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Stable Harmonic Maps from Riemann Surfaces to Compact Hermitian Symmetric Spaces

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Introduction

A harmonic map f from a compact Riemannian manifold N to a Riemannian manifold M is called *stable* if the second variation of the energy is nonnegative for every variation of the map f. A beautiful class of compact Kaehler manifolds is a class of irreducible Hermitian symmetric spaces of compact type ($P_m(C)$, $Q_m(C)$, $G_{p,q}(C)$, $Sp(k)/U(k)$, $SO(2k)/U(k)$, $E_6/Spin(10)\cdot T^1$, $E_7/E_6\cdot T^1$). We consider stable harmonic maps from or to compact Hermitian symmetric spaces. In this note we show the following.

THEOREM. *Let M be a compact irreducible Hermitian symmetric space of complex dimension m and Σ be a compact Riemann surface. Then any stable harmonic map f from Σ to M is holomorphic or anti-holomorphic.*

In the case where Σ is a Riemann sphere, the above result was obtained by Siu [S-1, Z] (see also [B-R-S]). In the case where M is a complex projective space and $|\deg f|\geqq m(p-1)/(m+1)$ where p denotes the genus of Σ, the above result was obtained by Eells and Wood [E-W]. They used algebraic geometric arguments (theorems of Riemann-Roch and Grothendieck). Recently, by using a twistor space over the domain manifold, Burns and de Bartolomeis have shown the above result in the case where M is a complex projective space (cf. Remark 6 of [B-R-S]). We show the above theorem by a simple argument for the second variations used in [L-S].

1. Proof of Theorem.

Let $f: N\rightarrow M$ be a harmonic map from an n-dimensional compact

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Riemannian manifold N to a Riemannian manifold M. Let $f^{-1}TM$ be the pull-back vector bundle of the tangent bundle TM by f. We denote by $\langle\ ,\ \rangle$ and ∇ the induced inner product and the induced connection of $f^{-1}TM$. The metric $\langle\ ,\ \rangle$ extends to the complexified tangent space as a complex bilinear form $(\ ,\ )$ or a Hermitian inner product $\langle\!\langle\ ,\ \rangle\!\rangle$. The curvature tensor R^M of M is defined by $R^M(X,Y)=\nabla_{[X,Y]}-[\nabla_X, \nabla_Y]$. By the second variational formula for harmonic maps (cf. [E-L]), for any variation f_t of $f=f_0$ with the variational vector field $V=(\partial/\partial t)f_t|_{t=0}$ on $C^\infty(f^{-1}TM)$, the second variation for energy is given as follows:

$$\frac{\partial^2}{\partial t^2}E(f_t)|_{t=0}=\int_N \langle \mathscr{J}_f(V), V\rangle dv_N\ ,$$

$$\mathscr{J}_f(V)=-\sum_{i=1}^n \nabla^2_{e_i,e_i}V-\sum_{i=1}^n R^M(df(e_i), V)df(e_i)\ .$$

Here $\{e_i\}$ is an orthonormal basis at the tangent space of N.

Suppose that M is a compact Hermitian symmetric space of complex dimension m and with the complex structure J. Let $\mathfrak{H}$ be the Lie algebra of all holomorphic vector fields on M and $\mathfrak{K}$ the Lie algebra of all Killing vector fields on M. Then, by the theorem of Matsushima, we have a decomposition $\mathfrak{H}=\mathfrak{K}+J\mathfrak{K}$. Since M is a symmetric space, we can define an ad($\mathfrak{K}$)-inner product on $\mathfrak{K}$ compatible with the symmetric metric of M. Any $V\in\mathfrak{K}$ satisfies $\langle\nabla_X V, Y\rangle=-\langle X, \nabla_Y V\rangle$ and $\nabla^2 V(X,Y)=R^M(V,Y)X$. We define a quadratic form Q_f on $\mathfrak{K}$ by

$$Q_f(V)=\int_N \langle \mathscr{J}_f(JV), JV\rangle dv_N$$

for any $V\in\mathfrak{K}$. We deform the harmonic map f along the holomorphic vector field JV and compute its second variation. For any $V\in\mathfrak{K}$, we compute

$$\begin{aligned}(1.1)\qquad \mathscr{J}_f(JV)&=-\sum_{i=1}^n \nabla^2_{e_i,e_i}JV-\sum_{i=1}^n R^M(df(e_i), JV)df(e_i)\\ &=-\sum_{i=1}^n (\nabla^2 JV)(e_i, e_i)-\sum_{i=1}^n R^M(df(e_i), JV)df(e_i)\\ &=\sum_{i=1}^n (JR^M(df(e_i), V)df(e_i)-R^M(df(e_i), JV)df(e_i))\ .\end{aligned}$$

Next we take the trace of Q_f on $\mathfrak{K}$ with respect to the inner product. We have

$$(1.2)\qquad \operatorname{trace} Q_f=\int_N \sum_{i=1}^n \sum_{\alpha=1}^{2m} (\langle JR^M(df(e_i), v_\alpha)df(e_i), Jv_\alpha\rangle$$

$$-\langle R^M(df(e_i), Jv_\alpha)df(e_i), Jv_\alpha\rangle)dv_N$$
$$=0\ ,$$

where $\{v_\alpha\}$ denotes an orthonormal basis at the tangent space of M.

Suppose that f is a stable harmonic map. By (1.2) and the stability of f, we have $Q_f(V)=\int_N \langle \mathcal{J}_f(JV), JV\rangle dv_N=0$ for any $V\in\mathfrak{K}$. Since $\mathcal{J}$ has no negative eigenvalue, it follows that, for all $V\in\mathfrak{K}$, $\mathcal{J}_f(JV)$ vanishes identically along f. Hence by (1.1) we get the following equation which any stable harmonic map f to a compact Hermitian symmetric space satisfies: For any $X\in T_{f(p)}M$ and $p\in N$,

$$\sum_{i=1}^{n}(JR^M(df(e_i), X)df(e_i)-R^M(df(e_i), JX)df(e_i))=0\ . \tag{1.3}$$

Now we recall a curvature operator acting on the symmetric square $T^{1,0}M\cdot T^{1,0}M$ of the (1, 0)-tangent space of a Kaehler manifold M. The curvature operator $\mathcal{Q}$ is defined by

$$\langle\!\langle \mathcal{Q}(X\cdot Y), Z\cdot W\rangle\!\rangle=(R^M(X, \bar{Z})Y, \bar{W})$$

for $X, Y, Z, W\in T^{1,0}M$. Then we can express (1.3) in terms of $\mathcal{Q}$ as follows:

PROPOSITION 1.

$$\mathcal{Q}\Big(\sum_{i=1}^{n} d^{1,0}f(e_i)\cdot d^{1,0}f(e_i)\Big)=0\ , \tag{1.4}$$

at any $f(p)$, $p\in N$. *Here* $d^{1,0}f(X)$ *denotes the* (1, 0)*-component of* $df(X)$.

When M is a Hermitian symmetric space, the eigenvalues of the curvature operator $\mathcal{Q}$ were determined by Calabi-Vesentini [C-V], Borel [B]. Itoh [I] studied properties of the curvature operator for Kaehlerian C-spaces. According to their results we know that a Hermitian symmetric space is irreducible if and only if its curvature operator $\mathcal{Q}$ has no zero-eigenvalue.

Moreover suppose that M is an irreducible Hermitian symmetric space of compact type. By the above fact and (1.4), we have

$$\sum_{i=1}^{n}(d^{1,0}f(e_i)\cdot d^{1,0}f(e_i))=0\ . \tag{1.5}$$

If N is a compact Riemann surface Σ, then (1.5) becomes

$$\Big(\sum_{\alpha=1}^{m} f_1^\alpha u_\alpha\Big)\cdot\Big(\sum_{\beta=1}^{m} f_1^\beta u_\beta\Big)=0\ . \tag{1.6}$$

Here $\{u_\alpha\}$ is a unitary basis at $f(p)\in M$, $\{z\}$ is a local complex coordinate of Σ, and $f_1^\alpha = \langle\langle d^{1,0}f(\partial/\partial z), u_\alpha\rangle\rangle$, $f_{\bar 1}^\alpha = \langle\langle d^{1,0}f(\partial/\partial \bar z), u_\alpha\rangle\rangle$. (1.6) is equivalent to

$$f_1^\alpha f_{\bar 1}^\alpha = 0 \qquad \text{for } \alpha = 1, \cdots, m, \tag{1.7}$$

$$f_1^\alpha f_{\bar 1}^\beta + f_1^\beta f_{\bar 1}^\alpha = 0 \qquad \text{for } \alpha \neq \beta, \quad \alpha, \beta = 1, \cdots, m. \tag{1.8}$$

By (1.7), (1.8) and the smoothness of f, there is an open subset of Σ on which f is holomorphic or antiholomorphic. By Aronszajn's unique continuation theorem (cf. [E-L]), f is holomorphic or antiholomorphic. Therefore we obtain Theorem.

2. On stable harmonic maps from Hermitian symmetric spaces.

In this section we give an equation which any stable harmonic map from a compact Hermitian symmetric space to a Riemannian manifold satisfies. This equation seems to be useful to show the holomorphicity of a compact Hermitian symmetric space to a specific Kaehler manifold.

We recall another curvature operator of a Kaehler manifold acting on (1, 1)-forms. The curvature operator $\mathscr{R}: \wedge^2 TM \to \wedge^2 TM$ is defined by $\langle \mathscr{R}(\omega_i \wedge \omega_j), \omega_k \wedge \omega_l\rangle = \langle R(e_i, e_j)e_k, e_l\rangle$, where $\{e_i\}$ is an orthonormal basis of T_xM and $\{\omega_i\}$ is its dual basis. Given a vector bundle E over M, we can extend the curvature operator $\mathscr{R}$ to a linear operator $\mathscr{R}: (\wedge^2 TM)\otimes E \to (\wedge^2 TM)\otimes E$ in a natural manner. We denote also by $\mathscr{R}$ its complex extension. We have a decomposition $\wedge^2 TM^C = \wedge^{(2,0)} TM + \wedge^{(1,1)} TM + \wedge^{(0,2)} TM$ via the complex structure of M. By the Kaehler identity, we have $\mathscr{R}(\wedge^{(2,0)} TM) = \mathscr{R}(\wedge^{(0,2)} TM) = 0$. We denote by $\mathscr{R}^{(1,1)}$ the restriction of $\mathscr{R}$ to $\wedge^{(1,1)} TM$. When M is a Hermitian symmetric space, the operator $\mathscr{R}^{(1,1)}$ is nonnegative. According to the theorem of Siu-Yau [S-Y], we know that if the operator $\mathscr{R}^{(1,1)}$ is positive, M is biholomorphic to a complex projective space $P_m(C)$.

Let M be a complex m-dimensional compact Hermitian symmetric space with the complex structure J. We use the same notation as in Section 1. Let $f: M \to N$ be a harmonic map from M to a Riemannian manifold N. By (1.4) of [O], for $V \in \mathfrak{R}$, we have

$$\begin{aligned} \mathscr{J}(df(JV)) &= -df\Big(\sum_{i=1}^{2m} \nabla^2_{e_i, e_i} JV + \mathrm{Ric}^M(JV)\Big) \\ &\quad -2\sum_{i=1}^{2m} (\nabla df)(e_i, \nabla^M_{e_i} JV) \\ &= -df\Big(\sum_{i=1}^{2m} JR^M(V, e_i)e_i + \mathrm{Ric}^M(JV)\Big) \\ &\quad -2\sum_{i=1}^{2m} (\nabla df)(e_i, \nabla^M_{e_i} JV) \end{aligned} \tag{2.1}$$

$$= -df(-J\,\mathrm{Ric}^M(V) + \mathrm{Ric}^M(JV))$$
$$-2\sum_{i=1}^{2m} (\nabla df)(e_i, \nabla^M_{e_i} JV)$$
$$= -2\sum_{i,j=1}^{2m} \langle J\nabla^M_{e_i} V, e_j\rangle (\nabla df)(e_i, e_j)\ .$$

We define a quadratic form Q^f on $\mathfrak{K}$ associated with f as follows:

$$Q^f(V) = \int_M \langle \mathscr{J}_f(df(JV)), df(JV)\rangle dv_M\ .$$

By (2.1) we have

$$Q^f(V) = -2\int_M \sum_{i,j=1}^{2m} \langle J\nabla^M_{e_i} V, e_j\rangle \langle (\nabla df)(e_i, e_j), df(JV)\rangle dv_M\ .$$

Hence we get $\mathrm{trace} Q^f = 0$.

Suppose that f is stable. Then for any $V \in \mathfrak{K}$ we get $\mathscr{J}_f(df(JV)) = 0$. It follows from (2.1) that

$$\sum_{i,j=1}^{2m} \langle J\nabla^M_{e_i} V, e_j\rangle (\nabla df)(e_i, e_j) = 0 \tag{2.2}$$

for any $x \in M$ and any $V \in \mathfrak{K}_x$. Here $\mathfrak{K}_x = \{V \in \mathfrak{K}; V_x = 0\}$ is the Lie algebra of the isotropy subgroup at x of the isometry group. Let $\mathfrak{K} = \mathfrak{K}_x + \mathfrak{m}$ be the canonical decomposition of $\mathfrak{K}$ as a symmetric space. Identifying $\mathfrak{m}$ with $T_x M$, we have $\nabla^M V = -\mathrm{ad}\, V$ for $V \in \mathfrak{K}_x$. Hence (2.2) becomes

$$\sum_{i,j=1}^{2m} \langle JR^M(X, Y)e_i, e_j\rangle (\nabla df)(e_i, e_j) = 0 \tag{2.3}$$

for any $x \in M$ and any $X, Y \in T_x M$. We can write the equation (2.3) in terms of the curvature operator $\mathscr{R}$ as follows:

PROPOSITION 2. *Let M be a compact Hermitian symmetric space and $f: M \to N$ be a stable harmonic map from M to a Riemannian manifold N. Then the second fundamental form of the map f satisfies the following equation*

$$\mathscr{R}^{(1,1)}\Big(\sum_{i,j=1}^{m} (\nabla df)\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\Big) dz_i \wedge d\bar{z}_j\Big) = 0\ .$$

We know that a compact Hermitian symmetric space is a complex projective space if and only if the kernel of $\mathscr{R}^{(1,1)}$ is zero. Therefore by the above proposition we get that any stable harmonic map from a complex projective space with the Fubini-Study metric to a Riemannian manifold is pluriharmonic (cf. [O]).

REMARK. At a conference of Osaka university on May 15-16 in 1987, Professor J. Eells told us that our theorem was also proved by Burstall, Rawnsley, Burns and de Bartolomeis at the same time as the authors did it.

References

[B-R-S] F. BURSTALL, J. RAWNSLEY and S. SALAmon, Stable harmonic 2-spheres in symmetric spaces, preprint.

[B] A. BOREL, On the curvature tensor of the Hermitian symmetric manifolds, Ann. of Math., **71** (1960), 508-521.

[C-V] E. CALABI and E. VESENTINI, On compact, locally symmetric Kähler manifolds, Ann. of Math., **71** (1960), 472-507.

[E-L] J. EELLS and L. LEMAIRE, Selected Topics in Harmonic Maps, CBMS Regional Conference Series in Math., **50**, Amer. Math. Soc., 1983.

[E-W] J. EELLS and J. C. WOOD, Harmonic maps from surfaces to complex projective spaces, Advances in Math., **49** (1983), 217-263.

[I] M. ITOH, On curvature properties of Kähler C-spaces, J. Math. Soc. Japan, **30** (1978), 39-71.

[L-S] H. B. LAWSON, JR. and J. SIMONS, On stable currents and their application to global problems in real and complex geometry, Ann. of Math., **98** (1973), 427-450.

[O] Y. OHNITA, Pluriharmonicity of stable harmonic maps, J. London Math. Soc., **35** (1987), 563-568.

[S-1] Y.-T. SIU, Curvature characterization of hyperquadrics, Duke Math. J., **47** (1980), 641-654.

[S-2] Y.-T. SIU, Strong rigidity of compact quotients of exceptional bounded symmetric domains, Duke Math. J., **48** (1981), 857-871.

[S-Y] Y.-T. SIU and S.-T. YAU, Compact Kähler manifolds of positive bisectional curvature, Invent. Math., **59** (1980), 189-204.

[U-1] S. UDAGAWA, Minimal immersions of Kaehler manifolds into complex space forms, Tokyo J. Math., **10** (1987), 227-239.

[U-2] S. UDAGAWA, Pluriharmonic maps and minimal immersions of Kaehler manifolds, to appear in J. London Math. Soc.

[U-3] S. UDAGAWA, Holomorphicity of certain stable harmonic maps and minimal immersions, preprint.

[Z] J.-Q. ZHONG, The degree of strong nondegeneracy of the bisectional curvature of exceptional bounded symmetric domains, Several Complex Variables, Proc. 1981 Hangzhou Conf., 1984, 127-139.

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