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# On Some Hypersurfaces of High-Dimensional Tori Related with the Riemann Zeta-Function

Kohji MATSUMOTO and Tetsuro MIYAZAKI

*Rikkyo University and University of Tokyo*
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## §1. Background of the problem.

Let $s=\sigma+it$ be a complex variable, and $\zeta(s)$ the Riemann zeta-function. Bohr-Jessen [1] [2] discussed the value-distribution of $\zeta(s)$ on the line $\sigma=\sigma_0$ $(>1/2)$, and proved the following result:

Let $R$ be any closed rectangle in the complex plane with the edges parallel to the axes, and $L(T)$ the measure of the set

$$\{t\in[0,\ T]\,|\log\zeta(\sigma_0+it)\in R\}\ .$$

Then, there exists the limit $W=\lim(L(T)/T)$ as $T$ tends to infinity, which depends only on $\sigma_0$ and $R$. (In case $1/2<\sigma_0\leqq 1$, caused by the possibility of the existence of the zeros of $\zeta(s)$, there must be a slight modification.)

For proving this result, Bohr-Jessen introduced the set

$$\Omega(R)=\Omega_N(R)=\{(\theta_1,\ \cdots,\ \theta_N)\in[0,\ 1)^N\,|\,S_N(\theta_1,\ \cdots,\ \theta_N)\in R\}\ ,$$

where $N$ is a large positive integer, and

$$S_N(\theta_1,\ \cdots,\ \theta_N)=-\sum_{n=1}^{N}\log(1-p_n^{-\sigma_0}\exp(2\pi i\theta_n))$$

($p_n$ denotes the $n$-th prime number). In fact they showed that, if we denote the measure of the set $\Omega_N(R)$ by $W_N$, then $W_N$ tends to $W$, as $N$ tends to infinity.

Recently, the first-named author has tried to refine Bohr-Jessen's argument, and obtained, in case $\sigma_0>1$, the asymptotic formula

$$L(T)=WT+O(T(\log\log(T))^{-(\sigma_0-1)/7+\varepsilon})$$

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for any $\varepsilon>0$ (see [5]). Furthermore, a closer investigation leads to a similar result on the line $\sigma=\sigma_0$ for $1/2<\sigma_0\leqq 1$ ([6]). To prove these results, the study of the geometric behaviour of $\Omega(R)$ is indispensable. In this article, we discuss such properties of $\Omega(R)$.

NOTATIONS. We shall denote the $N$-dimensional torus $[0,1)^N$ by $T^N$. The symbol $\partial X$ signifies the boundary of the set $X$. For any subset $Y$ of the complex plane $\boldsymbol{C}$,

$$\Omega(Y)=\{(\theta_1,\cdots,\theta_N)\in T^N \mid S_N(\theta_1,\cdots,\theta_N)\in Y\} .$$

In particular, if $Y$ is the line $\{z \mid \mathrm{Re}(z)=k\}$ (resp. $\{z \mid \mathrm{Im}(z)=k\}$), we write $\Omega(k)$ (resp. $\Omega^*(k)$) instead of $\Omega(Y)$. For any small positive $\varepsilon$ and any subset $Z$ of $T^N$,

$$Z_\varepsilon=\{(\theta_1,\cdots,\theta_N)\in T^N \mid \mathrm{dist}((\theta_1,\cdots,\theta_N),Z)\leqq\varepsilon\} .$$

We write the $n$-dimensional volume of a set $A$ ($\subset T^N$) as $\mathrm{vol}_n(A)$ ($1\leqq n\leqq N$). Throughout the following sections, the $O$-constants depend only on $\sigma_0$.

## §2. Statement of results. Some reductions.

Our main purpose in this paper is the estimation of $\mathrm{vol}_N((\partial\Omega(R))_\varepsilon)$. At first, we discuss some reductions of the problem. Since $S_N$ is continuous, it can be easily checked that $\partial\Omega(R)\subset\Omega(\partial R)$. Hence we have

$$\mathrm{vol}_N((\partial\Omega(R))_\varepsilon)\leqq\mathrm{vol}_N((\Omega(\partial R))_\varepsilon) . \tag{2.1}$$

Let $A_1+iB_1$, $A_1+iB_2$, $A_2+iB_1$ and $A_2+iB_2$ be the four vertices of the rectangle $R$. Then it is obvious (see Fig. 1) that

$$\mathrm{vol}_N((\Omega(\partial R))_\varepsilon)\leqq\sum_{p=1}^{2}\mathrm{vol}_N((\Omega(A_p))_\varepsilon)+\sum_{q=1}^{2}\mathrm{vol}_N((\Omega^*(B_q))_\varepsilon) . \tag{2.2}$$

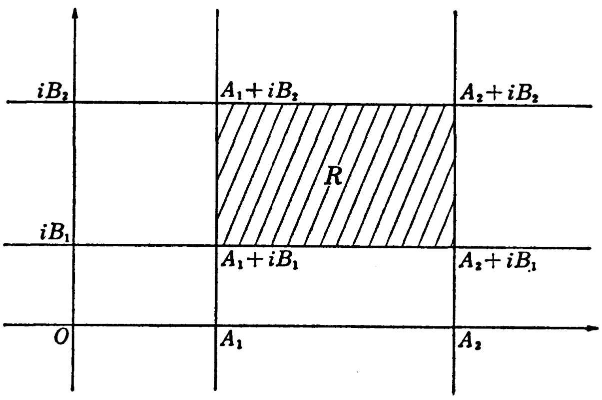



FIGURE 1

Hence it is sufficient to estimate $\mathrm{vol}_N((\Omega(k))_\varepsilon)$ and $\mathrm{vol}_N((\Omega^*(k))_\varepsilon)$ for real $k$.

We now calculate the Jacobian of $S_N$:

$$\frac{\partial S_N}{\partial \theta_n}=\frac{2\pi i p_n^{-\sigma_0}\exp(2\pi i\theta_n)}{1-p_n^{-\sigma_0}\exp(2\pi i\theta_n)}$$
$$(=\Theta_n,\ \text{say})\ ,$$

and therefore,

$$\frac{\partial(\mathrm{Re}(S_N))}{\partial\theta_n}=\mathrm{Re}(\Theta_n) \tag{2.3}$$
$$=\frac{-2\pi p_n^{-\sigma_0}\sin(2\pi\theta_n)}{1-2p_n^{-\sigma_0}\cos(2\pi\theta_n)+p_n^{-2\sigma_0}}$$

for $1\leqq n\leqq N$. So we see that, on $\Omega(k)$, the differentiation of $\mathrm{Re}(S_N)$ does not vanish except for the points $(\theta_1,\cdots,\theta_N)$ for which $\mathrm{Re}(\Theta_n)=0$ holds for every $n\leqq N$. Hence, if we put

$$T_a^N=\left\{(\theta_1,\cdots,\theta_N)\in T^N \,\middle|\, \left[\sum_{n=1}^{N}(\mathrm{Re}(\Theta_n))^2\right]^{1/2}>a\right\}$$

for small positive $a$, then $T(k)=\Omega(k)\cap T_a^N$ is a smooth submanifold of $T_a^N$ (see [7], Chap. II, §10, Corollary of Theorem 1). Concerning the geometric properties of $T(k)$, we will show the following lemmas in the sections below.

LEMMA 1. $\mathrm{vol}_{N-1}(T(k))\leqq 2N$ *for any real* $k$.

LEMMA 2.

$$\mathrm{vol}_N((\Omega(k))_\varepsilon\cap T_a^N)=O(a^{-1}\varepsilon\cdot\sup_{-\infty<k<\infty}\mathrm{vol}_{N-1}(T(k)))\ .$$

Next we discuss the volume of $T^N-T_a^N$. Since the denominator of the right-hand side of (2.3) is not larger than $(1+p_n^{-\sigma_0})^2$, we have

$$T^N-T_a^N\subset\{(\theta_1,\cdots,\theta_N)\in T^N\mid |\mathrm{Re}(\Theta_n)|\leqq a \text{ for any } n\}$$
$$\subset\{(\theta_1,\cdots,\theta_N)\in T^N\mid |\sin(2\pi\theta_n)|\leqq((1+p_n^{-\sigma_0})^2/2\pi p_n^{-\sigma_0})a \text{ for any } n\}\ .$$

In the interval $0\leqq\theta_n<1/4$, we have $4\theta_n\leqq\sin(2\pi\theta_n)$, so it follows that if

$$|\sin(2\pi\theta_n)|\leqq\frac{(1+p_n^{-\sigma_0})^2}{2\pi p_n^{-\sigma_0}}a\ , \tag{2.4}$$

then $\theta_n\leqq(1/4)\cdot((1+p_n^{-\sigma_0})^2/2\pi p_n^{-\sigma_0})a$. Furthermore, if $\theta_n$ satisfies (2.4), then $(1/2)-\theta_n$, $(1/2)+\theta_n$ and $1-\theta_n$ also satisfy (2.4). Hence, the measure of the set $\{\theta_n\in[0,1)\mid\theta_n$ satisfies (2.4)$\}$ is not larger than $((1+p_n^{-\sigma_0})^2/2\pi p_n^{-\sigma_0})a$. Therefore,

$$\mathrm{vol}_N(T^N - T_a^N) \leqq \Big(\prod_{n=1}^{N} \frac{(1+p_n^{-\sigma_0})^2}{2\pi p_n^{-\sigma_0}}\Big) a^N = O\Big(a^N \prod_{n=1}^{N} p_n^{\sigma_0}\Big) . \tag{2.5}$$

Now we note that, for $N \geqq 2$, by using a result of Rosser-Schoenfeld [8], we can easily obtain that

$$\prod_{n=1}^{N} p_n^{\sigma_0} = \exp(\sigma_0 \sum_{n=1}^{N} \log(p_n)) \leqq \exp(2\sigma_0 N \cdot \log(N)) = N^{2\sigma_0 N} . \tag{2.6}$$

We combine (2.5), (2.6) and the above two lemmas to get

$$\begin{aligned} \mathrm{vol}_N((\Omega(k))_\varepsilon) &\leqq \mathrm{vol}_N((\Omega(k))_\varepsilon \cap T_a^N) + \mathrm{vol}_N(T^N - T_a^N) \\ &= O(a^{-1}\varepsilon N + (aN^{2\sigma_0})^N) . \end{aligned}$$

The same estimate holds for $\mathrm{vol}_N((\Omega^*(k))_\varepsilon)$, and therefore, by (2.1) and (2.2) we now obtain the following

THEOREM. *For any $\sigma_0 > 1/2$ and any small positive $a$ and $\varepsilon$,*

$$\mathrm{vol}_N((\partial\Omega(R))_\varepsilon) = O(a^{-1}\varepsilon N + (aN^{2\sigma_0})^N) . \tag{2.7}$$

This result is used essentially in [5] and [6]. (In those papers, we put $\varepsilon = r^{-1}N^{1/2}$ for some $r \gg N$. If we choose $a = (r^{-1}N^{3/2})^{1/(N+1)}N^{-2\sigma_0}$, then the right-hand side of (2.7) is surpassed by $r^{-1+1/(N+1)}N^{(3/2)+2\sigma_0}$.)

The following two sections are devoted to the proofs of Lemma 1 and Lemma 2, respectively.

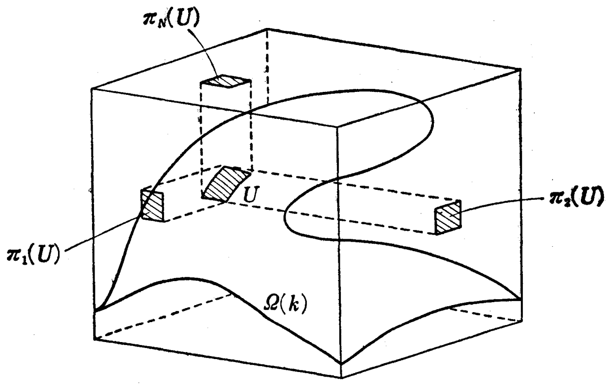
## §3. Proof of Lemma 1.

Let $(\theta_1, \cdots, \theta_N)$ be a point of $T(k)$, and we denote a unit normal vector of $T(k)$ at $(\theta_1, \cdots, \theta_N)$ by $\sum_{n=1}^{N} a_n(\partial/\partial\theta_n)$. Then, the corresponding volume form is

$$\omega = \sum_{n=1}^{N} (-1)^n a_n d\theta_1 \wedge \cdots \wedge d\theta_{n-1} \wedge d\theta_{n+1} \wedge \cdots \wedge d\theta_N$$

(see [4], Appendix 6, and [7], Chap. V, §5, Problem 1). We put $P_n = \{(\theta_1, \cdots, \theta_N) \in T^N \mid \theta_n = 0\}$ $(1 \leqq n \leqq N)$, and by $\pi_n$ we mean the projection from $T(k)$ to $P_n$. If we can take a sufficiently small neighbourhood $U$ of $(\theta_1, \cdots, \theta_N)$ in $T(k)$ which is mapped diffeomorphically onto $\pi_n(U)$ by $\pi_n$, we can evaluate $\mathrm{vol}_{N-1}(U)$, using the naturality of the integral, as follows (see Fig. 2):

$$\begin{aligned} \mathrm{vol}_{N-1}(U) &= \int_U \omega \\ &= \sum_{n=1}^{N} (-1)^n \int_U a_n d\theta_1 \wedge \cdots \wedge d\theta_{n-1} \wedge d\theta_{n+1} \wedge \cdots \wedge d\theta_N \\ &= \sum_{n=1}^{N} (-1)^n \int_{\pi_n(U)} a_n \circ \pi_n^{-1} d\theta_1 \wedge \cdots \wedge d\theta_{n-1} \wedge d\theta_{n+1} \wedge \cdots \wedge d\theta_N . \end{aligned}$$




$U=U^{(1)}_{j_1}\cap\cdots\cap U^{(N)}_{j_N}$

FIGURE 2

We note that, as $\sum a_n(\partial/\partial\theta_n)$ is a unit vector, $|a_n|\leqq 1$. Hence,

$$\text{(3.1)}\qquad \begin{aligned}\mathrm{vol}_{N-1}(U)&\leqq\sum_{n=1}^{N}\int_{\pi_n(U)}d\theta_1\cdots d\theta_{n-1}d\theta_{n+1}\cdots d\theta_N\\ &=\sum_{n=1}^{N}\mathrm{vol}_{N-1}(\pi_n(U))\,.\end{aligned}$$

These arguments are based on the assumption that $\pi_n$ maps $U$ diffeomorphically onto $\pi_n(U)$. Now we show that we can take such a neighbourhood $U$ for any point with exceptions of a measure zero subset of $T(k)$. In fact, the point of which we can not take such a neighbourhood is characterized by the condition that the $\theta_n$-component of the (unit) normal vector of the tangent space of $T(k)$ at that point is equal to zero. Let $\boldsymbol{n}$ be a unit normal vector field of $T(k)$. Since $\mathrm{Re}(S_N)$ is identically $k$ on $T(k)$ and $\partial(\mathrm{Re}(S_N))/\partial\theta_n=\mathrm{Re}(\Theta_n)$, we can take

$$\text{(3.2)}\qquad \begin{aligned}\boldsymbol{n}&=\left(\sum_{n=1}^{N}\frac{\partial(\mathrm{Re}(S_N))}{\partial\theta_n}\cdot\frac{\partial}{\partial\theta_n}\right)\Big/\left\|\sum_{n=1}^{N}\frac{\partial(\mathrm{Re}(S_N))}{\partial\theta_n}\cdot\frac{\partial}{\partial\theta_n}\right\|\\ &=\left(\sum_{n=1}^{N}(\mathrm{Re}(\Theta_n))^2\right)^{-1/2}\sum_{n=1}^{N}\mathrm{Re}(\Theta_n)\cdot\frac{\partial}{\partial\theta_n}\,,\end{aligned}$$

where the symbol $\|\ \ \|$ denotes the standard norm. Hence, the set of the points we now consider is characterized by the relation $\mathrm{Re}(\Theta_n)=0$, that is, $\theta_n=0,\ 1/2$. We define the two subtori $T_0$ and $T_{1/2}$ of $T^N$ by the equations $\theta_n=0$ and $\theta_n=1/2$, respectively. We apply the argument analogous to that of $T(k)$ on $T_0$, $T_{1/2}$, then we get that the two sets $T_0\cap T(k)$ and $T_{1/2}\cap T(k)$ are submanifolds of $T_0$ and $T_{1/2}$, respectively, with codimension one. Also, those sets are the codimension one submanifolds of $T(k)$, so we have

$$\mathrm{vol}_{N-1}(T_0 \cap T(k)) = \mathrm{vol}_{N-1}(T_{1/2} \cap T(k)) = 0 .$$

Since this is true for any $n$, we ignore the union of those sets for $n=1, \cdots, N$ in the remainder of this section, by removing it from $T(k)$ if necessary.

Next we fix real numbers $\theta_1^0, \cdots, \theta_{n-1}^0, \theta_{n+1}^0, \cdots, \theta_N^0$ belonging to the interval $[0, 1)$, and consider

$$L_n = \{(\theta_1^0, \cdots, \theta_{n-1}^0, \theta_n, \theta_{n+1}^0, \cdots, \theta_N^0) \in T^N \mid 0 \leqq \theta_n < 1\} .$$

This is a line segment in $T^N = [0, 1)^N$, parallel to some axis. The image $S_N(L_n)$ is a closed convex curve in $C$ (see Bohr-Jessen [3]), so the intersection of $S_N(L_n)$ and the line $\{z \mid \mathrm{Re}(z) = k\}$ consists of at most two points. Since the mapping $S_N$, when restricted onto $L_n$, is injective, the set $L_n \cap T(k)$ also consists of at most two points (see Fig. 3).

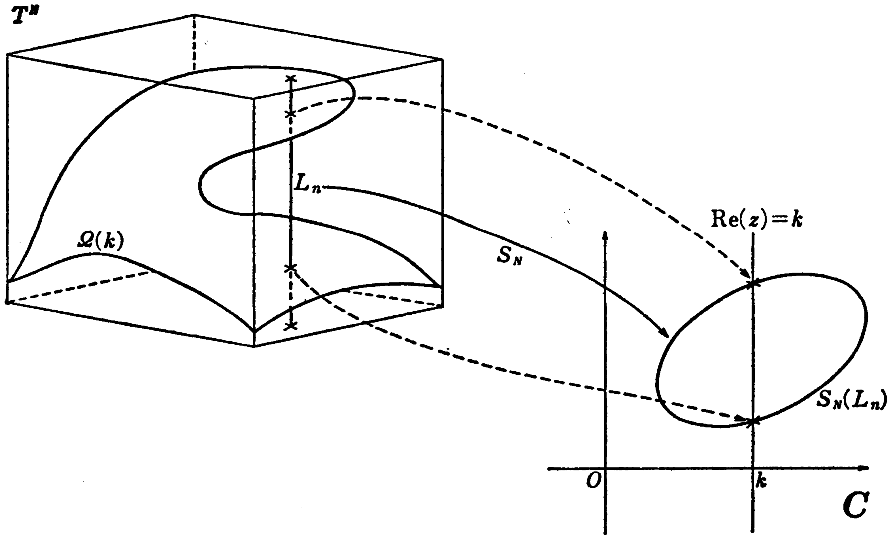



FIGURE 3

We denote such points by $(\theta_1^{(j)}, \cdots, \theta_N^{(j)})$ $(j=1, 2)$. If we put

$$L_n(\eta) = \{(\theta_1, \cdots, \theta_N) \in T^N \mid \theta_m^0 - \eta \leqq \theta_m \leqq \theta_m^0 + \eta \ (m \neq n), 0 \leqq \theta_n < 1\}$$

for sufficiently small $\eta$, then $L_n(\eta) \cap T(k)$ consists of at most two connected components $U_1^{(n)}$ and $U_2^{(n)}$, where $U_j^{(n)}$ is a neighbourhood of $(\theta_1^{(j)}, \cdots, \theta_N^{(j)})$ $(j=1, 2)$. We cover $T(k)$ by the neighbourhoods of the type $U_{j_1}^{(1)} \cap \cdots \cap U_{j_N}^{(N)}$ ($j_n$ takes the value 1 or 2 for $n=1, \cdots, N$), and apply (3.1) to each of these neighbourhoods. The union of those

$$\pi_n(U_{j_1}^{(1)} \cap \cdots \cap U_{j_N}^{(N)})$$

covers $P_n$ at most two times, so it follows that

$$\begin{aligned}\mathrm{vol}_{N-1}(T(k)) &\leqq \mathrm{vol}_{N-1}(\cup (U_{j_1}^{(1)} \cap \cdots \cap U_{j_N}^{(N)})) \\ &\leqq \sum_{n=1}^{N} \sum (\mathrm{vol}_{N-1}(\pi_n(U_{j_1}^{(1)} \cap \cdots \cap U_{j_N}^{(N)}))) \\ &\leqq 2 \sum_{n=1}^{N} \mathrm{vol}_{N-1}(P_n) = 2N .\end{aligned}$$

This proves Lemma 1.

## §4. Proof of Lemma 2.

At first we show that

$$\left[\sum_{n=1}^{N} (\mathrm{Re}(\Theta_n))^2\right]^{1/2} \leqq C = C(\sigma_0)$$

for any $(\theta_1, \cdots, \theta_N) \in T^N$. In fact, by using the inequality

$$1 - 2p_n^{-\sigma_0} \cos(2\pi\theta_n) + p_n^{-2\sigma_0} \geqq (1 - p_n^{-\sigma_0})^2 ,$$

we have (see (2.3))

$$\begin{aligned}\left[\sum_{n=1}^{N} (\mathrm{Re}(\Theta_n))^2\right]^{1/2} &\leqq \left[\sum_{n=1}^{N} \left(\frac{2\pi p_n^{-\sigma_0}}{1 - p_n^{-\sigma_0}}\right)^2\right]^{1/2} \\ &\ll \left(\sum_{n=1}^{N} p_n^{-2\sigma_0}\right)^{1/2} \leqq \left(\sum_{n=1}^{\infty} p_n^{-2\sigma_0}\right)^{1/2} = C(\sigma_0) .\end{aligned}$$

Now we prove the following

LEMMA 3.

$$(\Omega(k))_\varepsilon \cap T_a^N \subset \bigcup_{k - C\varepsilon \leqq t \leqq k + C\varepsilon} T(t) \quad (= T[k, C\varepsilon], \ \mathit{say}) .$$

First we show

$$(\Omega(k))_\varepsilon \subset \bigcup_{k - C\varepsilon \leqq t \leqq k + C\varepsilon} \Omega(t) . \tag{4.1}$$

We consider the isometric path $\gamma: [0, \varepsilon] \to (\Omega(k))_\varepsilon$ which satisfies $\gamma(0) \in \Omega(k)$. We write the differentiation of $\gamma$ as

$$\dot{\gamma}(u) = \sum_{n=1}^{N} b_n \left(\frac{\partial}{\partial \theta_n}\right)_{\gamma(u)} .$$

Then $\sum_{n=1}^{N} b_n^2 = 1$, and

$$(\mathrm{Re}(S_N))_* \dot{\gamma}(u) = \sum_{n=1}^{N} b_n (\mathrm{Re}(S_N))_* \left(\frac{\partial}{\partial \theta_n}\right)_{\gamma(u)}$$

$$=\sum_{n=1}^{N} b_n \left(\frac{\partial(\mathrm{Re}(S_N))}{\partial\theta_n}\right)_{\gamma(u)} \left(\frac{d}{dt}\right)_{\mathrm{Re}(S_N)\circ\gamma(u)}$$
$$=\left(\sum_{n=1}^{N} b_n \,\mathrm{Re}(\Theta_n)_{\gamma(u)}\right)\cdot\left(\frac{d}{dt}\right)_{\mathrm{Re}(S_N)\circ\gamma(u)} ,$$

where $(\mathrm{Re}(S_N))_*$ is the differentiation of $\mathrm{Re}(S_N)$. By using Schwarz' inequality, we have

$$\int_0^\varepsilon \|(\mathrm{Re}(S_N))_*\dot{\gamma}(u)\|\,du = \int_0^\varepsilon \left|\sum_{n=1}^{N} b_n \,\mathrm{Re}(\Theta_n)_{\gamma(u)}\right| du$$
$$\leqq \int_0^\varepsilon \left(\sum_{n=1}^{N} b_n^2\right)^{1/2} \left[\sum_{n=1}^{N} (\mathrm{Re}(\Theta_n)_{\gamma(u)})^2\right]^{1/2} du$$
$$\leqq \varepsilon \max_{0\leqq u\leqq \varepsilon} \left[\sum_{n=1}^{N} (\mathrm{Re}(\Theta_n)_{\gamma(u)})^2\right]^{1/2} \leqq C\varepsilon .$$

This means that the image of $\mathrm{Re}(S_N)\circ\gamma$ is included in $[k-C\varepsilon, k+C\varepsilon]$. In other words,

$$\gamma([0, \varepsilon]) \subset \bigcup_{k-C\varepsilon\leqq t\leqq k+C\varepsilon} \Omega(t) .$$

Since this is true for any such paths, (4.1) holds. Taking the intersection of the both sides of (4.1) with $T_a^N$, Lemma 3 follows.

Now, to prove Lemma 2, it is sufficient to evaluate $\mathrm{vol}_N(T[k, C\varepsilon])$. Let $\boldsymbol{n}$ be a unit normal vector field of $T(t)$, and $\xi$ the dual form of $\boldsymbol{n}$. We recall that $\boldsymbol{n}$ can be given by (3.2). By the calculation similar to that of $(\mathrm{Re}(S_N))_*\dot{\gamma}(u)$, we get

$$(\mathrm{Re}(S_N))_*(\boldsymbol{n}) = \left[\sum_{n=1}^{N} (\mathrm{Re}(\Theta_n))^2\right]^{1/2} \frac{d}{dt} .$$

The dual version of this formula is

$$\xi = \left[\sum_{n=1}^{N} (\mathrm{Re}(\Theta_n))^2\right]^{-1/2} (\mathrm{Re}(S_N))^*(dt) ,$$

where $(\mathrm{Re}(S_N))^*$ is the dual mapping of $(\mathrm{Re}(S_N))_*$.

We denote the volume form of $T(t)$ by $\omega_t$. Since

$$\left[\sum_{n=1}^{N} (\mathrm{Re}(\Theta_n))^2\right]^{1/2} > a$$

for any $(\theta_1, \cdots, \theta_N) \in T(t)$, we have

$$\mathrm{vol}_N(T[k, C\varepsilon]) = \int_{T[k,C\varepsilon]} \omega_t \wedge \xi$$
$$= \int_{T[k,C\varepsilon]} \omega_t \wedge \left[\sum_{n=1}^{N} (\mathrm{Re}(\Theta_n))^2\right]^{-1/2} (\mathrm{Re}(S_N))^*(dt)$$

$$
\begin{aligned}
&=\int_{k-C\varepsilon}^{k+C\varepsilon} dt \int_{T(t)} \left[\sum_{n=1}^{N} (\mathrm{Re}(\Theta_n))^2\right]^{-1/2} \omega_t \\
&\leqq a^{-1}\int_{k-C\varepsilon}^{k+C\varepsilon} dt \int_{T(t)} \omega_t \\
&\leqq 2a^{-1}C\varepsilon \cdot \max_{k-C\varepsilon\leqq t\leqq k+C\varepsilon} \mathrm{vol}_{N-1}(T(t)) \,.
\end{aligned}
$$

This inequality, with Lemma 3, completes the proof of Lemma 2.

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*Present Address*:
DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION
IWATE UNIVERSITY
UEDA, MORIOKA 020
AND
KAISEI GAKUEN
NISHI-NIPPORI, ARAKAWA-KU, TOKYO 116