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A Decomposition Theorem for Simple Lie Groups Associated with Parahermitian Symmetric Spaces

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Introduction

Let G/H be a semisimple affine symmetric space and let σ be the associated involutive automorphism of G. Let K be a σ-stable maximal compact subgroup of G. Then it is known (Flensted-Jensen [1], Rossmann [9]) that G admits the decomposition $G=KCH$ (with intersection), where C is a so-called split Cartan subgroup of G. In this paper we are mainly concerned with a simple parahermitian symmetric space M whose Weyl group $W(M)$ coincides with the Weyl group $W(M^*)$ of the fiber M^* of the Berger fibration of M ([5]). We then obtain a decomposition theorem for the simple Lie group G which arises as the automorphism group of M (Theorem 3.6). More precisely, we have the decomposition (with intersection) $G=KCH_l$ $(0\leqq l\leqq r=\dim C)$, where H_0 is the isotropy subgroup of G at a point in M, and H_l $(1\leqq l\leqq r)$ is the isotropy subgroup of G at a point on the boundary of M in a certain compactification of M. This is a partial generalization of the above-mentioned decomposition due to Flensted-Jensen and Rossmann; actually, when $l=0$, our decomposition is theirs. In Appendix, we give the table of the rank of the operator $\mathscr{K}(x)$ for each simple parahermitian symmetric space. That operator played an essential role in our previous paper [4].

§1. Basic facts.

Throughout this paper we shall use the terminologies in the previous papers [4], [2], [3]. Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be a simple symmetric triple satisfying the condition:

(C) there exists an element $Z\in\mathfrak{g}$ such that $\operatorname{ad} Z$ is a semisimple operator with eigenvalues 0, ± 1 only and that $\mathfrak{h}$ is the centralizer of Z in $\mathfrak{g}$.

We denote by $\mathfrak{m}^{\pm}$ the eigenspaces in $\mathfrak{g}$ under the operator $\operatorname{ad} Z$, and put

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$\mathfrak{m}=\mathfrak{m}^+ + \mathfrak{m}^-$. Then the decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ is the decomposition into $+1$ and -1 eigenspaces under σ. Let τ be a Cartan involution of $\mathfrak{g}$ which commutes with σ, and let

$$\mathfrak{g}=\mathfrak{k}+\mathfrak{p} \tag{1.1}$$

be the corresponding Cartan decomposition, where $\mathfrak{k}$ and $\mathfrak{p}$ are $+1$ and -1 eigenspaces under τ. Then we have the decomposition:

$$\mathfrak{g}=\mathfrak{k}^* + \mathfrak{m}_{\mathfrak{k}} + \mathfrak{h}_{\mathfrak{p}} + \mathfrak{m}_{\mathfrak{p}} , \tag{1.2}$$

where $\mathfrak{k}^*=\mathfrak{h}\cap\mathfrak{k}$, $\mathfrak{m}_{\mathfrak{k}}=\mathfrak{m}\cap\mathfrak{k}$, $\mathfrak{h}_{\mathfrak{p}}=\mathfrak{h}\cap\mathfrak{p}$, $\mathfrak{m}_{\mathfrak{p}}=\mathfrak{m}\cap\mathfrak{p}$. For any subspace $\mathfrak{b}$ of $\mathfrak{p}$ we always identify a linear form λ on $\mathfrak{b}$ with an element in $\mathfrak{b}$ with respect to the inner product $(\ ,\ )$ on $\mathfrak{b}$ defined by the Killing form of $\mathfrak{g}$. We have $Z\in\mathfrak{h}_{\mathfrak{p}}$ ([4]). Let us choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ which contains Z. Then $\mathfrak{a}$ is contained in $\mathfrak{h}_{\mathfrak{p}}$. Let $\Sigma(\mathfrak{g}, \mathfrak{a})$ be a (restricted) root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. By our convention, $\Sigma(\mathfrak{g}, \mathfrak{a})$ is viewed as a subset of $\mathfrak{a}$. Let $\Sigma_1(\mathfrak{g}, \mathfrak{a})$ denote the subsystem of roots $\alpha\in\Sigma(\mathfrak{g}, \mathfrak{a})$ satisfying $(\alpha, Z)=0$. $\Sigma_1(\mathfrak{g}, \mathfrak{a})$ is the set of roots of $\mathfrak{h}$. Let us choose a linear order in $\Sigma(\mathfrak{g}, \mathfrak{a})$ in such a way that if $\alpha\in\Sigma(\mathfrak{g}, \mathfrak{a})$ is positive, then (α, Z) is nonnegative. $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ denotes the set of positive roots in $\Sigma(\mathfrak{g}, \mathfrak{a})$. A root in the set $\Sigma(\mathfrak{g}, \mathfrak{a})-\Sigma_1(\mathfrak{g}, \mathfrak{a})$ is called a complementary root. Note that $\mathfrak{m}^+$ (resp. $\mathfrak{m}^-$) is spanned by the root vectors for complementary positive (resp. negative) roots. Let us choose a maximal system of strongly orthogonal roots $\{\beta_1, \cdots, \beta_r\}$ in $\Sigma^+(\mathfrak{g}, \mathfrak{a})-\Sigma_1(\mathfrak{g}, \mathfrak{a})$ satisfying the following two conditions ([11]): (1) β_1 is the highest root in $\Sigma(\mathfrak{g}, \mathfrak{a})$, and $\beta_1>\beta_2>\cdots>\beta_r$, (2) all β_i's have the same length. Let $\mathfrak{g}(\mathfrak{a}; \alpha)$ denote the root space for a root $\alpha\in\Sigma(\mathfrak{g}, \mathfrak{a})$. Now choose a non-zero vector $E_i\in\mathfrak{g}(\mathfrak{a}; \beta_i)$, $1\leqq i\leqq r$ in such a way [4] that

$$(E_i, \tau E_i)=-2d^{-1} , \tag{1.3}$$

where $d=(\beta_1, \beta_1)$. Put $E_{-i}=-\tau E_i$. Then we see that $E_{-i}\in\mathfrak{g}(\mathfrak{a}; -\beta_i)$, and

$$[E_i, E_{-i}]=2d^{-1}\beta_i . \tag{1.4}$$

We set $X_i=E_i+E_{-i}$. Then

$$\mathfrak{c}=\sum_{i=1}^{r} \boldsymbol{R}X_i \tag{1.5}$$

is a maximal abelian subspace in $\mathfrak{m}_{\mathfrak{p}}$ [4], which is called a *(split) Cartan subalgebra* of $(\mathfrak{g}, \mathfrak{h}, \sigma)$. The dimension of $\mathfrak{c}$ is called the *split rank* of $(\mathfrak{g}, \mathfrak{h}, \sigma)$.

§ 2. Weyl groups.

We preserve notations and conventions in the previous section. Let G be the adjoint group of the Lie algebra $\mathfrak{g}$, and let us consider the *Cayley transformation*

$$c = \mathrm{Ad}\exp\frac{\pi}{4}\sum_{i=1}^{r}(E_{-i}-E_i)\ . \tag{2.1}$$

Let $\mathfrak{a}_0$ be the real span of $\beta_1, \cdots, \beta_r$ in $\mathfrak{a}$. We then have $\mathfrak{a}=\mathfrak{a}_0^{\perp}+\mathfrak{a}_0$, where $\mathfrak{a}_0^{\perp}$ is the orthogonal complement of $\mathfrak{a}_0$ in $\mathfrak{a}$ with respect to the inner product (,). By the same reason as in Moore [7], we have

LEMMA 2.1. *$c(\beta_i)=2h_i$ $(1\leqq i\leqq r)$ holds, where $h_i=(d/4)X_i$. In particular we have $c(\mathfrak{a}_0)=\mathfrak{c}$. Furthermore c is the identity on $\mathfrak{a}_0^{\perp}$.*

Let $\hat{\mathfrak{a}}=c(\mathfrak{a})$. Then $\hat{\mathfrak{a}}=\mathfrak{a}_0^{\perp}+\mathfrak{c}$ is a maximal abelian subspace of $\mathfrak{p}$. The coset space $M=G/C(Z)$ is a simple parahermitian symmetric space of adjoint type [4] corresponding to the symmetric triple $(\mathfrak{g}, \mathfrak{h}, \sigma)$, where $C(Z)$ is the centralizer of Z in G. Let G^* (resp. K^*) be the analytic subgroup of G generated by the subalgebra $\mathfrak{g}^*=\mathfrak{k}^*+\mathfrak{m}_{\mathfrak{p}}$ (resp. $\mathfrak{k}^*$) of $\mathfrak{g}$. The coset space $M^*=G^*/K^*$ is a noncompact Riemannian symmetric space which is dual to the symmetric R-space $M^-=G/U^-=K/K'$, where K is the maximal compact subgroup of G generated by $\mathfrak{k}$, $U^-=C(Z)\exp\mathfrak{m}^-$ and $K'=K\cap U^-=K\cap C(Z)$. Note that M is diffeomorphic to the cotangent bundle T^*M^- of M^-. For a non-zero linear form λ on $\mathfrak{c}$, let

$$\mathfrak{g}(\mathfrak{c};\lambda)=\{X\in\mathfrak{g} : (\mathrm{ad}\, H)X=(\lambda, H)X,\ H\in\mathfrak{c}\}\ . \tag{2.2}$$

Let $\Sigma(\mathfrak{g}, \mathfrak{c})$ denote the totality of non-zero linear forms λ on $\mathfrak{c}$ with $\mathfrak{g}(\mathfrak{c};\lambda)\neq(0)$. It is known [9], [8] that $\Sigma(\mathfrak{g}, \mathfrak{c})$ satisfies the axiom of a root system in $\mathfrak{c}$. Let $\Sigma(\mathfrak{g}, \hat{\mathfrak{a}})$ be the root system of $\mathfrak{g}$ with respect to $\hat{\mathfrak{a}}$. As was remarked in [9], the relation

$$\Sigma(\mathfrak{g}, \mathfrak{c})=\{\alpha|_{\mathfrak{c}} : \alpha|_{\mathfrak{c}}\neq 0,\ \alpha\in\Sigma(\mathfrak{g}, \hat{\mathfrak{a}})\} \tag{2.3}$$

is valid. The following proposition is a special case of a result of Oshima-Sekiguchi [8], but our proof is rather classification-free.

PROPOSITION 2.2. *Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be a simple symmetric triple of split rank r satisfying the condition* (C). *Then the root system $\Sigma(\mathfrak{g}, \mathfrak{c})$ is given by*

$$\{\pm(h_i\pm h_j)\ (1\leqq i<j\leqq r),\ \ \pm 2h_i\ (1\leqq i\leqq r)\} \tag{2.4}$$

or

$$(2.5) \qquad \{\pm(h_i\pm h_j)\,(1\leqq i<j\leqq r),\ \ \pm 2h_i,\ \ \pm h_i\ (1\leqq i\leqq r)\}\,.$$

In particular $\Sigma(\mathfrak{g}, \mathfrak{c})$ *is of type* C_r *or of type* BC_r.

PROOF. Let $\varpi: \mathfrak{a}\to\mathfrak{a}_0$ and $\hat{\varpi}: \hat{\mathfrak{a}}\to\mathfrak{c}$ be the orthogonal projections with respect to the inner products induced by the Killing form of $\mathfrak{g}$. Then we have [11]

$$(2.6) \qquad \begin{aligned} &\varpi(\Sigma(\mathfrak{g}, \mathfrak{a}))-(0)\\ &\quad=\{\pm(\beta_i\pm\beta_j)/2\,(1\leqq i<j\leqq r),\ \ \pm\beta_i\,(1\leqq i\leqq r)\}\end{aligned}$$

or

$$=\{\pm(\beta_i\pm\beta_j)/2\,(1\leqq i<j\leqq r),\ \ \pm\beta_i,\ \ \pm\beta_i/2\ (1\leqq i\leqq r)\}\,.$$

Also we have the commutative diagram

$$(2.7) \qquad \begin{array}{ccc} \mathfrak{a} & \xrightarrow{\ \varpi\ } & \mathfrak{a}_0 \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\ \hat{\mathfrak{a}} & \xrightarrow{\ \hat{\varpi}\ } & \mathfrak{c} \end{array}\,.$$

Since the Cayley transformation c sends $\Sigma(\mathfrak{g}, \mathfrak{a})\subset\mathfrak{a}$ to $\Sigma(\mathfrak{g}, \hat{\mathfrak{a}})\subset\hat{\mathfrak{a}}$, we have

$$\hat{\varpi}(\Sigma(\mathfrak{g}, \hat{\mathfrak{a}}))-(0)=c(\varpi(\Sigma(\mathfrak{g}, \mathfrak{a})))-(0)\,.$$

Hence, by Lemma 2.1 and (2.3) we get the assertion of the proposition.

The maximal abelian subspace $\mathfrak{c}$ of $\mathfrak{m}_{\mathfrak{p}}$ is also a maximal abelian subspace for the symmetric pair $(\mathfrak{g}^*, \mathfrak{k}^*)$. One can consider the root system $\Sigma(\mathfrak{g}^*, \mathfrak{c})$ of $\mathfrak{g}^*$ with respect to $\mathfrak{c}$, which is a subsystem of the root system $\Sigma(\mathfrak{g}, \mathfrak{c})$. Let $W(M)$ (resp. $W(M^*)$) denote the Weyl group of M (resp. M^*), or equivalently, the Weyl group of the root system $\Sigma(\mathfrak{g}, \mathfrak{c})$ (resp. $\Sigma(\mathfrak{g}^*, \mathfrak{c})$). $W(M^*)$ is obviously a subgroup of $W(M)$.

PROPOSITION 2.3. *Let* $(\mathfrak{g}, \mathfrak{h}, \sigma)$ *be a simple symmetric triple of split rank* r *satisfying the condition* (C), *and let* M *be the corresponding para-hermitian symmetric space of adjoint type. Then* $W(M)=W(M^*)$ *holds if and only if* M *is one of the following coset spaces*:

1) $SL(p+q, \boldsymbol{F})/S(GL(p, \boldsymbol{F})\times GL(q, \boldsymbol{F}))$*), $\boldsymbol{F}=\boldsymbol{R}$, $\boldsymbol{C}$ *or* $\boldsymbol{H}$, *where* $p\leqq q$ *for* $\boldsymbol{F}\neq\boldsymbol{R}$, *and* $p<q$ *for* $\boldsymbol{F}=\boldsymbol{R}$,
2) $SO(2n+1, 2n+1)/GL(2n+1, \boldsymbol{R})$,
3) $Sp(n, n)/GL(n, \boldsymbol{H})$,
4) $SO(2n, \boldsymbol{C})/GL(n, \boldsymbol{C})$,
5) $Sp(n, \boldsymbol{C})/GL(n, \boldsymbol{C})$,

*) For the notation, see [2].

6) *$SO(p+1, 1)/SO(p)\boldsymbol{R}^*$,*
7) *$SO(n+2, \boldsymbol{C})/SO(n, \boldsymbol{C})\boldsymbol{C}^*$,*
8) *the space of adjoint type corresponding to the pair $(E_6^1, \mathfrak{so}(5, 5)+\boldsymbol{R})$, $(E_6^4, \mathfrak{so}(1, 9)+\boldsymbol{R})$, $(E_6^C, \mathfrak{so}(10, \boldsymbol{C})+\boldsymbol{C})$ or $(E_7^C, E_6^C+\boldsymbol{C})$.*

PROOF. As is known in [4], [2], the above coset spaces are diffeomorphic respectively to the cotangent bundle of the following symmetric R-spaces: 1) the Grassmannian $G_{p,q}(\boldsymbol{F})$ ($p\leqq q$ for $\boldsymbol{F}\neq\boldsymbol{R}$, $p<q$ for $\boldsymbol{F}=\boldsymbol{R}$), 2) $SO(2n+1)$, 3) $Sp(n)$, 4) $SO(2n)/U(n)$, 5) $Sp(n)/U(n)$, 6) the Möbius space in the real projective space $P_{p+1}(\boldsymbol{R})$, 7) $SO(n+2)/SO(n)\times SO(2)$, 8) $G_{2,2}(\boldsymbol{H})/\boldsymbol{Z}_2$, the octanion projective plane $P_2(\boldsymbol{O})$, $E_6/Spin(10)T^1$ or E_7/E_6T^1. The root systems $\Sigma(\mathfrak{g}^*, \mathfrak{c})$ are determined in [6] for the non-compact duals M^* of the symmetric R-spaces M^-. Hence we can see by inspection that M^* is the dual to one of the aforementioned symmetric R-spaces if and only if $\Sigma(\mathfrak{g}^*, \mathfrak{c})$ is either one of

$$(2.8)\qquad \{\pm(h_i\pm h_j)\,(1\leqq i<j\leqq r),\ \pm h_i\,(1\leqq i\leqq r)\}\ (\text{type } \mathrm{B}_r)\ ,$$
$$\{\pm(h_i\pm h_j)\,(1\leqq i<j\leqq r),\ \pm h_i,\ \pm 2h_i\,(1\leqq i\leqq r)\}\ (\text{type } \mathrm{BC}_r)\ ,$$

or

$$\{\pm(h_i\pm h_j)\,(1\leqq i<j\leqq r),\ \pm 2h_i\,(1\leqq i\leqq r)\}\ (\text{type } \mathrm{C}_r)\ .$$

Furthermore, only in that case the Weyl group $W(M^*)$ consists of all signed permutations $h_i\mapsto\pm h_{\rho(i)}$, $\rho\in\mathfrak{S}_r$ (=the symmetric group of degree r). Therefore, in view of Proposition 2.2, we get the assertion of the proposition.

Let A be a subgroup of G, and $\mathfrak{b}$ be an abelian subspace of $\mathfrak{p}$. We denote by $N_A(\mathfrak{b})$ (resp. $C_A(\mathfrak{b})$) the normalizer (resp. centralizer) of $\mathfrak{b}$ in A. We know [9]

$$(2.9)\qquad W(M)\cong N_K(\mathfrak{c})/C_K(\mathfrak{c})\ .$$

Also we have

LEMMA 2.4. *$W(M^*)\cong N_{K'}(\mathfrak{c})/C_{K'}(\mathfrak{c})$.*

PROOF. By the general theory of symmetric spaces, we have

$$(2.10)\qquad W(M^*)\cong N_{K^*}(\mathfrak{c})/C_{K^*}(\mathfrak{c})\ ,$$

since K^* is the analytic subgroup of G^* generated by $\mathfrak{k}^*$. Let us put $Y_i=E_i-E_{-i}$ $(1\leqq i\leqq r)$, and let $\mathfrak{a}'$ be the real span of $Y_1, \cdots, Y_r$. Then $\mathfrak{a}'$ is a maximal abelian subspace of $\mathfrak{m}_{\mathfrak{k}}$ ([11]) and so it is a Cartan subalgebra of the compact symmetric pair $(\mathfrak{k}, \mathfrak{k}^*)$. Let us now consider the para-complex structure $I=\mathrm{ad}_{\mathfrak{m}}Z$ on $\mathfrak{m}$. Then $\mathfrak{m}^{\pm}$ are ± 1 eigenspaces of I. Since $E_i \in \mathfrak{m}^+$ and $E_{-i} \in \mathfrak{m}^-$, we have $IY_i = X_i$ $(1 \leqq i \leqq r)$. In particular I sends $\mathfrak{a}'$ to $\mathfrak{c}$. Since $K'=K\cap C(Z)$, it follows that I commutes with each operator in $\mathrm{Ad}_{\mathfrak{m}}K'$. Hence, for each $a\in K'$, we have the commutative diagram

$$\begin{array}{ccc} \mathfrak{a}' & \xrightarrow{\mathrm{Ad}\,a} & \mathfrak{a}' \\ I\big\downarrow & & \big\downarrow I \\ \mathfrak{c} & \xrightarrow{\mathrm{Ad}\,a} & \mathfrak{c} \end{array} . \tag{2.11}$$

Using this diagram, we have

$$N_{K'}(\mathfrak{c})/C_{K'}(\mathfrak{c}) \cong N_{K'}(\mathfrak{a}')/C_{K'}(\mathfrak{a}') . \tag{2.12}$$

Let $A'=\exp \mathfrak{a}'$. It is known [12] that for the compact symmetric pair (K, K'), one has

$$K'=K^*(K'\cap A') . \tag{2.13}$$

Therefore an easy argument shows that $N_{K'}(\mathfrak{a}')=N_{K^*}(\mathfrak{a}')(K'\cap A')$ and $C_{K'}(\mathfrak{a}')=C_{K^*}(\mathfrak{a}')(K'\cap A')$. From these two relations and (2.11) again, we have

$$N_{K'}(\mathfrak{a}')/C_{K'}(\mathfrak{a}') \cong N_{K^*}(\mathfrak{a}')/C_{K^*}(\mathfrak{a}') \cong N_{K^*}(\mathfrak{c})/C_{K^*}(\mathfrak{c}) . \tag{2.14}$$

The lemma now follows from (2.10), (2.14) and (2.12).

§ 3. Orbit structure.

Throughout this section, we will assume that a symmetric triple $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is simple of split rank r and satisfies the two conditions (C) and $W(M)=W(M^*)$. Let $M^+=G/U^+$, where $U^+=C(Z)\exp \mathfrak{m}^+$, and let us consider the compact manifold $\tilde{M}=M^-\times M^+$. We denote the origins of $M^{\pm}$ by $o^{\pm}$. We define the mapping ξ of $\mathfrak{m}$ to $\tilde{M}$ by putting

$$\xi(X, Y)=(\exp X\cdot o^-, \exp Y\cdot o^+) , \qquad X\in\mathfrak{m}^+, Y\in\mathfrak{m}^- . \tag{3.1}$$

Then ξ is an imbedding of $\mathfrak{m}$ and the image $\xi(\mathfrak{m})$ is open dense in $\tilde{M}$ [4]. G acts on $\tilde{M}$ by the rule

$$g(p, q)=(gp, gq) , \qquad (p, q)\in M^-\times M^+ , \tag{3.2}$$

which is called the *diagonal G-action*. The ξ-equivariant action of G on $\mathfrak{m}$ for the diagonal G-action is birational ([6]). Let $C=\exp \mathfrak{c}\subset G$, and let

(o_1, o_2) be the origin of $\mathfrak{m}$. The ξ-equivariant action of C is defined at the point (o_1, o_2) and its orbit $C(o_1, o_2)$ is given by ([4])

$$C(o_1, o_2)=\left\{\sum_{i=1}^{r} t_i X_i \in \mathfrak{c} : |t_i|<1\ (1\leqq i\leqq r)\right\} . \tag{3.3}$$

Let I_l be a subset of $\{1, 2, \cdots, r\}$ consisting of l integers, and let $\varepsilon=(\varepsilon_i)_{i\in I_l}$ be an l-tuple of the numbers $\varepsilon_i=1$ or -1. And put

$$P(I_l, \varepsilon)=\{\sum_{i\in I_l} \varepsilon_i X_i+\sum_{j\notin I_l} t_j X_j : |t_j|<1\ (j\notin I_l)\} . \tag{3.4}$$

Then $P(I_l, \varepsilon)$ is an open $(r-l)$-face of the closed r-cube $\overline{C(o_1, o_2)}$, and every open $(r-l)$-face of $\overline{C(o_1, o_2)}$ can be described in that way by means of suitable I_l and ε.

LEMMA 3.1. *For given I_l and $\varepsilon=(\varepsilon_i)_{i\in I_l}$, the group C has the ξ-equivariant action at the point $\sum_{i\in I_l}\varepsilon_i X_i$, and we have*

$$P(I_l, \varepsilon)=C(\sum_{i\in I_l}\varepsilon_i X_i) . \tag{3.5}$$

PROOF. The ξ-equivariant action of $\exp\sum_{i=1}^{r} t_i X_i$ at a point $\sum_{i=1}^{r}\lambda_i X_i\in\mathfrak{c}$ is given by ([4])

$$\left(\exp\sum_{i=1}^{r} t_i X_i\right)\left(\sum_{i=1}^{r}\lambda_i X_i\right)=\sum_{i=1}^{r}\frac{\lambda_i \operatorname{ch} t_i+\operatorname{sh} t_i}{\lambda_i \operatorname{sh} t_i+\operatorname{ch} t_i}X_i . \tag{3.6}$$

Therefore it follows that

$$\left(\exp\sum_{i=1}^{r} t_i X_i\right)(\sum_{i\in I_l}\varepsilon_i X_i)=\sum_{i\in I_l}\varepsilon_i X_i+\sum_{j\notin I_l}(\operatorname{th} t_j)X_j , \tag{3.7}$$

which implies (3.5).

LEMMA 3.2. *For given I_l and $\varepsilon=(\varepsilon_i)_{i\in I_l}$, there exists $k\in N_{K'}(\mathfrak{c})$ such that*

$$(\operatorname{Ad} k)\left(\sum_{i=1}^{l} X_i\right)=\sum_{j\in I_l}\varepsilon_j X_j . \tag{3.8}$$

PROOF. By Proposition 2.2, the Weyl group $W(M)$ consists of all signed permutations $h_i\mapsto\pm h_{\rho(i)}$, $\rho\in\mathfrak{S}_r$. Therefore, noting that $X_i=(d/4)h_i$, one can find $\eta\in W(M)$ such that $\eta(\sum_{i=1}^{l}X_i)=\sum_{j\in I_l}\varepsilon_j X_j$. By the assumption $W(M)=W(M^*)$ and Lemma 2.4, there exists an element $k\in N_{K'}(\mathfrak{c})$ such that $\eta=(\operatorname{Ad} k)|_{\mathfrak{c}}$.

Let

$$o_l=\sum_{i=1}^{l} X_i \quad (1\leqq l\leqq r)\ , \qquad o_0=(o_1,\ o_2)\ ,$$
$$\bar{o}_l=\xi(o_l) \quad (0\leqq l\leqq r)\ . \tag{3.9}$$

Note that $\bar{o}_0=(o^-, o^+)$. Let us consider the orbit

$$M_l=G\bar{o}_l \qquad (0\leqq l\leqq r) \tag{3.10}$$

under the diagonal G-action in $\tilde{M}$.

LEMMA 3.3. $M_l\cap M_{l'}=\varnothing$ $(l\neq l')$.

PROOF. We can assume that $l<l'$. Let us recall the operator $\mathscr{K}$: $\mathfrak{m}\to \mathrm{End}\,\mathfrak{m}^+$ ([4]):

$$\mathscr{K}(x)=\mathrm{id}-\mathrm{ad}_{\mathfrak{m}^+}[x^+,\ x^-]+\frac{1}{4}(\mathrm{ad}\ x^+)^2(\mathrm{ad}\ x^-)^2\ , \tag{3.11}$$

where $x=x^++x^-$, $x^\pm\in\mathfrak{m}^\pm$. Let $i_l=\mathrm{rank}\,\mathscr{K}(o_l)$. Then, by a result of [4], we have $i_l>i_{l'}$, since $l<l'$. Now suppose that $M_l\cap M_{l'}\neq\varnothing$. Then there exists an element $g\in G$ such that $g\bar{o}_l=\bar{o}_{l'}$, or equivalently, $\xi^{-1}g\xi(o_l)=o_{l'}$. That means that g has the ξ-equivariant action at the point o_l. The G-covariance ([4]) of the operator $\mathscr{K}$ now implies that $i_l=i_{l'}$ which is a contradiction.

LEMMA 3.4. *The orbit M_0 is open dense in $\tilde{M}$.*

PROOF. We only give a sketch of the proof, since it is similar to the arguments in Tanaka [13] p. 313. Let π be the projection of $G\times G$ onto $\tilde{M}$ defined by $\pi(a, b)=(ao^-, bo^+)$, and let λ: $G\times G\to G$ be the map defined by $\lambda(a, b)=b^{-1}a$. Let Ω denote the subset $(\exp\mathfrak{m}^+)C(Z)(\exp\mathfrak{m}^-)=U^+U^-$ in G. Then Ω is open dense in G ([10]). It can be proved that $\pi^{-1}(M_0)=\lambda^{-1}(\Omega)$. From these facts it follows that M_0 is open dense in $\tilde{M}$.

Let $P^{(l)}$ be the union of all open $(r-l)$-faces of $\overline{C(o_1,\ o_2)}$. Then one has

$$\overline{C(o_1,\ o_2)}=\coprod_{l=0}^{r} P^{(l)}\ . \tag{3.12}$$

LEMMA 3.5. $\tilde{M}=\bigcup_{l=0}^{r} K(\xi(P^{(l)}))$.

PROOF. We have

$$\tilde{M}=K\xi(\overline{C(o_1,\ o_2)})\ . \tag{3.13}$$

The proof of this fact is done by the same way as that of (4.48) in [4].

Only one distinct point is that we use now Lemma 3.4 instead of using Theorem 3.1 in [4]. The assertion of the lemma is an easy consequence of (3.12) and (3.13).

Finally we have the following decomposition for the group G.

THEOREM 3.6. *Let* $(\mathfrak{g}, \mathfrak{h}, \sigma)$ *be a simple symmetric triple of split rank* r, *satisfying the condition* (C). *Let* G *be the adjoint group of* $\mathfrak{g}$, *and* K *be the maximal compact subgroup of* G *generated by the subalgebra* $\mathfrak{k}$ in (1.1). *Let* C *be the analytic subgroup generated by the split Cartan subalgebra* $\mathfrak{c}$ *of* $\mathfrak{g}$ *given in* (1.5). *Let* H_l $(0\leqq l\leqq r)$ *be the isotropy subgroup of* G *at the point* $\bar{o}_l$ (*cf.* (3.9)). *Suppose that the two Weyl groups* $W(M)$ *and* $W(M^*)$ *coincides. Then* G *can be expressed as*

$$G=KCH_l \qquad (0\leqq l\leqq r)\,. \tag{3.14}$$

PROOF. Let us consider an open $(r-l)$-face $P(I_l, \varepsilon)$ of the closed r-cube $\overline{C(o_1, o_2)}$. Then, by Lemmas 3.1 and 3.2, we have $P(I_l, \varepsilon)=C((\mathrm{Ad}\, k)o_l)$, where $k\in N_{K'}(\mathfrak{c})$. The right-hand side is the C-orbit under the ξ-equivariant action. As is known in [4], Ad k coincides with the ξ-equivariant action of k, since $k\in C(Z)$. Since k normalizes C, we have $P(I_l, \varepsilon)=(\xi^{-1}C\xi)(\xi^{-1}k\xi)(o_l)=\xi^{-1}Ck(\bar{o}_l)=\xi^{-1}kC(\bar{o}_l)$, which implies that

$$\xi(P(I_l, \varepsilon))=kC(\bar{o}_l)\,. \tag{3.15}$$

Consequently,

$$\xi(P^{(l)})\subset KC(\bar{o}_l)\subset G\bar{o}_l=M_l\,. \tag{3.16}$$

Therefore we have

$$K\xi(P^{(l)})\subset M_l\,. \tag{3.17}$$

So, by Lemmas 3.5 and 3.3, we conclude

$$\tilde{M}=\coprod_{l=0}^{r} M_l\,, \tag{3.18}$$

and hence, again by Lemma 3.5 and (3.17) we obtain $M_l=K\xi(P^{(l)})$. In view of (3.16) we have $M_l=K\xi(P^{(l)})\subset KC(\bar{o}_l)\subset M_l$. Therefore we get $G\bar{o}_l=KC\bar{o}_l$, which implies the assertion of the theorem.

In the course of the proof, we have shown

COROLLARY 3.7. *Under the same assumption as in the theorem, the* G-*orbit* M_l *in* $\tilde{M}$ *is written as*

$$M_l = KC\bar{o}_l\ , \qquad 0 \leqq l \leqq r\ . \tag{3.19}$$

REMARK 3.8. The G-orbit decomposition (3.18) of $\tilde{M}$ was originally obtained in [4], while the above arguments give an alternative proof for it under the assumption that $W(M) = W(M^*)$.

Let $a_l = \exp \sum_{i=1}^{l} E_i$ and $b_l = \exp \sum_{i=1}^{l} E_{-i}$ $(1 \leqq l \leqq r)$, and let $a_0 = b_0 = 1$. Then one can write

$$\bar{o}_l = (a_l o^-, b_l o^+)\ , \qquad 0 \leqq l \leqq r\ . \tag{3.20}$$

The following corollary is a special case of a result of Takeuchi [11].

COROLLARY 3.9. *Under the assumption of Theorem 3.6, we have the* (U^-, U^+)*-coset decomposition of* G:

$$G = \coprod_{l=0}^{r} U^- a_l^{-1} b_l U^+\ . \tag{3.21}$$

PROOF. Let $g \in G$, and let us write it in the form $g = a^{-1}b$, where $a, b \in G$. From (3.18) it follows that the point $(ao^-, bo^+) \in \tilde{M}$ is sent to one of the reference points, $\bar{o}_l$, under the diagonal G-action on $\tilde{M}$. An easy argument shows that $g \in U^- a^{-1} b U^+ = U^- a_l^{-1} b_l U^+$. The disjointness of the union in (3.21) also follows from (3.18).

Appendix

We give here the table of the numbers $i_l = \operatorname{rank} \mathscr{K}(o_l)$, $0 \leqq l \leqq r$, for each simple parahermitian symmetric pair $(\mathfrak{g}, \mathfrak{h})$ of split rank r. Note that by the G-covariance of the operator $\mathscr{K}$ ([4]), all possible values of the rank of the operator $\mathscr{K}(x)$ are $\{i_0, i_1, \cdots, i_r\}$, as long as x varies in $\mathfrak{m}$.

Type	$(\mathfrak{g}, \mathfrak{h})$	$i_l\ (0 \leqq l \leqq r)$
AI	$(\mathfrak{sl}(r+q, \boldsymbol{R}), \mathfrak{sl}(r, \boldsymbol{R}) + \mathfrak{sl}(q, \boldsymbol{R}) + \boldsymbol{R})$ $1 < r \leqq q$	$(r-l)(q-l)$
AII	$(\mathfrak{sl}(r+q, \boldsymbol{H}), \mathfrak{sl}(r, \boldsymbol{H}) + \mathfrak{sl}(q, \boldsymbol{H}) + \boldsymbol{R})$ $1 < r \leqq q$	$4(r-l)(q-l)$
AIII	$(\mathfrak{su}(r, r), \mathfrak{sl}(r, \boldsymbol{C}) + \boldsymbol{R}),\ r \geqq 2$	$(r-l)^2$
BDI	$(\mathfrak{so}(p, q), \mathfrak{so}(p-1, q-1) + \boldsymbol{R}),\ 2 \leqq p \leqq q$	$i_0 = p+q-2,\ i_1 = 1,\ i_2 = 0$
BDII	$(\mathfrak{so}(1, q), \mathfrak{so}(q-1) + \boldsymbol{R}),\ 2 \leqq q$	$i_0 = q-1,\ i_1 = 0$
CI	$(\mathfrak{sp}(r, \boldsymbol{R}), \mathfrak{gl}(r, \boldsymbol{R})),\ r \geqq 2$	$(r-l)(r-l+1)/2$
CII	$(\mathfrak{sp}(r, r), \mathfrak{gl}(r, \boldsymbol{H})),\ r \geqq 2$	$(r-l)(2r-2l+1)$
DI	$(\mathfrak{so}(2r, 2r), \mathfrak{gl}(2r, \boldsymbol{R})),\ r \geqq 2$	$(r-l)(2r-2l-1)$
	$(\mathfrak{so}(2r+1, 2r+1), \mathfrak{gl}(2r+1, \boldsymbol{R})),\ r \geqq 2$	$(r-l)(2r-2l+1)$
DIII	$(\mathfrak{so}(2r, \boldsymbol{H}), \mathfrak{gl}(r, \boldsymbol{H})),\ r \geqq 2$	$(r-l)(2r-2l-1)$

EI	$(E_6^1, \mathfrak{so}(5,5)+\boldsymbol{R})$	$i_0=16,\ i_1=5,\ i_2=0$
EIV	$(E_6^4, \mathfrak{so}(1,9)+\boldsymbol{R})$	$i_0=16,\ i_1=0$
EV	$(E_7^1, E_6^1+\boldsymbol{R})$	$i_0=27,\ i_1=10,\ i_2=1,\ i_3=0$
EVII	$(E_7^3, E_6^4+\boldsymbol{R})$	$i_0=27,\ i_1=10,\ i_2=1,\ i_3=0$
AI^C	$(\mathfrak{sl}(r+q, \boldsymbol{C}), \mathfrak{sl}(r, \boldsymbol{C})+\mathfrak{sl}(q, \boldsymbol{C})+\boldsymbol{C})$ $1<r\leqq q$	$2(r-l)(q-l)$
BDI^C	$(\mathfrak{so}(n+2, \boldsymbol{C}), \mathfrak{so}(n, \boldsymbol{C})+\boldsymbol{C}),\ n\geqq 3$	$i_0=2n,\ i_1=2,\ i_2=0$
CI^C	$(\mathfrak{sp}(r, \boldsymbol{C}), \mathfrak{gl}(r, \boldsymbol{C})),\ r\geqq 2$	$(r-l)(r-l+1)$
DI^C	$(\mathfrak{so}(4r, \boldsymbol{C}), \mathfrak{gl}(2r, \boldsymbol{C})),\ r\geqq 2$	$2(r-l)(2r-2l-1)$
	$(\mathfrak{so}(4r+2, \boldsymbol{C}), \mathfrak{gl}(2r+1, \boldsymbol{C})),\ r\geqq 2$	$2(r-l)(2r-2l+1)$
EI^C	$(E_6^C, \mathfrak{so}(10, \boldsymbol{C})+\boldsymbol{C})$	$i_0=32,\ i_1=10,\ i_2=0$
EV^C	$(E_7^C, E_6^C+\boldsymbol{C})$	$i_0=54,\ i_1=20,\ i_2=2,\ i_3=0$

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