\in\Delta_{\binom{a_1\cdots a_l}{b_1\cdots b_l}}$ such that $\begin{pmatrix}a_2\cdots a_l\\ b_2\cdots b_l\end{pmatrix}\in B_i$. It is easy to verify that for $\Delta_{\binom{a_1\cdots a_l}{b_1\cdots b_l}}$ such that $\begin{pmatrix}a_2\cdots a_l\\ b_2\cdots b_l\end{pmatrix}\in B_1$, $\bar{T}_2(\Delta_{\binom{a_1\cdots a_l}{b_1\cdots b_l}})=Y_{\binom{a_1}{b_1}}\times X_{\binom{a_2\cdots a_l}{b_2\cdots b_l}}\in\mathscr{Q}_1$ and for $\Delta_{\binom{a_1\cdots a_l}{b_1\cdots b_l}}$ such that $\begin{pmatrix}a_2\cdots a_l\\ b_2\cdots b_l\end{pmatrix}\in B_2$, $\bar{T}_2(\Delta_{\binom{a_1\cdots a_l}{b_1\cdots b_l}})=Z_{\binom{a_1}{b_1}}\times X_{\binom{a_2\cdots a_l}{b_2\cdots b_l}}\in\mathscr{Q}_2$. Thus $\bar{T}_2$ is a one to one and onto map from $\Delta_{\binom{a_1\cdots a_l}{b_1\cdots b_l}}$ to $\Gamma_{\binom{a_1\cdots a_l}{b_1\cdots b_l}}$, that is, the map $\bar{T}_2$ is a natural extension of T_2.

THEOREM 6. *The natural extension $\bar{T}_2$ defined above has an invariant measure $\bar{\mu}$ such that*

$$\frac{d\bar{\mu}}{d\bar{\lambda}}=\frac{1}{(1-\xi\theta)^3}\ ,$$

where $\bar{\lambda}$ is the Lebesgue measure on M; furthermore, $\bar{T}_2$ is ergodic.

COROLLARY 6. *The map T_2 of X has the following invariant density function:*

$$\frac{d\mu_2}{d\lambda}=\begin{cases}\dfrac{2-\theta}{2(1-\theta)^2} & \text{if }\ \theta+\varphi<1\\[2ex] \dfrac{1}{2(1-\theta)} & \text{if }\ \theta+\varphi>1\ .\end{cases}$$

The proofs of Theorem 6 and Corollary 6 are similar to the proofs of Theorem 4 and Corollary 4 and hence are omitted.

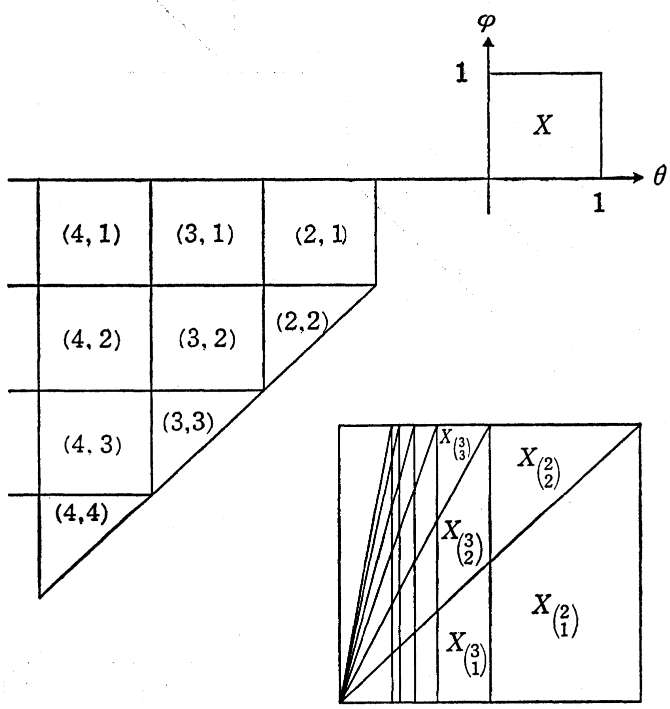


FIGURE 1

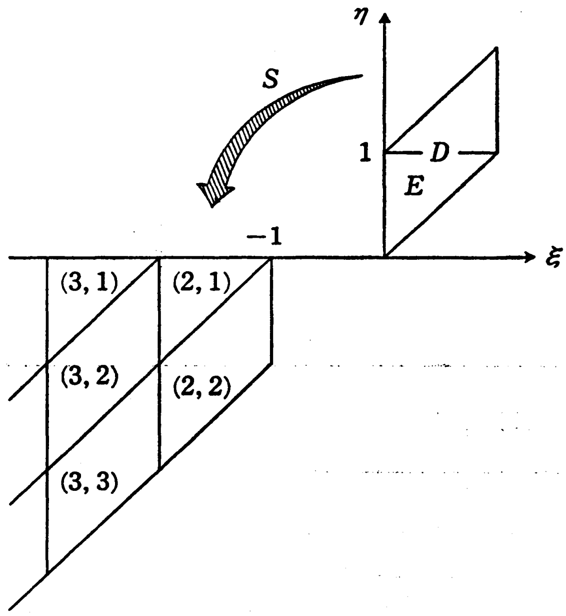


FIGURE 2

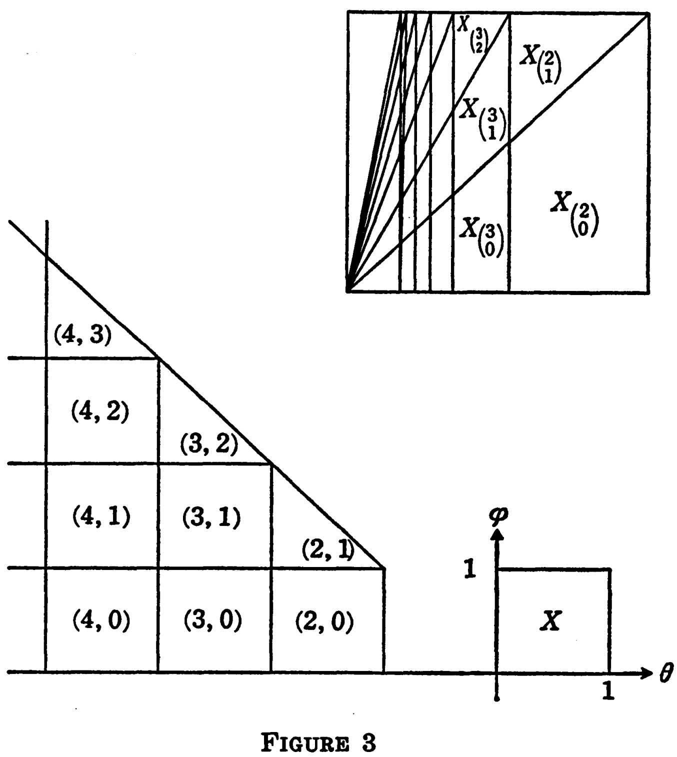


FIGURE 3

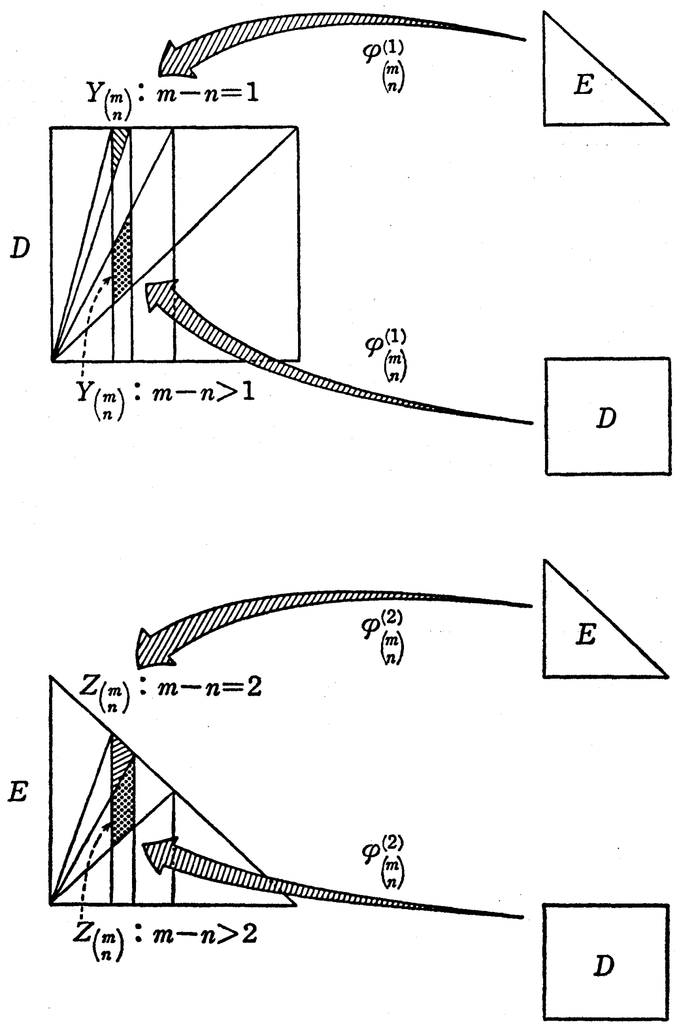

$\varphi^{(1)}_{\binom{m}{n}}$
E
$Y_{\binom{m}{n}}$: $m-n=1$
D
$\varphi^{(1)}_{\binom{m}{n}}$
$Y_{\binom{m}{n}}$: $m-n>1$
D
$\varphi^{(2)}_{\binom{m}{n}}$
E
$Z_{\binom{m}{n}}$: $m-n=2$
E
$\varphi^{(2)}_{\binom{m}{n}}$
$Z_{\binom{m}{n}}$: $m-n>2$
D

FIGURE 4

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