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Minimal Affine Boundaries of Convex Sets

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Introduction.

Let M be a compact convex set in a real locally convex linear topological space V and denote by $A(M)$ the set of restrictions on M of all real affine and continuous functionals in V, i.e. $f\in A(M)$ iff $f(tx+(1-t)y)=tf(x)+(1-t)f(y)$ for any $t\in \boldsymbol{R}$. Remind that a subset N of M is called an *end subset* of M iff it consists of points z that satisfy the following condition: z can not be represented as $z=\lambda x+\mu y$ with $\lambda>0$, $\mu>0$, $\lambda+\mu=1$, unless x and y belong to N. *Extreme points* of M are the points that are end subsets of M. Let $E(M)$ stand for the closure of extreme points of M. This is the smallest closed subset of M within which any positive element of $A(M)$ attains its minimum. Indeed, let $f\in A(M)$, $f>0$, and let $\min_{x\in M} f(x)=a<b=\min_{x\in E(M)} f(x)$. Since f is affine, the set $M\cap\{f(x)\geqq b\}$ is a compact convex set that contains $E(M)$ and consequently it contains also the closed convex hull of $E(M)$, i.e., it contains the whole set M according to the Krein-Milman's theorem (e.g. [1]). Hence $f(x)\geqq b>a$ on M, that is a contradiction. So every positive element of $A(M)$ attains its minimum within $E(M)$. If a closed subset N of M possesses the same property, then its closed convex hull $[\langle N\rangle]$ will coincide with M. In fact, if $[\langle N\rangle]\neq M$ we can find a positive continuous affine functional $f\in A(M)$ for which $f(x)\geqq a>0$ on $[\langle N\rangle]$ but $f(x_0)<a$ for some point $x_0\in M$ in contradiction with our supposition on N. But the equality $[\langle N\rangle]=M$ implies that $N\supset E(M)$ since the latter is the smallest closed subset of M for which $[\langle N\rangle]=M$ (e.g. [2]). Here we introduce n-dimensional analogues to the closure $E(M)$ of extreme points of a compact convex set M.

§1. Affine n-boundaries.

Denote by $A^n(M)$ the set of all n-tuples $(f_1, \cdots, f_n)$ of elements of

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$A(M)$, by $Z(f_1, \cdots, f_n)$ the zero set of $(f_1, \cdots, f_n)$, i.e. $Z(f_1, \cdots, f_n)=\{x \in M: f_1(x)=f_2(x)=\cdots=f_n(x)=0\}$ and by $A_*^n(M)$ the set of all *regular* n-tuples over $A(M)$, i.e. $(f_1, \cdots, f_n) \in A_*^n(M)$ iff $Z(f_1, \cdots, f_n)=\emptyset$. $A_*^0(M)$ will stand for all constant elements of $A(M)$. Let $\|(f_1, \cdots, f_n)\|$ be the following function on M:

$$\|(f_1, \cdots, f_n)(x)\|=\left(\sum_{j=1}^{n} f_j^2(x)\right)^{1/2} . \tag{1}$$

DEFINITION 1. A subset E of a compact convex subset M of a real locally convex linear topological space V is called an *affine n-boundary* of M iff for every regular n-tuple $(f_1, \cdots, f_n)$ of affine continuous functionals on M there exists a point x_0 belonging to E such that for any $x \in M$ it holds:

$$\|(f_1, \cdots, f_n)(x_0)\| \leqq \|(f_1, \cdots, f_n)(x)\| , \tag{2}$$

i.e. iff the minimum of the function $\|(f_1, \cdots, f_n)\|$ is attained within E for every regular n-tuple $(f_1, \cdots, f_n) \in A_*^n(M)$.

DEFINITION 2. The intersection $E_n(M)$ of all closed affine n-boundaries of a compact convex subset M of V is called the *minimal affine n-boundary* of M.

It is clear that $E_1(M) \subset E_2(M) \subset \cdots \subset E_n(M) \subset \cdots$. According to the remark from the Introduction, we have that $E_1(M)=E(M) \neq \emptyset$. The next theorem shows that minimal affine n-boundaries of M are nonempty subsets of M for every $n>1$ and, moreover, it gives a description of them.

THEOREM 1. *The sets*

$$[\cup\{E(Z(f_1, \cdots, f_{n-1})): (f_1, \cdots, f_{n-1}) \in A^{n-1}(M)\}] \tag{3}$$

coincide with the minimal affine n-boundaries $E_n(M)$ of compact convex subsets M of V, where $[N]$ denotes the closure of N for a subset N in V.

PROOF. First we shall prove that the set (3) is an affine n-boundary of M. Let $(f_1, \cdots, f_n) \in A_*^n(M)$ and $x_0 \in M$. Without loss of generality (applying, if necessary, certain orthogonal transformation in $\boldsymbol{R}^n$) we can assume that $f_j(x_0)=0$ for any $j>1$, so that $(f_1(x_0), f_2(x_0), \cdots, f_n(x_0))=(f_1(x_0), 0, \cdots, 0)$ and $\|(f_1, \cdots, f_n)(x_0)\|^2=f_1^2(x_0)$. The set $Z_1=Z(f_2, \cdots, f_n)$ is an affine manifold, i.e. a translated linear subspace of V. Because f_1 does not vanish at Z_1 and $x_0 \in M \cap Z_1$, $f_1^2(x_0) \geqq \min_{E(Z_1)} f_1^2(x)$ according to our remark in the Introduction, applied to Z_1 and $f_1|_{Z_1}$. Consequently

$$\|(f_1, \cdots, f_n)(x_0)\|^2 = \sum_{j=1}^{n} f_j^2(x_0) = f_1^2(x_0) \geqq \min_{E(Z_1)} f_1^2(x) = \min_{E(Z_1)} \Big(\sum_{j=1}^{n} f_j^2(x)\Big)$$
$$\geqq \inf\Big\{\sum_{j=1}^{n} f_j^2(x) : x \in \cup\{E(Z(g_1, \cdots, g_{n-1})) : (g_1, \cdots, g_{n-1}) \in A^{n-1}(M)\}\Big\} .$$

Hence the continuous function $\|(f_1, \cdots, f_n)(x)\|$ attains its minimum within the set $[\cup\{E(Z(f_1, \cdots, f_{n-1}) : (f_1, \cdots, f_{n-1}) \in A^{n-1}(M)\}]$ for any regular n-tuple $(f_1, \cdots, f_n) \in A_*^n(M)$, i.e. (3) is an affine boundary of M. But (3) is the smallest affine n-boundary of M. Indeed, let $E \subset M$ be a closed affine n-boundary of M, i.e. let the minimum of the function $\|(f_1, \cdots, f_n)(x)\|$ is attained within E for any regular n-tuple over $A(M)$. Let $(g_1, \cdots, g_{n-1})$ be a fixed $(n-1)$-tuple over $A(M)$ and suppose that for some $f \in A(M)$ the restriction $f|_{Z(g_1, \cdots, g_{n-1})}$ is positive on the set $Z(g_1, \cdots, g_{n-1})$ and that $f(x) \geqq r > 0$ for some positive r and for every $x \in Z(g_1, \cdots, g_{n-1}) \cap E$. We shall show that then $f(x) \geqq r$ on the whole $Z(g_1, \cdots, g_{n-1})$. For any $\varepsilon > 0$, $\varepsilon < r$ there exists a neighborhood $U_\varepsilon \subset M$ of the set $Z(g_1, \cdots, g_{n-1}) \cap E$ on which $f(x) \geqq r - \varepsilon$. Consequently for some positive constant C_ε, big enough, on E we will have:

$$C_\varepsilon^2 \sum_{j=1}^{n-1} g_j^2(x) + f^2(x) \geqq (r - \varepsilon)^2 . \tag{4}$$

Consequently (4) will hold on the whole M because the n-tuple $(C_\varepsilon g_1, \cdots, C_\varepsilon g_{n-1}, f)$ is regular and E is a closed affine n-boundary of M. In particular on $Z(g_1, \cdots, g_{n-1})$ we will have that $f^2(x) \geqq (r - \varepsilon)^2$, from where $f^2(x) \geqq r^2$ because of the liberty of the choice of ε. We obtain that all affine functionals of $A(Z(g_1, \cdots, g_{n-1}))$ that are positive attain their minimums within $Z(g_1, \cdots, g_{n-1}) \cap E$, wherefrom $Z(g_1, \cdots, g_{n-1}) \cap E \supset E(Z(g_1, \cdots, g_{n-1}))$ because the latter set is the smallest closed affine 1-boundary of $Z(g_1, \cdots, g_{n-1})$. Now $E \supset \cup\{Z(g_1, \cdots, g_{n-1}) \cap E : (g_1, \cdots, g_{n-1}) \in A^{n-1}(M)\} \supset \cup\{E(Z(g_1, \cdots, g_{n-1})) : (g_1, \cdots, g_{n-1}) \in A^{n-1}(M)\}$ and by taking the closures we obtain finally that E contains the set (3). Q.E.D.

§2. Properties of n-affine boundaries.

COROLLARY 1. *The range of the minimal affine n-boundary of a compact convex subset M of V through any n-tuple $(f_1, \cdots, f_n)$ of affine functionals from $A(M)$ contains the topological boundary of the range of M, i.e.*

$$(f_1, \cdots, f_n)(E_n(M)) \supset b((f_1, \cdots, f_n)(M)) , \qquad \forall (f_1, \cdots, f_n) \in A^n(M) . \tag{5}$$

PROOF. Supposing that $b((f_1, \cdots, f_n)(M)) \setminus (f_1, \cdots, f_n)(E_n(M)) \neq \emptyset$, let

x_0 be such a point of M that $(f_1, \cdots, f_n)(x_0) \in b((f_1, \cdots, f_n)(M)) \setminus (f_1, \cdots, f_n)(E_n(M))$ and let

$$\min_{E_n(M)} \|(f_1, \cdots, f_n)(x_0) - (f_1, \cdots, f_n)(x)\| = \delta > 0 .$$

The continuous function $\|(f_1, \cdots, f_n)(x) - X^0\|$ on M, where $X^0 = (x_1^0, \cdots, x_n^0)$ is a fixed point from $\boldsymbol{R}^n \setminus (f_1, \cdots, f_n)(M)$ with $\|X^0 - (f_1, \cdots, f_n)(x_0)\| < \delta/2$, satisfies the following inequality:

$$\begin{aligned} &\|(f_1 - x_1^0, \cdots, f_n - x_n^0)(x)\| = \|(f_1, \cdots, f_n)(x) - X^0\| \\ &\quad \geqq | \, \|(f_1, \cdots, f_n)(x) - (f_1, \cdots, f_n)(x_0)\| - \|(f_1, \cdots, f_n)(x_0) - X^0\| \, | \geqq \delta/2 \end{aligned}$$

for any $x \in E_n(M)$. Hence

$$\|(f_1 - x_1^0, \cdots, f_n - x_n^0)(x)\| = \|(f_1, \cdots, f_n)(x) - X^0\| \geqq \delta/2$$

for any $x \in M$ because $(f_1 - x_1^0, \cdots, f_n - x_n^0)$ is a regular n-tuple over $A(M)$ (since $X^0 \notin (f_1, \cdots, f_n)(M)$) in contradiction with the choice of X^0. Consequently $b((f_1, \cdots, f_n)(M)) \setminus (f_1, \cdots, f_n)(E_n(M)) = \varnothing$. Q.E.D.

THEOREM 2. *The minimal affine n-boundary of a compact convex subset M of V coincides with the intersection of all closed subsets E of M, such that $(f_1, \cdots, f_n)(E) \supset b((f_1, \cdots, f_n)(M))$ for every n-tuple $(f_1, \cdots, f_n) \in A^n(M)$, i.e.*

$$E_n(M) = \cap \{E\colon E = [E] \subset M,\ (f_1, \cdots, f_n)(E) \supset b((f_1, \cdots, f_n)(M)) \quad \text{for each } (f_1, \cdots, f_n) \in A^n(M)\} . \tag{6}$$

PROOF. Corollary 1 shows that $E_n(M)$ contains the right hand side set of (6). Let E be a closed subset of M such that $(f_1, \cdots, f_n)(E) \supset b((f_1, \cdots, f_n)(M))$ for every n-tuple $(f_1, \cdots, f_n) \in A^n(M)$ and let $(g_1, \cdots, g_n)$ be a fixed regular n-tuple over $A(M)$. Because of $(g_1, \cdots, g_n)(M) \not\ni (0, \cdots, 0)$, we can find a point $X^0 \in b((g_1, \cdots, g_n)(M))$ such that $\|X^0\| = \|(x_1^0, \cdots, x_n^0)\| = \min_{x \in M} \|(g_1, \cdots, g_n)(x)\|$. Now

$$\begin{aligned} \|(g_1, \cdots, g_n)(x)\| &\geqq \|X^0\| = \min_{x \in M} \|(g_1, \cdots, g_n)(x)\| = \min_{X \in (g_1, \cdots, g_n)(M)} \|X\| \\ &= \min_{X \in b((g_1, \cdots, g_n)(M))} \|X\| \geqq \min_{X \in (g_1, \cdots, g_n)(E)} \|X\| = \min_{x \in E} \|(g_1, \cdots, g_n)(x)\| , \end{aligned}$$

because $(g_1, \cdots, g_n)(E) \supset b((g_1, \cdots, g_n)(M))$ according to our supposition. Consequently the minimum of the function $\|(f_1, \cdots, f_n)(x)\|$ is attained within E for every regular n-tuple $(f_1, \cdots, f_n) \in A_*^n(M)$, i.e. E is an affine n-boundary of M. Hence $E \supset E_n(M)$ because the latter is the smallest closed affine n-boundary of M. Q.E.D.

COROLLARY 2. *Let $\|(f_1, \cdots, f_n)(x)\|$ be one of the following convex functions:*

$$\sum_{j=1}^{n} |f_j(x)| \ ; \qquad \max_{j=1}^{n} |f_j(x)| \ ; \qquad \Big(\sum_{j=1}^{n} |f_j(x)|^p\Big)^{1/p}, \quad p \geqq 2 \ .$$

Then $E_n(M)$ is the smallest closed subset E of M that satisfies one of the following equivalent conditions:

1) $\min_{x\in E}\|F(x)\| \leqq \min\{\|x\|: X\in bF(M)\}$ *for every* $F=(f_1, \cdots, f_n)\in A^n(M)$;

2) $B(\min\{\|X\|: X\in F(E)\cap bF(M)\})$ *is contained either entirely in* $F(M)$ *or entirely outside* $F(M)$ *for every* $F\in A^n(M)$, *where* $B(r)$ *is the open ball in* $\boldsymbol{R}^n$ *centered at the origin and with radius* r;

3) $\min_{x\in E}\|F(x)\| = \min_{x\in M}\|F(x)\|$ *for every regular n-tuple* $F\in A^n_*(M)$;

4) F *vanishes within* E *for every* $F\in A^n(M)$ *such that* $bF(M)\ni(0, \cdots, 0)$;

5) $B(\min_{x\in E}\|F(x)\|)\subset B(\min\{\|X\|: X\in F(E)\cap bF(M)\})\subset F(M)$ *for every* $F\in A^n(M)\setminus A^n_*(M)$.

PROOF. Actually every one of these conditions characterizes affine n-boundaries of M, as we shall see. 1) If E is an affine n-boundary of M, then according to Theorem 2 $F(E)\supset bF(M)$ for any $F\in A^n(M)$ and hence $\min_{x\in E}\|F(x)\| = \min_{X\in F(E)}\|X\| \leqq \min_{X\in bF(M)}\|X\|$. Conversely, if E is not an affine n-boundary of M, then according to Theorem 2 there will exist an n-tuple $F=(f_1, \cdots, f_n)\in A^n(M)$ so that $F(E)\not\supset bF(M)$. If $X^0\in bF(M)\setminus F(E)$ and $x_0\in F^{-1}(X^0)$, then for the n-tuple $H=(f_1-x_1^0, \cdots, f_n-x_n^0)\in A^n(M)$ we have: $H(x_0)=(0, \cdots, 0)\in bH(M)\setminus H(E)$ since the set $H(M)$ can be obtained from $F(M)$ by a translation with X^0 and it preserves the topological properties of $\boldsymbol{R}^n$. Hence $0=\min_{X\in bH(M)}\|X\| < \min_{x\in E}\|H(x)\|$, i.e. condition 1) is not satisfied for the n-tuple $H\in A^n(M)$. 2) If $F\in A^n_*(M)$ then $\rho(O, bF(M)) = \rho(O, F(M))$ and hence $B(\rho(O, bF(M)))\subset \boldsymbol{R}^n\setminus F(M)$ where $O=(0, \cdots, 0)$ and $\rho(O, N)=\inf_{X\in N}\|X\|$ is the distance in $\boldsymbol{R}^n$ from O to the set $N\subset M$ with respect to the metric $\rho(x, y)=\|x-y\|$. Condition 1) now says that $B(\rho(O, F(E)))\subset B(\rho(O, bF(M)))\subset \boldsymbol{R}^n\setminus F(M)$. If $F\in A^n(M)\setminus A^n_*(M)$ then $\rho(O, bF(M))=\rho(O, \boldsymbol{R}^n\setminus F(M))$ and hence $B(\rho(O, bF(M)))\subset F(M)$. Now 1) says that $B(\rho(O, F(E)))\subset B(\rho(O, bF(M)))\subset F(M)$, which proves the case 2). Because 1) implies 3), 4) and 5) for the corresponding n-tuples $F\in A^n(M)$, these conditions are fulfilled for every affine n-boundary E of M. If E is not an affine boundary then, as we saw above, condition 1) does not hold for the n-tuple H with $(0, \cdots, 0)\in bH(M)$. This completes the proof of cases 4) and 5). By a suitable translation with some point $Y^0\in \boldsymbol{R}^n\setminus H(M)$ we can obtain also a regular n-tuple $H-Y^0\in A^n_*(M)$ such that $\rho(O, b(H-$

$Y^0)(M)) < \rho(O, (H-Y^0)(E))$, i.e. $\min_{x\in E}\|H(x)-Y^0\| > \min_{(H-Y^0)(M)}\|X\|$ in contradiction with condition 3). Q.E.D.

The next corollary gives local characterizations of the points of the minimal affine n-boundary $E_n(M)$.

COROLLARY 3. *A point $x_0 \in M$ belongs to $E_n(M)$ iff for any neighborhood U of x_0 there exists an n-tuple $F \in A^n(M)$, such that:*

1) $F \in A^n_*(M)$ *and* $\min_U\|F(x)\| < \min_{M\setminus U}\|F(x)\|$;
2) $bF(M) \ni (0, \cdots, 0)$ *and* $\min_{M\setminus U}\|F(x)\| > 0$;
3) $F \in A^n(M)\setminus A^n_*(M)$ *and* $\rho(0, F(U)\cap bF(M)) < \rho(0, F(M\setminus U)\cap bF(M))$.

PROOF. If some of these properties fails to be true, then according to Theorem 1 or Corollary 2 $E_n(M) \subset M\setminus U$ in contradiction with $x_0 \in U$. If some of these properties holds for every $U \ni x_0$ this will imply that $U \cap E_n(M) \neq \emptyset$ so that every neighborhood of x_0 will contain points from $E_n(M)$, wherefrom $x_0 \in E_n(M)$ since $E_n(M)$ is closed. Q.E.D.

§ 3. Some applications.

The following is an affine version of classical Rouche's theorem for analytic functions and of its generalization for n-tuples of uniform algebra elements, due to Corach and Maestripieri, as well [5].

THEOREM 3. *Let M be a compact convex subset of V, and F and G be n-tuples from $A^n(M)$. If the inequality*

$$\|F(x)-G(x)\| < \|F(x)+G(x)\| \tag{7}$$

holds on $E_n(M)$ then F and G are simultaneously regular or irregular n-tuples of $A^n(M)$.

PROOF. Because the minimal affine n-boundary $E_n(M)$ is a compact subset of M, there will exist an integer m such that:

$$m \min_{E_n(M)}(\|F(x)+G(x)\| - \|F(x)-G(x)\|) > \max_M \|F(x)-G(x)\| .$$

Assume that the theorem is not true. Then the end members of the sequence $2mF$, $(2m-1)F+G$, $(2m-2)F+2G$, $\cdots$, $F+(2m-1)G$, $2mG$ are not simultaneously regular n-tuples over $A(M)$. Hence there are two neighboring members of this sequence, one of which is regular and the other is irregular. Suppose that k is an integer such that the n-tuple $(m-k)F+(m+k)G$ is regular but $(m-k+1)F+(m+k-1)G$ is an irregular n-tuple and let x_0 be a point of M for which $(m-k+1)F(x_0)+(m+k-1)G(x_0)=0$. Now

$$\begin{aligned}\max_{x\in M}\|F(x)-G(x)\| &< m\min_{E_n(M)}(\|F(x)+G(x)\|-\|F(x)-G(x)\|)\\ &\leqq \min_{E_n(M)}(m\|F(x)+G(x)\|-k\|F(x)-G(x)\|)\leqq \min_{E_n(M)}\|(m-k)F(x)+(m+k)G(x)\|\\ &=\min_{M}\|(m-k)F(x)+(m+k)G(x)\|\leqq\|(m-k)F(x_0)+(m+k)G(x_0)\|\\ &=\|(m-k)F(x_0)+(m+k)G(x_0)-[(m-k+1)F(x_0)+(m+k-1)G(x_0)]\|\\ &=\|G(x_0)-F(x_0)\|\leqq\max_{x\in M}\|F(x)-G(x)\|\ .\end{aligned}$$

The obtained contradiction proves the theorem. Q.E.D.

An other application of minimal affine n-boundaries is the proving of the following affine version of a theorem of Hartogs for analytic functions in the unit ball in C^n and its generalization for pairs of uniform algebra elements, due to Sibony [6], as well.

THEOREM 4. *Let M be a compact convex subset of V, and F and G be n-tuples of elements of $A(M)$. If the equality*

$$\|F(x)\|=\|G(x)\| \tag{8}$$

holds on $E_{2n}(M)$ then it holds everywhere in M.

It is interesting to know if the minimal affine n-boundary $E_n(M)$ coincides with the closure of these end subsets of M, that are contained in $(n-1)$-dimensional affine subspaces of V as in the case $n=1$.

NOTE. Recently M. Hayashi has proved positively this problem (private communication).

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