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On the Fundamental Units and the Class Numbers of Real Quadratic Fields II

Takashi AZUHATA

Science University of Tokyo
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Introduction

Let M be a positive square-free integer and $\boldsymbol{Q}(\sqrt{M})$ be a real quadratic field with discriminant D. Denote by $h(M)$ and ε_M the class number and the fundamental unit of $\boldsymbol{Q}(\sqrt{M})$ respectively. After the works of Ankeny-Chowla-Hasse [1] and Hasse [5], there appeared several results about the lower bound of $h(M)$ with some conditions when $\varepsilon_M=(t+u\sqrt{D})/2$ is small (see Lang [6], Takeuchi [7] and Yokoi [9], [10], [11]). They used the basic result that the Diophantine equation $x^2-Dy^2=\pm 4m$ has no solutions in $\boldsymbol{Z}$ for $m<(t-2)/u^2$ if $N\varepsilon_M=1$, and for $m<t/u^2$ if $N\varepsilon_M=-1$. As a special case, we have $h(M)>(\log(D-1)/\log 4)-1$ for $M=(4C)^2+1$ $(C>1)$ from it. In this note, we also consider the same problem using continued fractions. We will get $h(M)>(\log D/\log 4)-1$ for $M=(C^s+\mu(C^t-\lambda))^2+4\lambda C^t$ with $s>t\geqq 1$, λ, $\mu=\pm 1$ if C is even and is not a power of 2. For these types of M with $t=1$, Bernstein [3], [4] gave the continued fractional expansion (c.f.e.) of $\sqrt{M}$ and the explicit representation of ε_M. The special case of them was mentioned in Yamamoto [8]. We also give ε_M explicitly for the above types of M and the lower bound of ε_M for another types of M from the c.f.e. of $\omega_0=(M_0+\sqrt{M})/2$ $(M_0<\sqrt{M}<M_0+2$, $M_0\equiv 1 \pmod 2)$. The lower bounds of ε_M were also given in [8] for sufficiently large M with several conditions. Then we investigate $h(M)$ for the above types of M and give the lower bounds with some conditions as mentioned above as a special case.

§1. Preliminaries.

In this section, we describe some basic properties of quadratic irrationals and ideals in real quadratic fields, which we will need in later

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sections. We employ some symbols and fundamental terminologies without definitions which we already mentioned in [2]. Let $M=a^2+b$ be a positive square-free integer with $0<b\leqq 2a$, and $D=D(M)$ be the discriminant of a quadratic field $Q(\sqrt{M})$. We first note the following lemma about the c.f.e. of quadratic irrationals from [2].

LEMMA 1. *Let* $\alpha=\alpha_0=(a-r_0+\sqrt{M})/c_0$ $(c_0>0)$ *be a quadratic irrational with discriminant* D, *and let*

$$\alpha_i=k_i+1/\alpha_{i+1}\,, \quad k_i=[\alpha_i] \qquad (i\geqq 0)$$

be the c.f.e. of α *with* $\alpha_i=(a-r_i+\sqrt{M})/c_i$. *Then the integers* c_i, k_i *and* r_i *are given by the following recurrence formula:*

$$(*)_i \qquad 2a-r_i=c_ik_i+r_{i+1}\,, \qquad c_{i+1}=c_{i-1}+(r_{i+1}-r_i)k_i$$

where $0\leqq r_{i+1}<c_i$ *and* $c_{-1}=(b+2ar_0-r_0^2)/c_0$.

It is easily verified that the integers c_i and r_i are both even if $a\equiv 1 \pmod 2$ and $b\equiv 0 \pmod 4$. So we set $\alpha_i=(a-2r_i+\sqrt{M})/2c_i$ in this case. Then $(*)_i$ is equivalent to the following:

$$(**)_i \qquad a-r_i=c_ik_i+r_{i+1}\,, \qquad c_{i+1}=c_{i-1}+(r_{i+1}-r_i)k_i$$

where $0\leqq r_{i+1}<c_i$ and $c_{-1}=(b+4ar_0-4r_0^2)/4c_0$. The c.f.e. of a reduced quadratic irrational is purely periodic.

LEMMA 2. *Let* α *be a reduced quadratic irrational with discriminant* D. *If the period of* α *is* n, *then we have*

$$\varepsilon_M=\alpha_0\alpha_1\cdots\alpha_{n-1} \qquad \textit{with} \quad N\varepsilon_M=(-1)^n\,.$$

For a proof of this lemma, see [8]. Let $\alpha=(B+\sqrt{D})/2A$ be a quadratic irrational with discriminant D. Then $\mathfrak{A}(\alpha)=[A, (B+\sqrt{D})/2]$ is an ideal of $Q(\sqrt{M})$ with $N(\mathfrak{A})=A$. We call $\mathfrak{A}(\alpha)$ the ideal corresponding to a quadratic irrational α. In particular, when α is a reduced quadratic irrational with $A=N(\mathfrak{A}(\alpha))>1$, we say that $\mathfrak{A}(\alpha)$ is a reduced ideal.

LEMMA 3. *Let* $\mathfrak{A}$ *be an integral ideal of* $Q(\sqrt{M})$ *which is prime to its conjugate ideal* $\mathfrak{A}'$. *If* $N(\mathfrak{A})<\sqrt{D}/2$, *then* $\mathfrak{A}$ *is a reduced ideal.*

PROOF. Let $\mathfrak{A}=[A, (B+\sqrt{D})/2]$, $\alpha=(B+\sqrt{D})/2A$ and set $k=[-\alpha']$. Then it follows from $0<-\alpha'-k<1$ that $-1<k+\alpha'<0$. Since $\alpha-\alpha'=\sqrt{D}/A>2$, we get $k+\alpha>1$. Thus $k+\alpha$ is a reduced quadratic irrational and $\mathfrak{A}=[A, (2Ak+B+\sqrt{D})/2]$. Hence we get our assertion.

This lemma was also mentioned in [8]. Let $\omega_0=(D_0+\sqrt{D})/2$ be the reduced quadratic irrational with $D_0<\sqrt{D}<D_0+2$, $D_0\equiv D \pmod 2$. The following lemma is well-known.

LEMMA 4. *Let $\mathfrak{A}(\alpha)$ be an ideal corresponding to a quadratic irrational α. Then $\mathfrak{A}(\alpha)$ is principal iff α is equivalent to ω_0.*

§2. Continued fractions and fundamental units.

Let $M=T^2+4\lambda B$ be a positive square-free integer where $T=AB^m+\mu(B-\lambda)$, $A\geqq 1$, $B\geqq 1$, $AB^m>B$ and λ, $\mu=\pm 1$. We assume that $[\sqrt{M}]=T$ or $T-1$ according as $\lambda=1$ or -1. We first consider the c.f.e. of a reduced quadratic irrational $\omega_0=(T+\lambda-1+\sqrt{M})/2$ with discriminant $D=M$. We put $Q_1=1$ and $Q_2=B$ if $A=1$. When $A>1$, we define Q_i successively as follows:

1) $Q_1=1$,
2) if Q_i is defined and $AQ_i|B$ or $B|AQ_i|B^2$, then

$$Q_1=1\ ,\quad Q_{i+1}=\begin{cases}AQ_i & \text{if}\quad AQ_i\leqq B\\ AQ_i/B & \text{if}\quad AQ_i>B\end{cases}\qquad (i\geqq 0)\ .$$

Q_i is a divisor of B. We set $\omega_0^{(i)}=(T+(\lambda-1)P_i+\sqrt{M})/2P_i$ with $P_i=B/Q_i$. We have $\omega_1=\omega_0^{(1)}$ for $\lambda=1$ from $\omega_1=1/(\omega_0-T)=(T+\sqrt{M})/2B$. When $\lambda=-1$, it follows from $[\sqrt{M}]=T-1$ that $2T-1>4B$. Then we have

$$\omega_1=1/(\omega_0-T+2)=(T-2+\sqrt{M})/2(T-B-1)\ ,\quad [\omega_1]=1\ ,$$
$$\omega_2=1/(\omega_1-1)=(T-2B+\sqrt{M})/2B=\omega_0^{(1)}\ .$$

Suppose that Q_{i+1} is defined. We notice that $\omega_0^{(2)}=\omega_0$ if $A=1$ and $\omega_0^{(i+1)}=\omega_0$ if $A>1$ and $AQ_i=B$. Let

$$\omega_j^{(i)}=k_j+1/\omega_{j+1}^{(i)}\ ,\quad k_j=[\omega_j^{(i)}]\qquad (j\geqq 0)$$

be the c.f.e. of $\omega_0^{(i)}$. We set $\omega_j^{(i)}=(a-2r_j+\sqrt{M})/2c_j$ if $\lambda=1$, and $\omega_j^{(i)}=(a-r_j+\sqrt{M})/c_j$ if $\lambda=-1$. We put

$$m'=m'(i)=\begin{cases}m-1 & \text{if}\quad A=1\ ,\\ m & \text{if}\quad A>1 \text{ and } AQ_i\leqq B\ ,\\ m+1 & \text{if}\quad A>1 \text{ and } AQ_i>B\ .\end{cases}$$

PROPOSITION. *Suppose that Q_{i+1} is defined and let the notations be as above. We have $\omega_{m''}^{(i)}=\omega_0^{(i+1)}$ where $m''=m''(i)=3m'$, $3m'-1$, $2m'+1$ or*

$4m'$ according as $(\lambda, \mu)=(1, 1)$, $(1, -1)$, $(-1, 1)$ *or* $(-1, -1)$. *And* $\omega_j^{(i)}$ *are given as follows*:

(i) *if* $(\lambda, \mu)=(1, 1)$, *then*

$$r_{3\nu-2}=P_iB^{\nu-1}-1\ ,\qquad c_{3\nu-2}=AB^m-AQ_iB^{m-\nu}-P_iB^{\nu-1}+B+1\ ,$$
$$r_{3\nu-1}=AQ_iB^{m-\nu}-1\ ,\qquad c_{3\nu-1}=AQ_iB^{m-\nu}\ ,$$
$$r_{3\nu}=B\ ,\qquad c_{3\nu}=P_iB^{\nu}\quad (1\leqq 3\nu-2,\ 3\nu-1,\ 3\nu\leqq 3m'-1)\ ;$$

(ii) *if* $(\lambda, \mu)=(1, -1)$, *then*

$$r_{3\nu-2}=1\ ,\qquad c_{3\nu-2}=AQ_iB^{m-\nu}\ ,$$
$$r_{3\nu-1}=AQ_iB^{m-\nu}-B\ ,\qquad c_{3\nu-1}=AB^m-AQ_iB^{m-\nu}-P_iB^{\nu}+B+1\ ,$$
$$r_{3\nu}=P_iB^{\nu}-B\ ,\qquad c_{3\nu}=P_iB^{\nu}\quad (1\leqq 3\nu-2,\ 3\nu-1,\ 3\nu\leqq 3m'-2)\ ;$$

(iii) *if* $(\lambda, \mu)=(-1, 1)$, *then*

$$r_{2\nu-1}=1\ ,\qquad c_{2\nu-1}=2AQ_iB^{m-\nu}\ ,$$
$$r_{2\nu}=2B-1\ ,\qquad c_{2\nu}=2P_iB^{\nu}\quad (1\leqq 2\nu-1,\ 2\nu\leqq 2m'-1)\ ,$$
$$r_{2m'}=2Q_{i+1}-1\ ,\qquad c_{2m'}=2(AB^m-P_{i+1}-Q_{i+1}+B+1)\ ;$$

(iv) *if* $(\lambda, u)=(-1, -1)$, *then*

$$r_{4\nu-3}=2P_iB^{\nu-1}-3\ ,\qquad c_{4\nu-3}=2(AB^m-AQ_iB^{m-\nu}-P_iB^{\nu-1}-B+1)\ ,$$
$$r_{4\nu-2}=2AQ_iB^{m-\nu}-3\ ,\qquad c_{4\nu-2}=2AQ_iB^{m-\nu}\ ,$$
$$r_{4\nu-1}=2AQ_iB^{m-\nu}-2B-1\ ,\qquad c_{4\nu-1}=2(AB^m-AQ_iB^{m-\nu}-P_iB^{\nu}+B-1)\ ,$$
$$r_{4\nu}=2P_iB^{\nu}-2B-1\ ,\qquad c_{4\nu}=2P_iB^{\nu}$$
$$(1\leqq 4\nu-3,\ 4\nu-2,\ 4\nu-1,\ 4\nu\leqq 4m'-2)\ ,$$
$$r_{4m'-1}=2Q_{i+1}-1\ ,\qquad c_{4m'-1}=2(AB^m-P_{i+1}-Q_{i+1}-B-1)\ .$$

PROOF. (i) We put $M=a^2+b$ with $a=[\sqrt{M}]$. It follows from the assumption that $a=AB^m+B-1$ and $b=4B$. We get easily our assertion by the induction from the following calculation using Lemma 1:

$$a=AB^m+B-1\ ,\quad b=4B\ ,\quad c_0=P_i\ ,\quad r_0=0\ ,\quad c_{-1}=Q_i\ ,$$
$$(**)_0\qquad AB^m+B-1=c_0(AQ_iB^{m-1}+Q_i-1)+P_i-1\ ,$$
$$k_0=AQ_iB^{m-1}+Q_i-1\ ,\quad r_1=P_i-1\ ,$$
$$c_1=c_{-1}+(P_i-1)(AQ_iB^{m-1}+Q_i-1)=AB^m-AQ_iB^{m-1}-P_i+B+1\ .$$

Assume that

$$c_{3\nu-3}=P_iB^{\nu-1}\ ,\quad c_{3\nu-2}=AB^m-AQ_iB^{m-\nu}-P_iB^{\nu-1}+B+1\ ,\quad r_{3\nu-2}=P_iB^{\nu-1}-1\ .$$

Then we have

$(**)_{3\nu-2}$ $\quad AB^m-P_iB^{\nu-1}+B=c_{3\nu-2}+AQ_iB^{m-\nu}-1\ ,\quad k_{3\nu-2}=1\ ,$

$r_{3\nu-1}=AQ_iB^{m-\nu}-1\ ,\quad c_{3\nu-1}=AQ_iB^{m-\nu}\ ,$

$(**)_{3\nu-1}$ $\quad AB^m-AQ_iB^{m-\nu}+B=c_{3\nu-1}(P_iB^{\nu-1}-1)+B\ ,\quad k_{3\nu-1}=P_iB^{\nu-1}-1\ ,$

$r_{3\nu}=B\ ,\quad c_{3\nu}=c_{3\nu-2}-(AQ_iB^{m-\nu}-B-1)(P_iB^{\nu-1}-1)=P_iB^{\nu}\ ,$

$(**)_{3\nu}$ $\quad AB^m-1=c_{3\nu}(AQ_iB^{m-\nu-1}-1)+P_iB^{\nu}-1\ ,$

$k_{3\nu}=AQ_iB^{m-\nu-1}-1\ ,\quad r_{3\nu+1}=P_iB^{\nu}-1\ ,$

$$\begin{aligned}c_{3\nu+1}&=c_{3\nu-1}+(P_iB^{\nu}-B-1)(AQ_iB^{m-\nu-1}-1)\\&=AB^m-AQ_iB^{m-\nu-1}-P_iB^{\nu}+B+1\ .\end{aligned}$$

This sequence holds as long as $r_{3\nu-1}<c_{3\nu-2}$, that is, $1\leqq 3\nu-2,\ 3\nu-1,\ 3\nu\leqq 3m'-2$. Consequently, we find that

$$k_{3\nu-2}=1\ ,\quad k_{3\nu-1}=P_iB^{\nu-1}-1\ ,\quad k_{3\nu}=AQ_iB^{m-\nu-1}-1$$
$$(1\leqq 3\nu-2,\ 3\nu-1,\ 3\nu\leqq 3m'-2)\ .$$

We notice that $AQ_iB^{m-m'}=Q_{i+1}$ and $P_iB^{m'}=AP_{i+1}B^m$, so

$$c_{3m'-2}=AB^m-Q_{i+1}-AP_{i+1}B^{m-1}+B+1\ ,\quad c_{3m'-1}=Q_{i+1}\ ,\quad r_{3m'-1}=Q_{i+1}-1\ .$$

Then we finally obtain

$(**)_{3m'-1}$ $\quad AB^m+B-Q_{i+1}=c_{3m'-1}(AP_{i+1}B^{m-1}+P_{i+1}-1)\ ,$

$k_{3m'-1}=AP_{i+1}B^{m-1}+P_{i+1}-1\ ,\quad r_{3m'}=0\ ,$

$c_{3m'}=c_{3m'-2}-(Q_{i+1}-1)(AP_{i+1}B^{m-1}+P_{i+1}-1)=P_{i+1}\ .$

We also get the other cases in the same way as follows:

(ii) $\quad a=AB^m-B+1,\quad b=4B,\quad c_0=P_i\quad r_0=0,\quad c_{-1}=Q_i,$

$(**)_0$ $\quad AB^m-B+1=c_0(AQ_iB^{m-1}-Q_i)+1\ ,\quad k_0=AQ_iB^{m-1}-Q_i\ ,$

$r_1=1\ ,\quad c_1=AQ_iB^{m-1}\ ,$

$c_{3\nu-3}=P_iB^{\nu-1}\ ,\quad c_{3\nu-2}=AQ_iB^{m-\nu}\ ,\quad r_{3\nu-2}=1\ ,$

$(**)_{3\nu-2}$ $\quad AB^m-B=c_{3\nu-2}(P_iB^{\nu-1}-1)+AQ_iB^{m-\nu}-B\ ,$

$k_{3\nu-2}=P_iB^{\nu-1}-1\ ,\quad r_{3\nu-1}=AQ_iB^{m-\nu}-B\ ,$

$$\begin{aligned}c_{3\nu-1}&=c_{3\nu-3}+(AQ_iB^{m-\nu}-B-1)(P_iB^{\nu-1}-1)\\&=AB^m-AQ_iB^{m-\nu}-P_iB^{\nu}+B+1\ ,\end{aligned}$$

$(**)_{3\nu-1}$ $\quad AB^m-AQ_iB^{m-\nu}+1=c_{3\nu-1}+P_iB^{\nu}-B\ ,$

$k_{3\nu-1}=1\ ,\quad r_{3\nu}=P_iB^{\nu}-B\ ,\quad c_{3\nu}=P_iB^{\nu}\ ,$

$(**)_{3\nu}$ $\quad AB^m-P_iB^{\nu}+1=c_{3\nu}(AQ_iB^{m-\nu-1}-1)+1\ ,$

$k_{3\nu}=AQ_iB^{m-\nu-1}-1\ ,\quad r_{3\nu}=1\ ,$

$c_{3\nu+1}=c_{3\nu-1}-(P_iB^{\nu}-B-1)(AQ_iB^{m-\nu-1}-1)=AQ_iB^{m-\nu-1}\ ,$

$$k_{3\nu-2}=P_iB^{\nu-1}-1\ ,\quad k_{3\nu-1}=1\ ,\quad k_{3\nu}=AQ_iB^{m-\nu-1}-1$$
$$(1\leqq 3\nu-2,\ 3\nu-1,\ 3\nu\leqq 3m'-3)\ ,$$
$$c_{3m'-3}=AP_{i+1}B^{m-1}\ ,\quad c_{3m'-2}=Q_{i+1}\ ,\quad r_{3m'-2}=1$$

$(**)_{3m'-2}$ $$AB^m-B=c_{3m'-2}(AP_{i+1}B^{m-1}-P_{i+1})\ ,$$
$$k_{3m'-2}=AP_{i+1}B^{m-1}-P_{i+1}\ ,\quad r_{3m'-1}=0\ ,$$
$$c_{3m'-1}=c_{3m'-3}-(AP_{i+1}B^{m-1}-P_{i+1})=P_{i+1}\ ;$$

(iii) $$a=AB^m+B,\quad b=2AB^m-2B+1,\quad c_0=2P_i,\quad r_0=2P_i-1,$$
$$c_{-1}=(2AB^m-2B+1+2(AB^m+B)(2P_i-1)-(2P_i-1)^2)/2P_i$$
$$=2(AB^m-P_i-Q_i+B+1)\ ,$$

$(*)_0$ $$2AB^m+2B-2P_i+1=c_0(AQ_iB^{m-1}+Q_i-1)+1\ ,$$
$$k_0=AQ_iB^{m-1}+Q_i-1\ ,\quad r_1=1\ ,$$
$$c_1=c_{-1}-2(P_i-1)(AQ_iB^{m-1}+Q_i-1)=2AQ_iB^{m-1}\ ,$$
$$c_{2\nu-2}=2P_iB^{\nu-1}\ ,\quad c_{2\nu-1}=2AQ_iB^{m-\nu}\ ,\quad r_{2\nu-1}=1\ ,$$

$(*)_{2\nu-1}$ $$2AB^m+2B-1=c_{2\nu-1}P_iB^{\nu-1}+2B-1\ ,\quad k_{2\nu-1}=P_iB^{\nu-1}\ ,\quad r_{2\nu}=2B-1\ ,$$
$$c_{2\nu}=c_{2\nu-2}+2(B-1)P_iB^{\nu-1}=2P_iB^{\nu}\ ,$$

$(*)_{2\nu}$ $$2AB^m+1=c_{2\nu}AQ_iB^{m-\nu-1}+1\ ,\quad k_{2\nu}=AQ_iB^{m-\nu-1}\ ,\quad r_{2\nu+1}=1\ ,$$
$$c_{2\nu+1}=c_{2\nu-1}-2(B-1)AQ_iB^{m-\nu-1}=2AQ_iB^{m-\nu-1}\ ,$$
$$k_{2\nu-1}=P_iB^{\nu-1}\ ,\quad k_{2\nu}=AQ_iB^{m-\nu-1}\quad (1\leqq 2\nu-1,\ 2\nu\leqq 2m'-2)\ ,$$
$$c_{2m'-1}=2Q_{i+1}\ ,\quad c_{2m'}=2(AB^m-P_{i+1}-Q_{i+1}+B+1)\ ,\quad r_{2m'}=2Q_{i+1}-1$$

$(*)_{2m'}$ $$2AB^m+2B-2Q_{i+1}+1=c_{2m'}+2P_{i+1}-1\ ,\quad k_{2m'}=1\ ,$$
$$r_{2m'+1}=2P_{i+1}-1\ ,\quad c_{2m'+1}=2P_{i+1}\ ;$$

(iv) $$a=AB^m-B-2,\quad b=2AB^m-6B-3,\quad c_0=2P_i,\quad r_0=2P_i-1,$$
$$c_{-1}=(2AB^m-6B-3+2(AB^m-B-2)(2P_i-1)-(2P_i-1)^2)/2P_i$$
$$=2(AB^m-P_i-Q_i-B-1)\ ,$$

$(*)_0$ $$2AB^m-2B-2P_i-3=c_0(AQ_iB^{m-1}-Q_i-2)+2P_i-3\ ,$$
$$k_0=AQ_iB^{m-1}-Q_i-2\ ,\quad r_1=2P_i-3\ ,$$
$$c_1=c_{-1}-2(AQ_iB^{m-1}-Q_i-2)=2(AB^m-AQ_iB^{m-1}-P_i-B+1)\ ,$$
$$c_{4\nu-4}=2P_iB^{\nu-1}\ ,\quad c_{4\nu-3}=2(AB^m-AQ_iB^{m-\nu}-P_iB^{\nu-1}-B+1)\ ,$$
$$r_{4\nu-3}=2P_iB^{\nu-1}-3\ ,$$

$(*)_{4\nu-3}$ $$2AB^m-2P_iB^{\nu-1}-2B-1=c_{4\nu-3}+2AQ_iB^{m-\nu}-3\ ,\quad k_{4\nu-3}=1\ ,$$
$$r_{4\nu-2}=2AQ_iB^{m-\nu}-3\ ,\quad c_{4\nu-2}=2AQ_iB^{m-\nu}\ ,$$

$(*)_{4\nu-2}$ $$2AB^m-2AQ_iB^{m-\nu}-2B-1=c_{4\nu-2}(P_iB^{\nu-1}-2)+2AQ_iB^{m-\nu}-2B-1\ ,$$
$$k_{4\nu-2}=P_iB^{\nu-1}-2\ ,\quad r_{4\nu-1}=2AQ_iB^{m-\nu}-2B-1\ ,$$

$$c_{4\nu-1}=c_{4\nu-3}-2(B-1)(P_iB^{\nu-1}-2)=2(AB^m-AQ_iB^{m-\nu}-P_iB^\nu+B-1)\ ,$$

$$(*)_{4\nu-1}\qquad 2AB^m-2AQ_iB^{m-1}-3=c_{4\nu-1}+2P_iB^\nu-2B-1\ ,\quad k_{4\nu-1}=1\ ,$$

$$r_{4\nu}=2P_iB^\nu-2B-1\ ,\quad c_{4\nu}=2P_iB^\nu\ ,$$

$$(*)_{4\nu}\qquad 2AB^m-2P_iB^\nu-3=c_{4\nu}(AQ_iB^{m-\nu-1}-2)+2P_iB^\nu-3,$$

$$k_{4\nu}=AQ_iB^{m-\nu-1}-2\ ,\quad r_{4\nu+1}=2P_iB^\nu-3\ ,$$

$$c_{4\nu+1}=c_{4\nu-1}+2(B-1)(AQ_iB^{m-\nu-1}-2)$$
$$=2(AB^m-AQ_iB^{m-\nu-1}-P_iB^\nu-B+1)\ ,$$

$$k_{4\nu-3}=1\ ,\quad k_{4\nu-2}=P_iB^{\nu-1}-2\ ,\quad k_{4\nu-1}=1\ ,\quad k_{4\nu}=AQ_iB^{m-\nu-1}-2$$
$$(1\leqq 4\nu-3,\ 4\nu-2,\ 4\nu-1,\ 4\nu\leqq 4m'-3)\ ,$$

$$c_{4m'-3}=2(AB^m-Q_{i+1}-AP_{i+1}B^{m-1}-B+1)\ ,\quad c_{4m'-2}=2Q_{i+1}-3\ ,$$

$$r_{4m'-2}=2Q_{i+1}-3\ ,$$

$$(*)_{4m'-2}\qquad 2AB^m-2B-2Q_{i+1}-1=c_{4m'-2}(AP_{i+1}B^{m-1}-P_{i+1}-2)+2Q_{i+1}-1\ ,$$

$$k_{4m'-2}=AP_{i+1}B^{m-1}-P_{i+1}-2\ ,\quad r_{4m'-1}=2Q_{i+1}-1\ ,$$

$$c_{4m'-1}=c_{4m'-3}+2(AP_{i+1}B^{m-1}-P_{i+1}-2)=2(AB^m-P_{i+1}-Q_{i+1}-B-1)\ ,$$

$$(*)_{4m'-1}\qquad 2AB^m-2B-2Q_{i+1}-3=c_{4m'-1}+2P_{i+1}-1\ ,\quad k_{4m'-1}=1\ ,$$

$$r_{4m'}=2P_{i+1}-1\ ,\quad c_{4m'}=2P_{i+1}\ .$$

We notice that $c_{4m-7}=2(B^m-B^{m-1}-2B+1)>0$ in (iv) if $A=1$ and $m=2$. In fact, it follows from $2T-1=2B^2-2B-3>4B$ that $B^2-3B+1>0$. This completes the proof.

If $A=1$ or $AQ_k=B$ for some k, then we get the whole c.f.e. of ω_0 from this Proposition. However, we get some beginning part in the other cases as long as Q_i is defined. Then we have the following

THEOREM 1. *Let* $M=T^2+4\lambda B$ *and* $T=AB^m+\mu(B-\lambda)$ *be as above,*
(i) *if* $A=C^u$, $B=C^t$, $s=tm+u$ *with* $C>1$, $t>u\geqq 0$, $(t,u)=1$, *then*

$$\varepsilon_M=\left(\frac{T+\sqrt{M}}{2B}\right)^s\left(\frac{T+2\lambda\mu+\sqrt{M}}{2}\right)^t\qquad with\quad N\varepsilon_M=(-\lambda)^s(\lambda\mu)^t\ ,$$

(ii) *if* $B=A^{l-1}C$ *with* $l\geqq 1$, $A\nmid C$, $C>1$, *then*

$$\varepsilon_M>A^l\left(\frac{T+\sqrt{M}}{2B}\right)^{2ml}\left(\frac{T+2\lambda\mu+\sqrt{M}}{2}\right)^{2l}\ .$$

PROOF. (i) We assume that $A>1$ since our assertion for $A=1$ and $B=C$ was already shown in [4]. Setting $Q_i=C^{q_i}$, we find that

$$q_1=1\ ,\qquad q_{i+1}=\begin{cases}q_i+u & \text{if}\quad q_i+u\leqq t\ ,\\ q_i+u-t & \text{if}\quad q_i+u>t\ ,\end{cases}$$

$$q_i \equiv (i-1)u \pmod t\ , \quad 0<q_2,\ q_3,\cdots,q_t<t\ , \quad q_{t+1}=t\ ,$$

and that $\sum_{i=1}^{t} m'(i)=mt+u-1=s-1$ since $AQ_i>B$ iff $t<q_i+u<t+u$. It follows from Proposition that $\omega_{m''(t)}^{(t)}=\omega_0^{(t+1)}=\omega_0$ and $\omega_j^{(i)}\neq\omega_0$ if $1\leqq i\leqq t$, $(i, j)\neq(t, m''(t))$. So we may write $\varepsilon_M=\omega_1\eta_1\cdots\eta_t$ (if $\lambda=1$) or $\omega_1\omega_2\eta_1\cdots\eta_t$ (if $\lambda=-1$) with $\eta_i=\prod_{j=1}^{m''(i)}\omega_j^{(i)}$ from Lemma 2. We note that $\omega_j^{(i)}\omega_{j+1}^{(i)}=k_j\omega_{j+1}^{(i)}+1$ from $\omega_j^{(i)}=k_j+1/\omega_{j+1}^{(i)}$. If $(\lambda, \mu)=(1, 1)$, then $[\omega_{3\nu-2}^{(i)}]=1$. Therefore we get

$$\begin{aligned}
&\omega_{3\nu-2}^{(i)}\omega_{3\nu-1}^{(i)}=\omega_{3\nu-1}^{(i)}+1=(T+2+\sqrt{M})/2c_{3\nu-1}\ ,\\
&\omega_{3\nu-2}^{(i)}\omega_{3\nu-1}^{(i)}\omega_{3\nu}^{(i)}=(T+2+\sqrt{M})(T-2B+\sqrt{M})/4c_{3\nu-1}c_{3\nu}\\
&\qquad=(2T(T-B+1)+2(T-B+1)\sqrt{M})/4AB^{m+1}=(T+\sqrt{M})/2B\ ,
\end{aligned}$$

for $1\leqq\nu\leqq m'(i)-1$ and

$$\omega_{3m'-2}^{(i)}\omega_{3m'-1}^{(i)}=(T+2+\sqrt{M})/2Q_{i+1}\ , \quad \omega_{3m'}^{(i)}=\omega_0^{(i+1)}=(T+\sqrt{M})/2P_{i+1}\ .$$

So,

$$\eta_i=\omega_1^{(i)}\omega_2^{(i)}\cdots\omega_{3m'}^{(i)}=\left(\frac{T+\sqrt{M}}{2B}\right)^{m'}\left(\frac{T+2+\sqrt{M}}{2}\right).$$

Hence we get

$$\varepsilon_M=\omega_1\eta_1\eta_2\cdots\eta_t=\left(\frac{T+\sqrt{M}}{2B}\right)^{s}\left(\frac{T+2+\sqrt{M}}{2}\right)^{t}.$$

The other cases are shown similarly. If $(\lambda, \mu)=(1, -1)$, then $[\omega_{3\nu-1}^{(i)}]=1$. So,

$$\begin{aligned}
&\omega_{3\nu-1}^{(i)}\omega_{3\nu}^{(i)}=\omega_{3\nu}^{(i)}+1=(T+2B+\sqrt{M})/2c_{3\nu}\ ,\\
&\omega_{3\nu-2}^{(i)}\omega_{3\nu-1}^{(i)}\omega_{3\nu}^{(i)}=(T-2+\sqrt{M})(T+2B+\sqrt{M})/4c_{3\nu-2}c_{3\nu}\\
&\qquad=(T+\sqrt{M})/2B \qquad (1\leqq\nu\leqq m'(i)-1)\ ,\\
&\omega_{3m'-2}^{(i)}=(T-2+\sqrt{M})/2Q_{i+1}\ , \quad \omega_{3m'-1}^{(i)}=(T+\sqrt{M})/2P_{i+1}\ ,\\
&\eta_i=\omega_1^{(i)}\omega_2^{(i)}\cdots\omega_{3m'-1}^{(i)}=\left(\frac{T+\sqrt{M}}{2B}\right)^{m'}\left(\frac{T-2+\sqrt{M}}{2}\right).
\end{aligned}$$

If $(\lambda, \mu)=(-1, 1)$, then $[\omega_{2\nu-1}^{(i)}]=P_iB^{\nu-1}$. So,

$$\begin{aligned}
&\omega_{2\nu-1}^{(i)}\omega_{2\nu}^{(i)}=P_iB^{\nu-1}\omega_{2\nu}^{(i)}+1=(T+\sqrt{M})/2B \qquad (1\leqq\nu\leqq m'(i)-1)\ ,\\
&\omega_{2m'-1}^{(i)}=(T-2+\sqrt{M})/2Q_{i+1}\ , \quad \omega_{2m'}^{(i)}\omega_{2m'+1}^{(i)}=\omega_{2m'+1}^{(i)}+1=(T+\sqrt{M})/2P_{i+1}\ ,\\
&\eta_i=\omega_1^{(i)}\omega_2^{(i)}\cdots\omega_{2m'+1}^{(i)}=\left(\frac{T+\sqrt{M}}{2B}\right)^{m'}\left(\frac{T-2+\sqrt{M}}{2}\right),
\end{aligned}$$

$$\omega_1\omega_2=\omega_2+1=(T+\sqrt{M})/2B\ .$$

If $(\lambda, \mu)=(-1, -1)$, then, $[\omega^{(i)}_{4\nu-3}]=[\omega^{(i)}_{4\nu-1}]=1$. So,

$$\omega^{(i)}_{4\nu-3}\omega^{(i)}_{4\nu-2}=\omega^{(i)}_{4\nu-2}+1=(T+2+\sqrt{M})/c_{4\nu-2}\ ,$$
$$\omega^{(i)}_{4\nu-1}\omega^{(i)}_{4\nu}=\omega^{(i)}_{4\nu}+1=(T+2B+\sqrt{M})/c_{4\nu}\ ,$$
$$\omega^{(i)}_{4\nu-3}\omega^{(i)}_{4\nu-2}\omega^{(i)}_{4\nu-1}\omega^{(i)}_{4\nu}=(T+2+\sqrt{M})(T+2B+\sqrt{M})/4AB^{m+1}$$
$$=(T+\sqrt{M})/2B \qquad (1\leqq i\leqq m'(i)-1)\ ,$$
$$\omega^{(i)}_{4m'-3}\omega^{(i)}_{4m'-2}=(T+2+\sqrt{M})/2Q_{i+1}\ ,\quad \omega^{(i)}_{4m'-1}\omega^{(i)}_{4m'}=(T+\sqrt{M})/2P_{i+1}\ ,$$
$$\eta_i=\omega^{(i)}_1\omega^{(i)}_2\cdots\omega^{(i)}_{4m'}=\left(\frac{T+\sqrt{M}}{2B}\right)^{m'}\left(\frac{T+2+\sqrt{M}}{2}\right)\ ,$$
$$\omega_1\omega_2=\omega_2+1=(T+\sqrt{M})/2B\ .$$

The signature of $N\varepsilon_M$ follows from

$$\operatorname{sgn}(N(T+\sqrt{M}))=-\mu\ ,\qquad \operatorname{sgn}(N(T+2\lambda\mu+\sqrt{M}))=\lambda\mu\ .$$

(ii) In this case, we find that $Q_i=A^{i-1}$, $m'(i)=m$ for each i with $1\leqq i\leqq l$ and that Q_{l+1} is not defined. Then we get the c.f.e. of ω_0 from Proposition only up to $\omega_{n'}=(T-A^l-\lambda\mu(A^l-2)+\sqrt{M})/2A^l$ where $n'=m''-1$ or $m''-2$ according as $\lambda=1$ or -1. We notice that $\omega_0=-1/\omega'_1$ and that if α is a reduced quadratic irrational equivalent to ω_0, then so is $-1/\alpha'$. It follows from Lemma 2 that $\varepsilon_M>\eta|\eta'|^{-1}$ for $\eta=\beta_1\beta_2\cdots\beta_k$ if $\beta_1,\cdots,\beta_k$ are reduced quadratic irrationals such that $\beta_i\neq-1/\beta'_j$ for i, j with $1\leqq i, j\leqq k$. Under the same notations as in Lemmas 1 and 2, it is known that $-1/\alpha'_i=(a-r_i+\sqrt{M})/c_{i-1}$ for $i\geqq 1$ if $\alpha_0=\omega_0$ and $\alpha_i=(a-r_i+\sqrt{M})/c_i$ (see [2], Prop. 2). Then we find easily from Proposition that there are no relations $\alpha=-1/\beta'$ among $\omega_1,\cdots,\omega_{n'}$ in each case. So we set $\eta=\omega_1\cdots\omega_{n'}$. If $(\lambda, \mu)=(1, 1)$, then

$$\omega_1\omega^{(l)}_{3m-2}\omega^{(l)}_{3m-1}=(T+\sqrt{M})(T+2+\sqrt{M})/4A^lB\ ,$$
$$\eta=\omega_1\eta_1\eta_2\cdots\eta_{l-1}\omega^{(l)}_1\cdots\omega^{(l)}_{3m-1}=\left(\frac{T+\sqrt{M}}{2B}\right)^{ml}\left(\frac{T+2+\sqrt{M}}{2}\right)^l\ .$$

If $(\lambda, \mu)=(1, -1)$, then

$$\omega_1\omega^{(l)}_{3m-2}=(T+\sqrt{M})(T-2+\sqrt{M})/4A^lB\ ,$$
$$\eta=\omega_1\eta_1\eta_2\cdots\eta_{l-1}\omega^{(l)}_1\cdots\omega^{(l)}_{3m-2}=\left(\frac{T+\sqrt{M}}{2B}\right)^{ml}\left(\frac{T-2+\sqrt{M}}{2}\right)^l\ .$$

If $(\lambda, \mu)=(-1, 1)$, then

$$\omega_1\omega_2\omega_{2m-1}^{(l)}=(T+\sqrt{M})(T-2+\sqrt{M})/4A^lB\,,$$

$$\eta=\omega_1\omega_2\eta_1\eta_2\cdots\eta_{l-1}\omega_1^{(l)}\cdots\omega_{2m-1}^{(l)}=\left(\frac{T+\sqrt{M}}{2B}\right)^{ml}\left(\frac{T-2+\sqrt{M}}{2}\right)^l.$$

If $(\lambda, \mu)=(-1, -1)$, then

$$\omega_1\omega_2\omega_{4m-3}^{(l)}\omega_{4m-2}^{(l)}=(T+\sqrt{M})(T+2+\sqrt{M})/4A^lB\,,$$

$$\eta=\omega_1\omega_2\eta_1\eta_2\cdots\eta_{l-1}\omega_1^{(l)}\cdots\omega_{4m-2}^{(l)}=\left(\frac{T+\sqrt{M}}{2B}\right)^{ml}\left(\frac{T+2+\sqrt{M}}{2}\right)^l.$$

It follows from

$$\left(\frac{T+\sqrt{M}}{2B}\right)^{-1}=\frac{T+\sqrt{M}}{2\lambda}\,,\quad \left(\frac{T+2\lambda\mu+\sqrt{M}}{2A}\right)^{-1}=(T+2\lambda\mu+\sqrt{M})/2\lambda\mu B^m$$

that

$$|\eta'|^{-1}=A^l\left(\frac{T+\sqrt{M}}{2B}\right)^{ml}\left(\frac{T+2\lambda\mu+\sqrt{M}}{2A}\right)^l.$$

Hence we get our assertion. This completes the proof.

§3. Application to class numbers.

Let ω_0 be a reduced quadratic irrational with discriminant D mentioned in section 1. Denote by $\mathrm{PR}(M)$ the set of all the norms of principal reduced ideals of $\boldsymbol{Q}(\sqrt{M})$. We can find all the principal reduced ideals of $\boldsymbol{Q}(\sqrt{M})$ if we know the whole c.f.e. of ω_0. Then we get the information about the class number from Proposition using Lemmas 3 and 4.

THEOREM 2. *Suppose that* $M=(C^s+\mu(C^t-\lambda))^2+4\lambda C^t=T^2+4\lambda C^t$ *is a positive square-free integer with* $[\sqrt{M}]=T$ *or* $T-1$ *according as* $\lambda=1$ *or* -1 *and* $D>4C^2$ *where* $C>1$, $s>t\geqq 1$, $(s, t)=1$, $\lambda, \mu=\pm 1$. *Denote by* R' *the minimal norm of principal reduced ideals of* $\boldsymbol{Q}(\sqrt{M})$ *prime to* C *and set* $R=\min\{R', \sqrt{D}/2\}$. *Then we have*

(i) *If* C *is neither prime nor prime power, whose minimal prime divisor is* p, *then* $h(M)>(\log D-\log 4)/2\log p$,

(i)′ *If in particular* C *is even and is not a power of* 2, *then* $h(M)>(\log D/\log 4)-1$,

(ii) *If* $C=p^n$ *for some prime* p, *then* $n|h(M)$,

(iii) *If* $C=B^n$ *for some integer* B *which is neither prime nor prime power, and if* q *is the minimal prime divisor of* B *such that* $q^2>B$, *then* $n|h(M)$ *and* $h(M)\geqq n[(\log D-\log 4)/2\log q+1]$,

(iv) *If there is a prime* p *such that* $p<R$, $(C/p)=1$, $(C, p)=1$, *then* $h(M)>(\log R)/(\log p)$.

PROOF. Setting $s=tm+u$ with $0\leqq u<t$, our situation is the same as (i) in Theorem 1 so that PR(M) is completely given by Proposition. We notice that C is in PR(M) since $P_i=C$ for some i and $C<\sqrt{D}/2$. Let $\mathfrak{C}$ be a principal ideal of $\boldsymbol{Q}(\sqrt{M})$ with $N(\mathfrak{C})=C$.

(i) If p is a prime divisor of C, then there is a prime ideal $\mathfrak{P}$ dividing $\mathfrak{C}$ with $N(\mathfrak{P})=p$. It follows from Proposition that any power of p is not in PR(M) since $C\neq p^i$ for any integer $i>0$. Then we find that $\mathfrak{P}^i$ is a reduced ideal and is not principal as long as $p^i<\sqrt{D}/2$ from Lemmas 3 and 4. Hence we get our assertion. (i)$'$ is the special case of (i).

(ii) By the same notations and argument as in (i), we find that $\mathfrak{P}^i$ is not principal for $1\leqq i<n$ and $\mathfrak{P}^n=\mathfrak{C}$ is principal. This proves our assertion.

(iii) We get $n|h(M)$ in the same way as in (ii). We take ideals $\mathfrak{B}$ and $\mathfrak{Q}$ with $N(\mathfrak{B})=B$, $N(\mathfrak{Q})=q$ so that $\mathfrak{B}^n=\mathfrak{C}$, $\mathfrak{Q}'|\mathfrak{B}$ and set $\mathfrak{B}=\mathfrak{A}\mathfrak{Q}'$, $N(\mathfrak{A})=A$. Note that neither $\mathfrak{B}^i$ nor $\mathfrak{Q}^j$ is principal for i, j with $1\leqq i<n$, $1\leqq j\leqq[\log(\sqrt{D}/2)/\log q]$ from (i) and (ii). We show that $\mathfrak{B}^i\mathfrak{Q}^j$ is not principal for the above i, j. It follows from $\mathfrak{B}\mathfrak{Q}=q\mathfrak{A}$ that $\mathfrak{B}^i\mathfrak{Q}^j$ is equivalent to $\mathfrak{A}^i\mathfrak{Q}'^{i-j}$ or $\mathfrak{A}^i\mathfrak{Q}^{j-i}$ according as $i\geqq j$ or $i<j$. Then we find easily that $A^iQ^{i-j}<B^i<\sqrt{D}/2$ or $A^iQ^{j-i}<q^i<\sqrt{D}/2$ from $A<q$ and that every A^iq^{i-j} or A^iq^{j-i} is not in PR(M). Hence the above assertion holds. Next we show that any two of the ideals $\mathfrak{B}^i\mathfrak{Q}^j$ with $0\leqq i<n$, $0\leqq j\leqq n'$ are not in the same ideal class. Suppose that $\mathfrak{B}^i\mathfrak{Q}^j$ and $\mathfrak{B}^k\mathfrak{Q}^l$ are equivalent for $(i, j)\neq(k, l)$ with $j\geqq l$. Then $\mathfrak{B}^{i-k}\mathfrak{Q}^{j-l}$ (if $i\geqq k$) or $\mathfrak{B}^{n+i-k}\mathfrak{Q}^{j-l}$ (if $i<k$) is principal, which contradicts the above argument. Hence we have (iii).

(iv) There is a prime ideal $\mathfrak{P}$ with $N(\mathfrak{P})=p$. Our assertion follows in the same way as in (i).

This completes the proof of Theorem 2.

REMARK. If $M=4C^2+1$, then we have PR$(M)=\{C\}$. Hence Theorem 2 also holds for this case. However this may be considered as a special case since

$$4C^2+1=(C+C-1)^2+4C=(C+C+1)^2-4C\,.$$

The whole c.f.e. of ω_0 for $M=(C^m+\mu(C-\lambda)/2)^2+\lambda C$ with $C>1$, $C\equiv1 \pmod 2$, $m>1$, λ, $\mu=\pm1$ were given in [3], [4]. Then the similar statements as in Theorem 2 except for (i)$'$ also holds for these types of square-free integer M.

EXAMPLES. We set $f_k(X, Y)=(X+\mu(Y-\lambda))^2+4\lambda Y$ where $k=1, 2, 3$ or 4 according as $(\lambda, \mu)=(1, 1), (1, -1), (-1, 1)$ or $(-1, -1)$, and $r(M)=[(\log M/\log 4)-1]$.

M	$h(M)$	$r(M)$
$1705=f_1(6^2, 6)$	8	4
$985=f_2(6^2, 6)$	6	3
$817=f_4(6^2, 6)$	5	3
$63145=f_1(6^3, 6^2)$	20	6
$32905=f_2(6^3, 6^2)$	16	6
$63865=f_3(6^3, 6^2)$	28	6
$31897=f_4(6^3, 6^2)$	9	6
$11921=f_1(10^2, 10)$	10	5
$8321=f_2(10^2, 10)$	10	5
$12281=f_3(10^2, 10)$	11	5
$7881=f_4(10^2, 10)$	12	5
$812201=f_2(10^3, 10^2)$	44	8
$807801=f_4(10^3, 10^2)$	66	8

M	$h(M)$
$5073=f_1(2^6, 2^3)$	6
$3281=f_2(2^6, 2^3)$	6
$5297=f_3(2^6, 2^3)$	3
$2993=f_4(2^6, 2^3)$	6
$201957=f_2(2^9, 2^6)$	12
$491293=f_4(3^6, 3^3)$	24
$73505=f_1(2^8, 2^4)$	16
$58145=f_2(2^8, 2^4)$	16
$74465=f_3(2^8, 2^4)$	16
$57057=f_4(2^8, 2^4)$	16
$986177=f_2(2^{10}, 2^5)$	55
$981953=f_4(2^{10}, 2^5)$	50

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Present Address:
DEPARTMENT OF MATHEMATICS
SCIENCE UNIVERSITY OF TOKYO
WAKAMIYA, SHINJUKU-KU, TOKYO 162