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On the Number of Cusps of Stable Perturbations of a Plane-to-Plane Singularity

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Introduction

Let $f: (\boldsymbol{R}^2, O) \to (\boldsymbol{R}^2, O)$ be a smooth map germ. By the theorem of Whitney [Wh], f can be approximated (in the semi-local sense) by a C^∞ stable mapping. In other words, f is a "degeneration" of neighboring stable mappings, which we call stable perturbations of f. Then it will be natural to expect that stable perturbations have several properties in common reflecting the structure of the "generating" map-germ f.

In this paper we concentrate on investigating the number of cusps of stable perturbations of a generic plane-to-plane singularity. For instance, we observe that the number $\kappa(\tilde{f})$ modulo 2 of cusps of a stable perturbation $\tilde{f}$ of a generic map-germ $f: (\boldsymbol{R}^2, O) \to (\boldsymbol{R}^2, O)$ is a topological invariant of f (Theorem 2.4). In fact $\kappa(\tilde{f}) \bmod 2$ is determined by the number of branches of the locus of critical points of f and the mapping degree of f (Theorem 2.1). Thus if two generic map-germs $f, g: (\boldsymbol{R}^2, O) \to (\boldsymbol{R}^2, O)$ are topologically equivalent, then the parities of $\kappa(\tilde{f})$ and $\kappa(\tilde{g})$ are coincident for any stable perturbations $\tilde{f}$ of f and $\tilde{g}$ of g.

This observation is obtained as an application of a global formula for singularities of maps between oriented 2-manifolds with boundary (Theorem 1.1), which is a modified form of Quine's formula [Q]. The topological invariant $\kappa(\tilde{f}) \bmod 2$ is algebraically calculable from f (Theorem 2.2).

In §1, our global formula is proved from Quine's formula. In §2, the genericity condition is explained and $\kappa(\tilde{f}) \bmod 2$ is investigated for stable perturbations $\tilde{f}$ of a generic map-germ $f: (\boldsymbol{R}^2, O) \to (\boldsymbol{R}^2, O)$. Another restriction for the number $\kappa(f_t)$ of cusps near the origin for a deformation $\{f_t\}$ of f is obtained in §3, using complex analytic geometry.

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Throughout this paper we use the following notations:

$C(f)$; the set (resp. set-germ) of critical points of a smooth mapping (resp. map-germ) f. (A point x is a critical point of f if the tangent mapping $T_x f$ is not surjective.)

$\deg f$; the mapping degree of a mapping (or map-germ) f.

$D_\delta^2 = \{(x_1, x_2) \in \boldsymbol{R}^2 \mid x_1^2 + x_2^2 \leqq \delta^2\}$.

$E_n = \{$smooth function germs: $(\boldsymbol{R}^n, 0) \to \boldsymbol{R}\}$.

$E(n, p) = \{$smooth map germs: $(\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^p, 0)\}$.

$Jf = J(f_1, f_2)$; the Jacobian determinant of a map-germ
$f = (f_1, f_2)$: $(\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$.

$\kappa(f)$; the number of cusps of a C^∞ stable mapping f.

$\chi(X)$; the Euler-Poincaré characteristic of a topological space X.

$\langle a, b, \cdots \rangle$; the ideal generated by $a, b, \cdots$.

$\sharp X$; the cardinal number of a set X.

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§1. A global formula.

On the number modulo 2 of cusps of a stable mapping f of a closed surface M into another surface N, there is a classical result that $\kappa(f) \equiv \chi(M) + \deg f (\mathrm{mod}\ 2) \chi(N)$ modulo 2 ([Th], [Wh], [L]). In this section we investigate the number modulo 2 of cusps of a stable mapping between compact oriented surfaces with boundary.

Let M (resp. N) be a compact oriented connected surface with boundary ∂M (resp. ∂N) and $f: M \to N$ be a C^∞ stable mapping such that $f^{-1}(\partial N) = \partial M$ and that $f|\partial M: \partial M \to \partial N$ is a C^∞ stable mapping (i.e., of Morse type).

After Quine [Q], we denote by M^- the closure in M of the set of regular points at which f is orientation-reversing, and, for each cusp point $q \in M$ of f, denote by $\mu(q)$ the local degree of $f: (M, q) \to (N, f(q))$. We set

$$\mathrm{cusp}\deg f = \sum \mu(q) ,$$

where the summation runs over all cusp points of f.

THEOREM 1.1. *Let M, N and f be as above with $\partial M \neq \emptyset$. Then*

$$\chi(M)-2\chi(M^-)+\frac{1}{2}\#(C(f\,|\,\partial M))+\text{cusp deg}\,f=(\deg f\,|\,\partial M)\chi(N)\ .$$

As cusp deg f is congruent to the number $\kappa(f)$ of cusps of f modulo 2, we have

COROLLARY 1.2. *Let M, N and f be as in Theorem* 1.1. *Then*

$$\kappa(f)\equiv\chi(M)+\deg(f\,|\,\partial M)\chi(N)+\frac{1}{2}\#(C(f\,|\,\partial M))\qquad \text{mod}\ 2\ .$$

Especially $\kappa(f)$ mod 2 *depends only on the topology of M, N and $f\,|\,\partial M$.*

PROOF OF THEOREM 1.1. As $f\,|\,\partial M$ is of Morse type, there exist collars

$$i:\partial M\times[0,1)\longrightarrow M\ ,\qquad j:\partial N\times[0,1)\longrightarrow N$$

such that $f(i(\partial M\times[0,1)))\subset j(\partial N\times[0,1))$ and that $j^{-1}\circ f\circ i:\partial M\times[0,1)\to\partial N\times[0,1)$ is equal to $(f\,|\,\partial M)\times\text{id}_{[0,1)}$.

Take two copies M_1, M_2 (resp. N_1, N_2) of M (resp. N), and make the double $\tilde{M}$ of M (resp. $\tilde{N}$ of N);

$$\tilde{M}=M_1\cup_{\text{id}_{\partial M}}M_2\qquad\text{with respect to the collar } i$$

(the same for $\tilde{N}$ with respect to j).

Define the double $\tilde{f}:\tilde{M}\to\tilde{N}$ of f by $\tilde{f}|M_1=f$ and $\tilde{f}|M_2=f$. Then $\tilde{f}$ is C^∞ stable. Denote by $(\partial M)^-$ (resp. M_i^-, $\tilde{M}^-$) the closure of the set of points $x\in\partial M$ (resp. M_i, $\tilde{M}$) such that x is a regular point of $f\,|\,\partial M$ (resp. f, $\tilde{f}$) and that $f\,|\,\partial M$ (resp. f, $\tilde{f}$) is orientation-reversing at x. By [Q], we have

$$(*)\qquad \chi(\tilde{M})-2\chi(\tilde{M}^-)+\text{cusp deg}\,\tilde{f}=(\deg\tilde{f})\chi(\tilde{N})\ .$$

As $\tilde{M}^-=M_1^-\cup M_2^-$, $M_1^-\cap M_2^-=(\partial M)^-$, we see

$$\chi(\tilde{M})=2\chi(M)\ ,\quad \chi(\tilde{N})=2\chi(N)\quad\text{and}\quad \chi(\tilde{M}^-)=2\chi(M^-)-\chi((\partial M)^-)\ .$$

On the other hand, we see

$$\chi((\partial M)^-)=\#(C(f\,|\,\partial M))\ ,\qquad \deg\tilde{f}=\deg(f\,|\,\partial M)$$
$$\text{cusp deg}\,\tilde{f}=2\,\text{cusp deg}\,f\ .$$

Substituting these quantities in (*) and dividing by 2 the both sides of (*), we have the required formula.

§2. A topological invariant.

Our main purpose is to show Theorems 2.1, 2.2 and 2.4 as an application of Theorem 1.1.

First we explain our genericity condition. Let $J^r(n, p)$ denote the space of r-jets of map-germs in $E(n, p)$. Define $\pi_r: E(n, p) \to J^r(n, p)$ and $\pi_{r,s}: J^r(n, p) \to J^s(n, p)$ $(r>s)$ by $\pi_r(f) = j^r f(0)$ and $\pi_{r,s}(j^r f(0)) = j^s f(0)$ respectively.

Generic map-germs in $E(n, p)$ mean map-germs in a fixed subset $G \subset E(n, p)$ with a system of semi-algebraic subset $\Sigma_r \subset J^r(n, p)$ $(r=1, 2, \cdots)$ satisfying

$$\pi_{r,s}^{-1}(\Sigma_s) \supset \Sigma_r\,, \qquad r>s\,,$$

$$\operatorname{codim} \Sigma_r \to \infty \qquad \text{as} \quad r \to \infty\,,$$

and

$$\cup\, \pi_r^{-1}(J^r(n, p) - \Sigma_r) \subset G\,.$$

In this paper, we fix G as the set of $f \in E(2, 2)$ which has a representative $f: D_\varepsilon^2 \to \boldsymbol{R}^2$ satisfying

(0) $f^{-1}(0) = \{0\}$,

(i) $f|(C(f) - \{0\})$ is injective and transverse to ∂D_δ^2 for sufficiently small $\delta > 0$,

(ii) each $x \in C(f) - \{0\}$ is a fold point,

(iii) $C(f)$ is transverse to $\partial D_{\varepsilon'}$ for sufficiently small $\varepsilon' > 0$.

Then $G \subset E(2, 2)$ has the required property for "genericity": the set of map-germs which does not satisfy (0)–(iii) is an ∞-codimensional subset of $E(2, 2)$ (cf. [F]). Thus we call $f \in E(2, 2)$ *generic* if f has a representative satisfying (0), (i), (ii) and (iii).

Let $f \in E(2, 2)$ be a generic smooth map-germ. Take a representative $f: D_\varepsilon^2 \to \boldsymbol{R}^2$ of f such that $f^{-1}(0) = \{0\}$. For a sufficiently small $\delta > 0$, set $\tilde{D}^2 = f^{-1}(D_\delta^2) \cap D_\varepsilon^2$. Then $\tilde{D}^2$ is diffeomorphic to $D^2 = D_1^2$. Furthermore $f = f|\tilde{D}^2: \tilde{D}^2 \to D_\delta^2$ is C^∞ stable outside the origin, and $f|\partial D^2: \partial \tilde{D}^2 \to \partial \tilde{D}_\delta^2$ is also C^∞ stable. Note that the C^∞ right-left equivalence class of $f|\tilde{D}^2$ is independent of $\delta > 0$ provided that δ is sufficiently small.

Let $\tilde{f}: \tilde{D}^2 \to D_\delta^2$ be a perturbation of f. Assume that the closure of $\{x \in \tilde{D}^2 \mid \tilde{f}(x) \neq f(x)\}$ is contained in the interior of $\tilde{D}^2$ and that $\tilde{f}$ is C^∞ stable. We call such perturbation $\tilde{f}$ of f a *stable perturbation* of f.

THEOREM 2.1. *Let $f: (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ be a generic smooth map-germ. Let $\tilde{f}: \tilde{D}^2 \to D_\delta^2$ be a stable perturbation of f. Then the number $\kappa(\tilde{f})$ of* *cusps of $\tilde{f}$ satisfies*

$$\kappa(\tilde{f}) \equiv 1 + \frac{1}{2}\#\{\textit{branches of } C(f) - \{0\}\} + \deg f \qquad \mathrm{mod}\ 2\ .$$

PROOF. By Theorem 1.1,

$$\kappa(\tilde{f}) \equiv \chi(\tilde{D}^2) + \frac{1}{2}\# C(\tilde{f}\,|\,\partial\tilde{D}^2) + \deg(\tilde{f}\,|\,\partial\tilde{D}^2)\chi(D_\delta^2) \qquad \mathrm{mod}\ 2\ .$$

Since

$$\chi(\tilde{D}^2) = \chi(D_\delta^2) = 1\ ,$$
$$\# C(\tilde{f}\,|\,\partial\tilde{D}^2) = \#\{\text{branches of } C(f) - \{0\}\}\ ,$$
$$\deg(\tilde{f}\,|\,\partial\tilde{D}^2) = \deg f\ ,$$

we have the result. Q.E.D.

THEOREM 2.2. *Let f and $\tilde{f}$ be as in Theorem 2.1. Then*

$$\kappa(\tilde{f}) \equiv 1 + \deg f + \deg(Jf, \Delta f) \qquad \mathrm{mod}\ 2\ .$$

Furthermore

$$\kappa(\tilde{f}) \equiv 1 + \dim_{\boldsymbol{R}} Q(f) + \dim_{\boldsymbol{R}} Q(Jf, \Delta f) \qquad \mathrm{mod}\ 2\ ,$$

provided that the right hand side is finite. Here

$Jf = J(f_1, f_2)$ *is the Jacobian determinant of f,*
$\Delta f = J(Jf, x_1^2 + x_2^2)\ , \qquad Q(f) = E_2/\langle f_1, f_2\rangle\ ,$
$Q(Jf, \Delta f) = E_2/\langle Jf, \Delta f\rangle\ .$

PROOF. We use the following lemma.

LEMMA ([FAS]). *Let $g\colon (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}, 0)$ be a function-germ such that 0 is a common isolated critical point of g and $J(g, x_1^2 + x_2^2)$. Then*

$$\#\{\textit{branches of } g^{-1}(0) - \{0\}\} = 2\,|\deg(g, J(g, x_1^2 + x_2^2))|\ .$$

$C(f)$ is the zero-locus of Jf. Applying the lemma to Jf, we have

$$\#\{\text{branches of } C(f) - \{0\}\} = 2\,|\deg(Jf, \Delta f)|\ .$$

By Theorem 2.1, the first half is shown.

Since $\deg f \equiv \dim_{\boldsymbol{R}} Q(f) \bmod 2$ (see [EL]), we have the second half of the theorem. Q.E.D.

EXAMPLE 2.3. Consider $f=z^n: (C, O)\to(C, O)$ as $f\in E(2, 2)$. Then f is generic. We see $\deg f=n$ and $\deg(Jf, \Delta f)=0$ because $Jf\geqq 0$. Thus, by Theorem 2.2, we see that the number of cusps of a stable perturbation of z^n is congruent to $n+1$ modulo 2. This can be seen also using the explicit perturbation $z^n+\varepsilon\bar{z}$ (cf. [Q], p. 312).

THEOREM 2.4. *Let f and $\tilde{f}$ be as those in Theorem 2.1. Then $\kappa(\tilde{f}) \bmod 2$ is a topological invariant of f: If generic smooth map-germs f and $g: (R^2, O)\to(R^2, O)$ are topologically (i.e., C^0 right-left) equivalent, then for any stable perturbations $\tilde{f}$ and $\tilde{g}$ of f and g respectively,*

$$\kappa(\tilde{f})\equiv\kappa(\tilde{g}) \quad \bmod 2 .$$

PROOF. Since generic map-germs f and g are topologically equivalent, there exists a homeomorphism-germ h of (R^2, O) such that $h(C(f))=C(g)$ (note that singular points of f and g are fold points except for the origin). Thus we have

$$\#\{\text{branches of } C(f)-\{O\}\} = \#\{\text{branches of } C(g)-\{O\}\} .$$

Furthermore, since f and g are topologically equivalent, we have $\deg f=\deg g$. By Theorem 2.1, we have the result.

§3. Estimates.

In this section we consider only analytic map-germs. For $f=(f_1, f_2)\in E(2, 2)$, set

$$J_1f=J(Jf, f_2) , \quad J_2f=J(f_1, Jf) ,$$
$$K(f)=\langle Jf, J_1f, J_2f\rangle \quad \text{in} \quad E_2 ,$$
$$Q=E_2/K(f) .$$

Let $F\in E(3, 2)$ be an analytic deformation of f. Set

$$f_t=F(\ \ , t) , \quad t\in R .$$

THEOREM 3.1. *Let F be an analytic deformation of an analytic map-germ f as above. Assume f_t is C^∞ stable near the origine O for a sufficiently small $t\neq 0$. Then the number $\kappa(f_t)$ of cusps of f_t near O satisfies*

$$\kappa(f_t)\leqq\dim_R Q ,$$
$$\kappa(f_t)\equiv\dim_R Q \quad (\bmod 2) ,$$

provided $\dim_R Q<+\infty$.

REMARK 3.2. (i) The condition that $\dim_R Q < \infty$ is a generic condition in the sense that the set of map-germs which do not satisfy the condition is an ∞-codimensional subset of $E(2, 2)$.

(ii) If $\dim_R Q < +\infty$, then the Thom-Boardman singularity $\overline{\Sigma^{1,1}f} \subset \{0\}$ as germ. For a representative $F: D_\varepsilon^2 \times \boldsymbol{R} \to \boldsymbol{R}^2$ and for any $\varepsilon' > 0$, there exists $\delta > 0$ such that if $|t| < \delta$, the cusp points of $f_t | D_\varepsilon^2$ are contained in $D_{\varepsilon'}^2$.

EXAMPLE 3.3. Consider again $f = z^n \in E(2, 2)$. Then $\dim_R Q = 3$ if $n = 2$, and $\dim_R Q = +\infty$ if $n > 2$.

PROOF OF THEOREM 3.1. We use the same notation $F: (\boldsymbol{C}^2 \times \boldsymbol{C}, (0, 0)) \to (\boldsymbol{C}^2, 0)$ for the complexification of F. Here we need some results in complex analytic geometry. Denote by $J^2(\boldsymbol{C}^2, \boldsymbol{C}^2)$ the space of 2-jets of holomorphic mappings of $\boldsymbol{C}^2$ into $\boldsymbol{C}^2$. Define polynomial functions

$$J,\ J_1 \text{ and } J_2: J^2(\boldsymbol{C}^2, \boldsymbol{C}^2) \longrightarrow \boldsymbol{C}$$

by

$$J(j^2f(x)) = Jf(x)\ , \qquad J_1(j^2f(x)) = J(Jf, f_2)(x)\ ,$$
$$J_2(j^2f(x)) = J(f_1, Jf)(x)\ .$$

The following lemma is a special form of results in [B], [Mo, p. 15], [M].

LEMMA 3.4. (i) $J^{-1}(0) \cap J_1^{-1}(0) \cap J_2^{-1}(0) = \overline{\Sigma^{1,1}}$, *where* Σ^I *is the Thom-Boardman singularity of type* I.

(ii) *At each point of* $\Sigma^{1,1} \subset J^2(\boldsymbol{C}^2, \boldsymbol{C}^2)$, *locally*

$$\langle J, J_1, J_2 \rangle = \langle J, J_1 \rangle \quad or \quad \langle J, J_1, J_2 \rangle = \langle J, J_2 \rangle\ .$$

(iii) *At each point of* $\Sigma^{1,1}$, $(J, J_1, J_2): J^2(\boldsymbol{C}^2, \boldsymbol{C}^2) \to \boldsymbol{C}^3$ *is of constant rank* 2.

LEMMA 3.5. $A = \mathcal{O}_{J^2(C^2, C^2)} / \langle J, J_1, J_2 \rangle$ *is Cohen-Macaulay for any* $z \in J^2(\boldsymbol{C}^2, \boldsymbol{C}^2)$.

PROOF. Let M_0, M_1, M_2 be the 2×2-minors of

$$\begin{pmatrix} x_{01} & x_{02} \\ x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}.$$

Then $B=\mathcal{O}_{C^6,w}/\langle M_0, M_1, M_2\rangle$ is Cohen-Macaulay for any $w\in C^6$ ([H], [Ful, p. 419]).

Define $\Phi: J^2(C^2, C^2)\to C^6$ by

$$\Phi(j^2f(x))=\begin{pmatrix} \begin{vmatrix} z^1_{11} & z^1_{22} \\ z^2_1 & z^2_2 \end{vmatrix}+\begin{vmatrix} z^1_1 & z^1_2 \\ z^2_{11} & z^2_{21} \end{vmatrix} & \begin{vmatrix} z^1_{12} & z^1_{22} \\ z^2_1 & z^2_2 \end{vmatrix}+\begin{vmatrix} z^1_1 & z^1_2 \\ z^2_{12} & z^2_{22} \end{vmatrix} \\ z^1_1 & z^1_2 \\ z^2_1 & z^2_2 \end{pmatrix}$$

where $z^i_j=(\partial f_i/\partial x_j)(x)$, $z^i_{jk}=(\partial^2 f_i/\partial x_j\partial x_k)(x)$ $(i, j, k=1, 2)$.

For the pull-back $\Phi^*: \mathcal{O}_{C^6,\Phi(z)}\to\mathcal{O}_{J^2(C^2,C^2)}$, we have

$$\Phi^*(\langle M_0, M_1, M_2\rangle)=\langle J, J_1, J_2\rangle .$$

Set

$$C=\mathcal{O}_{J^2(C^2,C^2)\times C^6,(z,\Phi(z))}/\langle M_0, M_1, M_2\rangle .$$

Then as B is Cohen-Macaulay, C is also Cohen-Macaulay. Set

$$I=\langle x_{ij}-\Phi_{ij} \mid i=0, 1, 2;\ j=1, 2\rangle \quad \text{in } C .$$

Then $C/I\cong A$. Now $\dim C-\dim A=6$. Hence

$$\text{the height of } I = 6 .$$

Therefore A is also Cohen-Macaulay ([Ma], (16; a, b)). Q.E.D.

Let us denote by $i(j^2f(0); \overline{\Sigma^{1,1}}\circ j^2f(C^2))$ the intersection multiplicity of $\overline{\Sigma^{1,1}}$ and $j^2f(C^2)$ at $j^2f(0)$. Then by [Ful; Proposition 7.1] and our Lemma 3.5,

$$i(j^2f(0); \overline{\Sigma^{1,1}}\circ j^2f(C^2))=\dim_C\tilde{Q} ,$$

where

$$\tilde{Q}=\mathcal{O}_{C^2,0}/(j^2f)^*\langle J, J_1, J_2\rangle$$
$$=\mathcal{O}_{C^2,0}/\langle Jf, J_1f, J_2f\rangle .$$

On the other hand

$$i(j^2f(0); \overline{\Sigma^{1,1}}\circ j^2f(C^2))=\sum i(j^2f_t(x); \overline{\Sigma^{1,1}}\circ j^2f_t(C^2)) ,$$

where the summation runs over points x in $\overline{\Sigma^{1,1}}f_t=(j^2f_t)^{-1}(\overline{\Sigma^{1,1}})$. If f_t is C^∞ stable, then by Lemma 3.4 (iii), the right hand side is equal to the number $\tilde{\kappa}(f_t)$ of (not necessarily real) cusp points of the holomorphic mapping f_t. Thus we see that

$$\tilde{\kappa}(f_t)=\dim_C \tilde{Q} .$$

Now

$$\dim_C \tilde{Q}=\dim_R Q , \qquad \kappa(f_t)\leqq \tilde{\kappa}(f_t) ,$$
$$\kappa(f_t)\equiv \tilde{\kappa}(f_t) \pmod 2 .$$

This completes the proof of Theorem 3.1.

REMARK 3.6. Let $f: (\boldsymbol{R}^2, 0)\to(\boldsymbol{R}^2, 0)$ be an analytic map-germ with $\dim_R Q=E_2/\langle Jf, J_1f, J_2f\rangle < +\infty$. Then $\dim_R Q(f) < +\infty$, where $Q(f)=E_2/\langle f_1, f_2\rangle$. Then f has a stable unfolding $F: (\boldsymbol{R}^2\times\boldsymbol{R}^r, (0, 0))\to(\boldsymbol{R}^2\times\boldsymbol{R}^r, (0, 0))$. Set $f_u=F(\ \ , u)$, $u\in(\boldsymbol{R}^r, 0)$. Then for generic u, f_u is C^∞ stable. Set

$$\kappa_u=\#\{\text{cusps of } f_u\} , \qquad k=\min_{u:\ \text{generic}} \kappa_u , \qquad K=\max_{u:\ \text{generic}} \kappa_u .$$

Then

(i) $k\equiv K\equiv\kappa_u \pmod 2$.

(ii) For any p with $p\equiv k \pmod 2$ and $k\leqq p\leqq K$, there exists a parameter $u\in(\boldsymbol{R}^r, 0)$ such that $p=\kappa_u$.

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