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On the Second Order Efficiency of Bootstrap Estimators of Sampling Distributions

Takeru SUZUKI

Waseda University

§1. Introduction.

Let $X_1, X_2, \cdots, X_n$ be independent identically distributed random variables with unknown distribution function (d.f.) F contained in a set Θ of d.f.'s on the real line $\boldsymbol{R}$. Let $g_n(\cdot, F)$ be a d.f. on $\boldsymbol{R}$ parametrized by $F \in \Theta$, which will be considered to be a sampling d.f. of an appropriately normalized statistic based on the sample $\boldsymbol{X}_n=(X_1, X_2, \cdots, X_n)$ under F. We consider in this paper the estimation problem of $g_n(\cdot, F)$ based on the sample $\boldsymbol{X}_n=(X_1, \cdots, X_n)$. In particular, we discuss some asymptotic properties of the bootstrap estimator $\hat{g}_{n,B}=g_n(\cdot, \hat{F}_n)$ of $g_n(\cdot, F)$ where $\hat{F}_n$ is the empirical (sample) d.f. based on $\boldsymbol{X}_n=(X_1, \cdots, X_n)$. Consistency of $\hat{g}_{n,B}$ has been proved by Efron [6] and by Bickel and Freedman [4]. In Bickel and Freedman [3] and in Singh [8] Edgeworth type expansions of $\hat{g}_{n,B}$ for some typical g_n (the sampling d.f. of normalized sample mean and sample quantile) has been discussed. Beran [2] has proved that $\hat{g}_{n,B}$ is locally asymptotically minimax for estimating g_n under some smoothness conditions with respect to F. In this paper we prove the second order asymptotic efficiency of appropriately corrected version of $\hat{g}_{n,B}$ under conditions about $g_n(\cdot, F)$ similar to Assumption 1 or Assumption 1′ of Beran [2]. The concept of second order asymptotic efficiency in our case is essentially due to Akahira and Takeuchi [1]. We note that, in general, locally asymptotically minimax property does not imply second order efficiency as the following example shows: Let each X_i obey the distribution with density

$$f(x, \theta)=2^{-1}\exp(-|x-\theta|) \qquad (\theta \in \boldsymbol{R},\ x \in \boldsymbol{R})\ .$$

In this case $\mathrm{med}_{1\le i\le n} X_i$ are locally asymptotically minimax, but not second order asymptotically efficient for estimating $\theta \in \Theta$ (cf. Akahira and Takeuchi [1], p. 96).

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In Section 2 we shall describe some conditions about g_n which will play an important role in the following sections. In Section 3 we try to get a bound of the second order asymptotic distributions of the second order asymptotically median unbiased estimator $\hat{g}_n$ of $g_n(\cdot, F)$, which is calculated in a similar way to the one developed in Akahira and Takeuchi [1]. In Section 4 it will be proved that the bound obtained in Section 3 is attained by the bootstrap estimator $\hat{g}_{n,B}$ with a correcting term of order n^{-1}, and so it is second order asymptotically efficient in this sense. The final section is devoted to describing a typical example which satisfies the conditions given in Section 2.

§ 2. Notations and assumptions.

Let $\mathscr{F}$ be the set of all d.f.'s on the real line $\boldsymbol{R}$ and Θ be a subset of $\mathscr{F}$. Let $\mathscr{B}$ be the set of all bounded functions on $\boldsymbol{R}$. We denote by $\|\cdot\|$ the sup norm in $\mathscr{B}$. We mean the topology of a subset $\mathscr{B}_1$ of $\mathscr{B}$ by the relative topology of $\mathscr{B}_1$ as a subset of the normed space $(\mathscr{B}, \|\cdot\|)$. Let $X_1, X_2, \cdots, X_n$ be independent identically distributed random variables with unknown d.f. F in Θ. Let μ be a σ-finite measure on $\boldsymbol{R}$, and for $k \in L^1(\mu)$ and $h \in \mathscr{B}$ let $\langle k, h\rangle = \int_{\boldsymbol{R}} k \cdot h d\mu$ where $L^1(\mu)$ is the set of all μ-integrable functions on $\boldsymbol{R}$. Let $\{g_n;\ n \geqq 1\}$ be a sequence of maps $g_n(\cdot, F)$ from Θ^* to $\mathscr{F}$, where Θ^* is an open set in $\mathscr{F}$ containing Θ. For each $F \in \Theta$ and $c>0$ define $B_n(F, c)$ as the set of $G \in \Theta^*$ satisfying $\|G-F\| \leqq c/n^{1/2}$. We consider the following conditions about $\{g_n\}$ on the second degree asymptotic differentiability of g_n as a function of F.

ASSUMPTION 1. (a) There exist sequences of maps $\{g_{n,i}(\cdot, F);\ n \geqq 1\}$, $i=0, 1, 2$, from Θ^* to $\mathscr{F}$ such that for each $c>0$ and each $F \in \Theta$

$$\sup_{G \in B_n(F,c)} \| g_n(\cdot, G) - g_{n,0}(\cdot, G) - n^{-1/2} g_{n,1}(\cdot, G) - n^{-1} g_{n,2}(\cdot, G) \| = o(n^{-1}) .$$

(b) There exist $\{w_F^{(i)};\ i=1, 2, 3\} \subset \mathscr{B}$, $\{\tilde{w}_F^{(2)}, \tilde{w}_F^{(3)}\} \subset \mathscr{B}$, $\{u_F, v_F, \tilde{v}_F\} \subset L^1(\mu)$ and $\{q_F, \tilde{q}_F\} \subset L^1(\mu) \times L^1(\mu)$ defined for each $F \in \Theta^*$ such that for each $F \in \Theta$ and each $c>0$

(i) $$\sup_{G \in B_n(F,c)} \| g_{n,0}(\cdot, G) - g_{n,0}(\cdot, F) - w_F^{(1)} \langle u_F, G-F\rangle - 2^{-1}\{ w_F^{(2)} \langle q_F(G-F), G-F\rangle + \tilde{w}_F^{(2)} \langle \tilde{q}_F(G-F), G-F\rangle \} \| = o(n^{-1}) ,$$

(ii) $$\sup_{G \in B_n(F,c)} \| g_{n,1}(\cdot, G) - g_{n,1}(\cdot, F) - w_F^{(3)} \langle v_F, G-F\rangle - \tilde{w}_F^{(3)} \langle \tilde{v}_F, G-F\rangle \| = o(n^{-1/2}) ,$$

(iii) $\sup_{G \in B_n(F,c)} \| g_{n,2}(\cdot, G) - g_{n,2}(\cdot, F) \| = o(1)$.

(c) For each $F \in \Theta$

(i) the d.f. of $\langle u_F, y_{F,1} \rangle$ under F is non-lattice,

(ii) $E_F(\langle u_F, y_{F,1} \rangle^2) > 0$,

where $y_{F,1}(t) = I_{(-\infty,t]}(X) - F(t)$ and $I_{(-\infty,t]}(X)$ denotes the indicator function of the set $(-\infty, t]$.

REMARK 1. The function u_F appeared in Assumption 1 is unique in the following sense: If Assumption 1 is satisfied with $\tilde{w}_F^{(1)} \in \mathscr{B}$ and $\tilde{u}_F \in L^1(\mu)$ instead of $w_F^{(1)}$ and u_F respectively, then for every $h \in \mathscr{B}_0$ and $F \in \Theta$

$$w_F^{(1)} \langle u_F, h - c(h)F \rangle = \tilde{w}_F^{(1)} \langle \tilde{u}_F, h - c(h)F \rangle ,$$

where $\mathscr{B}_0$ is the class of bounded functions on $\boldsymbol{R}$ such that $c(h) = \lim_{t\to\infty} h(t)$ exists and $\lim_{t\to-\infty} h(t) = 0$.

We have the following proposition which is an easy consequence of our assumption.

PROPOSITION 1. *Suppose that the conditions* (a) *and* (b) *in Assumption* 1 *are satisfied. Then we have for each* $c > 0$ *and each* $F \in \Theta$

$$\begin{aligned} \sup_{G \in B_n(F,c)} \| g_n(\cdot, G) - g_n(\cdot, F) - w_F^{(1)} \langle u_F, G - F \rangle \\ - 2^{-1}\{ w_F^{(2)} \langle q_F(G-F), G-F \rangle + \tilde{w}_F^{(2)} \langle \tilde{q}_F(G-F), G-F \rangle \} \\ - n^{-1/2}\{ w_F^{(3)} \langle v_F, G-F \rangle + \tilde{w}_F^{(3)} \langle \tilde{v}_F, G-F \rangle \} \| = o(n^{-1}) . \end{aligned}$$

We consider the following condition stronger than previous one, which will be used in Section 4 to prove second order asymptotic efficiency of the bootstrap estimators. This condition is almost the same as Assumption 1.

ASSUMPTION 2. (a) There exist sequences of maps $\{g_{n,i}(\cdot, F); n \geqq 1\}$, $i = 0, 1, 2$, from Θ^* to $\mathscr{F}$ such that for every $F \in \Theta$

$$\sup_{G \in B_n(F,c_n)} \| g_n(\cdot, G) - g_{n,0}(\cdot, G) - n^{-1/2} g_{n,1}(\cdot, G) - n^{-1} g_{n,2}(\cdot, G) \| = o(n^{-1}) ,$$

where $\{c_n\}$ is a sequence of positive numbers satisfying

$$\lim_{n\to\infty} \{4c_n^2 - \log n\} = \infty .$$

(b) There exist $\{w_F^{(i)}; i = 1, 2, 3\} \subset \mathscr{B}$, $\{\tilde{w}_F^{(2)}, \tilde{w}_F^{(3)}\} \subset \mathscr{B}$, $\{u_F, v_F, \tilde{v}_F\} \subset L^1(\mu)$ and $\{q_F, \tilde{q}_F\} \subset L^1(\mu) \times L^1(\mu)$ defined for each $F \in \Theta^*$ such that

(i) $\sup_{G \in B_n(F,c_n)} \| g_{n,0}(\cdot, G) - g_{n,0}(\cdot, F) - w_F^{(1)} \langle u_F, G - F \rangle$

$$-2^{-1}\{w_F^{(2)}\langle q_F(G-F),\ G-F\rangle+\tilde{w}_F^{(2)}\langle \tilde{q}_F(G-F),\ G-F\rangle\}\|=o(n^{-1})\ ,$$

(ii) $$\sup_{G\in B_n(F,c_n)}\|g_{n,1}(\cdot,\ G)-g_{n,1}(\cdot,\ F)-w_F^{(3)}\langle v_F,\ G-F\rangle-\tilde{w}_F^{(3)}\langle \tilde{v}_F,\ G-F\rangle\|=o(n^{-1/2})\ ,$$

(iii) $$\sup_{G\in B_n(F,c_n)}\|g_{n,2}(\cdot,\ G)-g_{n,2}(\cdot,\ F)\|=o(1)\ .$$

(c) For each $F\in\Theta$

(i) the d.f. of $\langle u_F,\ y_{F,1}\rangle$ under F is non-lattice,

(ii) $E_F(\langle u_F,\ y_{F,1}\rangle^2)>0$.

We have the following proposition which can be verified in the same way as Proposition 1.

PROPOSITION 2. *Suppose that the conditions* (a) *and* (b) *in Assumption* 2 *are satisfied. Then we have for each* $F\in\Theta$

$$\begin{aligned}\sup_{G\in B_n(F,c_n)}\|g_n(\cdot,\ G)-g_n(\cdot,\ F)-w_F^{(1)}\langle u_F,\ G-F\rangle\\ -2^{-1}\{w_F^{(2)}\langle q_F(G-F),\ G-F\rangle+\tilde{w}_F^{(2)}\langle \tilde{q}_F(G-F),\ G-F\rangle\}\\ -n^{-1/2}\{w_F^{(3)}\langle v_F,\ G-F\rangle+\tilde{w}_F^{(3)}\langle \tilde{v}_F,\ G-F\rangle\}\|=o(n^{-1})\ .\end{aligned}$$

REMARK 2. The smoothed d.f. g_n of a second degree U-statistic satisfies Assumption 2 with any sequence $\{c_n\}$ of positive numbers satisfying

$$\lim_{n\to\infty}\{\log n-6\log c_n\}=\infty\ .$$

We discuss this example more precisely in Section 5.

§ 3. A bound of second order asymptotic distributions.

We mean by the estimator of g_n the measurable map $\hat{g}_n$ from $\mathscr{X}_n$ to $\mathscr{F}$, where $\mathscr{X}_n$ is the sample space of random vector $X_n=(X_1,\cdots,X_n)$ equipped with the Borel σ-field. For each $F\in\Theta$ we denote by $P_{F,n}$ the probability distribution of X_n provided that each X_i obeys the d.f. F. Let K be the set of all $k\in L^1(\mu)$ satisfying $\int_R|k|d\mu=1$. We denote by $B_n^*(F,c)$ the intersection of $B_n(F,c)$ and $\mathscr{C}(F):=\{G\in\mathscr{F};\ F$ is absolutely continuous with respect to $G\}$. Let $\mathscr{E}$ be the class of sequences $\{\hat{g}_n\}$ of estimators of $\{g_n\}$ such that for each $k\in K$, each $F\in\Theta$, each $c>0$ and each sequence $\{\varepsilon_n\}$ of real numbers satisfying $\varepsilon_n=o(n^{-1/2})$ we have

(3.1) $$\sup_{G\in B_n^*(F,c_n)}|P_{G,n}\{n^{1/2}\langle k,\ \hat{g}_n-g_n(\cdot,\ G)\rangle\leqq\varepsilon_n\}-2^{-1}|=o(n^{-1/2})\ .$$

We note that if (3.1) holds with $\varepsilon_n=0$, $n=1, 2, \cdots$, and $n^{1/2}\langle k, \hat{g}_n-g_n(\cdot, G)\rangle$ admits Edgeworth expansion uniformly in G over $B_n^*(F, c)$ up to order $n^{-1/2}$ for each $k\in K$ and each $F\in\Theta$, then $\{\hat{g}_n\}$ is an element of $\mathscr{E}$. Following Akahira and Takeuchi [1] we call in this paper the sequence $\{\hat{g}_n\}$ of estimators in $\mathscr{E}$ second order asymptotically median unbiased (or second order AMU). This definition is a modification of the concept of AMU estimator defined in Akahira and Takeuchi [1] to our situation. Before describing the theorem we define some notations here. Let $y_{F,i}(t)=I_{(-\infty,t]}(X_i)-F(t)$, $i=1, 2$. Define

$$c_i(F, k)=\langle w_F^{(i)}, k\rangle\ ,\quad i=1, 2, 3\ ,\qquad \tilde{c}_i(F, k)=\langle \tilde{w}_F^{(i)}, k\rangle\ ,\quad i=2, 3\ ,$$

$$\alpha(F)=E_F(\langle u_F, y_{F,1}\rangle\langle v_F, y_{F,1}\rangle)\ ,\qquad \tilde{\alpha}(F)=E_F(\langle u_F, y_{F,1}\rangle\langle \tilde{v}_F, y_{F,1}\rangle)\ ,$$

$$\beta(F)=E_F(\langle u_F, y_{F,1}\rangle^3)\ ,$$

$$\gamma(F)=E_F(\langle q_F, y_{F,1}\rangle)\ ,\qquad \tilde{\gamma}(F)=E_F(\langle \tilde{q}_F y_{F,1}, y_{F,1}\rangle)\ ,$$

$$\delta(F)=E_F(\langle u_F, y_{F,1}\rangle\langle u_F, y_{F,2}\rangle\langle q_F y_{F,1}, y_{F,2}\rangle)\ ,$$

$$\tilde{\delta}(F)=E_F(\langle u_F, y_{F,1}\rangle\langle u_F, y_{F,2}\rangle\langle \tilde{q}_F y_{F,1}, y_{F,2}\rangle)\ .$$

We state a theorem which gives a second order bound of asymptotic distributions of the estimators $\{\hat{g}_n\}$ in $\mathscr{E}$. In the following Φ denotes the standard normal distribution function and ϕ the density function of Φ.

Theorem 1. *Suppose that Assumption 1 is satisfied. Then for any sequence $\{\hat{g}_n\}$ of estimators of $\{g_n\}$ in $\mathscr{E}$, for every $F\in\Theta$ and for every $k\in K$, we have*

$$(3.2)\qquad \begin{aligned} &P_{F,n}\{n^{1/2}\langle k, \hat{g}_n-g_n(\cdot, F)\rangle\leqq t\}\\ &\quad\leqq\Phi(t/J^{1/2}(F, k))-n^{-1/2}\phi(t/J^{1/2}(F, k))\Psi(t, F, k)+o(n^{-1/2})\\ &\quad(\geqq)\end{aligned}$$

for every $t>0$ ($t<0$, respectively), where

$$\begin{aligned}\Psi(t, F, k)=&\,tc_1(F, k)(c_3(F, k)\alpha(F)+\tilde{c}_3(F, k)\tilde{\alpha}(F))/J^{3/2}(F, k)\\ &+t^2(c_1^3(F, k)\beta(F))+3c_1^2(F, k)(c_2(F, k)\delta(F)+\tilde{c}_2(F, k)\tilde{\delta}(F))/(6J^{5/2}(F, k))\ .\end{aligned}$$

Proof. We can prove this in a similar way to the one in Akahira and Takeuchi [1]. We face, however, some difficulties because parametric model is infinite dimensional. Take $t>0$, $F\in\Theta$ and $k\in K$ arbitrarily and then fix them. We define functions $a_{F,1}$, $a_{F,2}$, b_F in $\mathscr{B}$ as follows:

$$a_{F,1}(X_1)=c_1(F,\,k)\langle u_F,\,y_{F,1}\rangle/J(F,\,k)\ ,$$

$$a_{F,2}(X_1)=-\{c_3(F,\,k)\langle v_F,\,y_{F,1}\rangle+\tilde{c}_3(F,\,k)\langle\tilde{v}_F,\,y_{F,1}\rangle\}/J(F,\,k)\ ,$$

$$b_F(X_1)=-c_1(F,\,k)\Big\{\int_R(c_2(F,\,k)\langle u_F,\,y_{F,2}\rangle\langle q_F y_{F,1},\,y_{F,2}\rangle$$
$$+\tilde{c}_2(F,\,k)\langle u_F,\,y_{F,2}\rangle\langle\tilde{q}_F y_{F,1},\,y_{F,2}\rangle)dF(X_2)\Big\}\Big/(2J^2(F,\,k))\ .$$

Using these functions we can construct a sequence $\{g_{t,n};\ n\geqq n_0\}$ of probability density functions on $\boldsymbol{R}$ with respect to dF as follows:

$$g_{t,n}(x)=1+n^{-1/2}t(a_{F,1}(x)+a_{F,2}(x)/n^{1/2})+n^{-1}t^2b_F(x)\ ,$$

where the integer n_0 depends only on t.

Let $G_{t,n}$ be the d.f. on $\boldsymbol{R}$ corresponding to the density $g_{t,n}$. Let $\phi_n^*=\phi_n^*(\boldsymbol{X}_n)$ be the most powerful test with asymptotic level $2^{-1}+o(n^{-1/2})$ for the problem of testing the hypothesis H_0: "true distribution is $G_{t,n}$" versus the alternative H_1: "true distribution is F". Define the random variables $Z_{in}=\log((dG_{t,n}/dF)(X_i))=\log g_{t,n}(X_i)$ and let $T_n=\sum_{i=1}^n Z_{in}$. We note that the test $\phi_n^*(\boldsymbol{X}_n)$ mentioned above has the following form: $\phi_n^*(\boldsymbol{X}_n)=1$ if $T_n<d_n$, $=0$ otherwise for appropriately chosen number d_n. By Taylor expansion we have the following results:

$$E_F(Z_{in})=-(2n)^{-1}t^2J_0(F,\,k)+n^{-3/2}t^3\{-E_F(a_{F,1}b_F)+3^{-1}E_F(a_{F,1}^3)$$
$$-t^{-1}E_F(a_{F,1}a_{F,2})\}+o(n^{-3/2})\ ,\qquad (J_0(F,\,k)=J^{-1}(F,\,k))\ ,$$

$$E_F(Z_{in}^2)=n^{-1}t^2J_0(F,\,k)+2n^{-3/2}t^3\{E_F(a_{F,1}b_F)-2^{-1}E_F(a_{F,1}^3)$$
$$+t^{-1}E_F(a_{F,1}a_{F,2})\}+o(n^{-3/2})\ ,$$

$$E_F(Z_{in}^3)=n^{-3/2}t^3E_F(a_{F,1}^3)+o(n^{-3/2})\ ,$$

$$E_{G_{t,n}}(Z_{in})=(2n)^{-1}t^2J_0(F,\,k)+n^{-3/2}t^3\{t^{-1}E_F(a_{F,1},\,a_{F,2})$$
$$+E_F(a_{F,1}b_F)-6^{-1}E_F(a_{F,1}^3)\}+o(n^{-3/2})\ ,$$

$$E_{G_{t,n}}(Z_{in}^2)=n^{-1}t^2J_0(F,\,k)+2n^{-3/2}t^3\{t^{-1}E_F(a_{F,1}a_{F,2})+E_F(a_{F,1}b_F)\}+o(n^{-3/2})\ ,$$

$$E_{G_{t,n}}(Z_{in}^3)=n^{-3/2}t^3E_F(a_{F,1}^3)+o(n^{-3/2})\ .$$

From these we have

$$E_F(T_n)=-2^{-1}t^2J_0(F,\,k)+n^{-1/2}t^3\{-E_F(a_{F,1}b_F)+3^{-1}E_F(a_{F,1}^3)$$
$$-t^{-1}E_F(a_{F,1}a_{F,2})\}+o(n^{-1/2})\ ,$$

$$V_F(T_n)=E_F((T_n-E_F(T_n))^2)=t^2J_0(F,\,k)+2n^{-1/2}t^3\{E_F(a_{F,1}b_F)$$

$$-2^{-1}E_F(a^3_{F,1})+t^{-1}E_F(a_{F,1}a_{F,2})\}+o(n^{-1/2}) ,$$

$$E_F((T_n-E_F(T_n))^3)=n^{-1/2}t^3E_F(a^3_{F,1})+o(n^{-1/2}) .$$

We also have

$$E_{G_{t,n}}(T_n)=2^{-1}t^2J_0(F,\ k)+n^{-1/2}t^3\{t^{-1}E_F(a_{F,1}a_{F,2})+E_F(a_{F,1}b_F)$$
$$-6^{-1}E_F(a^3_{F,1})\}+o(n^{-1/2}) ,$$

$$V_{G_{t,n}}(T_n)=t^2J_0(F,\ k)+2n^{-1/2}t^3\{t^{-1}E_F(a_{F,1}a_{F,2})+E_F(a_{F,1}b_F)\}+o(n^{-1/2}) ,$$

$$E_{G_{t,n}}((T_n-E_{G_{t,n}}(T_n))^3)=n^{-1/2}t^3E_F(a^3_{F,1})+o(n^{-1/2}) .$$

Thus, according to the Gram-Charlier (Edgeworth) expansion, we have

$$\begin{aligned}(3.3)\qquad P_{G_{t,n},n}\{T_n\leqq d_n\}&=P_{G_{t,n},n}\{(T_n-2^{-1}t^2J_0(F,\ k))/(tJ_0^{1/2}(F,\ k))\leqq\tilde{d}_n\}\\&=\Phi(\tilde{d}_n)-n^{-1/2}\phi(\tilde{d}_n)\{t^2J_0^{-1/2}(F,\ k)(t^{-1}E_F(a_{F,1}a_{F,2})+E_F(a_{F,1}b_F)\\&\quad-6^{-1}E_F(a^3_{F,1}))+tJ_0^{-1}(F,\ k)(t^{-1}E_F(a_{F,1}a_{F,2})+E_F(a_{F,1}b_F))\tilde{d}_n\\&\quad+(E_F(a^3_{F,1}))/(6J_0^{3/2}(F,\ k))(\tilde{d}_n^2-1)\}+o(n^{-1/2})\end{aligned}$$

where $\tilde{d}_n=(d_n-2^{-1}t^2J_0(F,\ k))/(tJ_0^{1/2}(F,\ k))$.

In fact, the validity of the expansion (3.3) can be verified by a similar method used in the proof of Theorem 1 in Feller [7], XVI. 4, page 512. We need the condition (c) in Assumption 1 to prove this. The proof is relatively easy but long, so it will be omitted here.

From (3.3) it follows that if we take $\tilde{d}_n=c_0(t,\ F)/n^{1/2}$ then we have

$$P_{G_{t,n},n}\{T_n\leqq d_n\}=2^{-1}+o(n^{-1/2})$$

where

$$c_0(t,\ F)=t^2\{t^{-1}E_F(a_{F,1}a_{F,2})+E_F(a_{F,1}b_F)-6^{-1}E_F(a^3_{F,1})\}/J_0^{1/2}(F,\ k)$$
$$-E_F(a^3_{F,1})/(6J_0^{3/2}(F,\ k)) .$$

Choosing such a sequence $\{\tilde{d}_n\}$ we can calculate the power function corresponding to the test sequence $\{I_{\{T_n\leqq d_n\}}(X_n)\}$. In a similar way to (3.3) we have

$$\begin{aligned}(3.4)\qquad &P_{F,n}\{T_n\leqq d_n\}\\&=\Phi(tJ^{-1/2}(F,\ k))-n^{-1/2}\phi(tJ^{-1/2}(F,\ k))[-tJ^{1/2}(F,\ k)E_F(a_{F,1}a_{F,2})\\&\quad+t^2J^{1/2}(F,\ k)(E_F(a^3_{F,1})/6-E_F(a_{F,1}b_F))]+o(n^{-1/2}) .\end{aligned}$$

We can check easily that

$$E_F(a^3_{F,1})=c_1^3(F,\ k)\beta(F)/J^3(F,\ k) ,$$

$$E_F(a_{F,1}a_{F,2}) = -c_1(F, k)(c_3(F, k)\alpha(F) + \tilde{c}_3(F, k)\tilde{\alpha}(F))/J^2(F, k) ,$$

$$E_F(a_{F,1}b_F) = -c_1^2(F, k)(c_2(F, k)\delta(F) + \tilde{c}_2(F, k)\tilde{\delta}(F))/(2J^3(F, k)) .$$

Hence the right hand side (R.H.S.) of the inequality (3.2) equals the R.H.S. of (3.4) up to the order $n^{-1/2}$.

Let $\{\hat{g}_n\}$ be any element of $\mathscr{E}$. We have by Proposition 1

$$P_{F,n}\{n^{1/2}\langle k, \hat{g}_n - g_n(\cdot, F)\rangle \leqq t\} = P_{F,n}\{n^{1/2}\langle k, \hat{g}_n - g_n(\cdot, G_{t,n})\rangle \leqq \varepsilon_n\} , \tag{3.5}$$

where $\{\varepsilon_n\}$ is a sequence of real numbers satisfying $\varepsilon_n = o(n^{-1/2})$. As $\{\hat{g}_n\}$ is a second order AMU estimator, we have

$$P_{G_{t,n},n}\{n^{1/2}\langle k, \hat{g}_n - g_n(\cdot, G_{t,n})\rangle \leqq \varepsilon_n\} = 2^{-1} + o(n^{-1/2}) .$$

Since the test sequence $\{I_{\{T_n \leqq d_n\}}\}$ is asymptotically most powerful with level $2^{-1} + o(n^{-1/2})$, it holds that

$$P_{F,n}\{n^{1/2}\langle k, \hat{g}_n - g_n(\cdot, G_{t,n})\rangle \leqq \varepsilon_n\} \leqq P_{F,n}\{T_n \leqq d_n\} + o(n^{-1/2}) . \tag{3.6}$$

From (3.4), (3.5) and (3.6) we have the inequality (3.2) for $t>0$. Substituting $-k$ for k in the inequality (3.2) for $t>0$ we have another inequality for $t<0$.

REMARK 3. Theorem 1 remains valid for $\mathscr{E}_0$ instead of $\mathscr{E}$, where $\mathscr{E}_0$ is the class of $\{\hat{g}_n\}$ satisfying the same conditions as imposed for $\mathscr{E}$ except for $B_n^*(F, c)$ being replaced by $B_n^{**}(F, c)$ in (3.1). Here $B_n^{**}(F, c)$ means the intersection of $B_n(F, c)$ and $\mathscr{C}_0(F) := \{G \in \mathscr{F};\ dG = (1 + k_n(x))dF,\ \sup\{|k_n(x)|;\ x \in \boldsymbol{R}\} = O(n^{-1/2})\}$.

The following definition of second order asymptotic efficiency owes essentially to Akahira and Takeuchi [1]. If a sequence $\{\hat{g}_n\}$ in $\mathscr{E}$ attains the bounds (3.2) for every $F \in \Theta$ for every $t \in \boldsymbol{R}$, and for every $k \in K$, then we call them second order asymptotically efficient AMU estimators.

§ 4. Second order asymptotic efficiency of bootstrap estimator.

Let $\hat{F}_n$ be the empirical (sample) d.f. based on the sample $X_1, X_2, \cdots, X_n$. In this section we study some second order asymptotic properties of bootstrap estimator $\hat{g}_{n,B}(\cdot) = g_n(\cdot, \hat{F}_n)$. We have the following result about the second order asymptotic distribution of $\{\hat{g}_{n,B}\}$.

THEOREM 2. *Suppose that Assumption 2 is satisfied. Then, for every $F \in \Theta$, every $t \in \boldsymbol{R}$ and every $k \in K$ we have*

$$\text{(4.1)}\qquad P_{F,n}\{n^{1/2}\langle k, \hat{g}_{n,B}-g_n(\cdot, F)\rangle \leqq t\} \\ =\Phi(tJ^{-1/2}(F, k))-n^{-1/2}\phi(tJ^{-1/2}(F, k))\Psi^*(t, F, k)+o(n^{-1/2}) ,$$

where

$$\Psi^*(t, F, k)=(c_2(F, k)\gamma(F)+\tilde{c}_2(F, k)\tilde{\gamma}(F))/(2J^{1/2}(F, k)) \\ -(c_1^3(F, k)\beta(F)+3c_1^2(F, k)(c_2(F, k)\delta(F)+\tilde{c}_2(F, k)\tilde{\delta}(F)))/(6J^{3/2}(F, k)) \\ +\Psi(t, F, k) .$$

PROOF. For $k\in K$ and $F\in\Theta$ let $S_n=n^{1/2}\langle k, \hat{g}_{n,B}-g_n(\cdot, F)\rangle$, $W_n=n^{1/2}(\hat{F}_n-F)$ and $A_n=\{\boldsymbol{x}_n\in\boldsymbol{X}_n;\ \hat{F}_n\in B_n(F, c_n)\}$. By Proposition 2 and by the property $P_{F,n}\{A_n^c\}=o(n^{-1/2})$ we can verify

$$\text{(4.2)}\qquad S_n=c_1(F, k)U_n+(c_2(F, k)Q_n+\tilde{c}_2(F, k)\tilde{Q}_n)/(2n^{1/2}) \\ +(c_3(F, k)V_n+\tilde{c}_3(F, k)\tilde{V}_n)/n^{1/2}+\tilde{\varepsilon}_n ,$$

where $U_n=\langle u_F, W_n\rangle$, $Q_n=\langle q_F W_n, W_n\rangle$, $\tilde{Q}_n=\langle\tilde{q}_F W_n, W_n\rangle$, $V_n=\langle v_F, W_n\rangle$ and $\tilde{V}_n=\langle\tilde{v}_F, W_n\rangle$. Here $\{\tilde{\varepsilon}_n\}$ is a sequence of random variables such that $\sup\{|\tilde{\varepsilon}_n|;\ \boldsymbol{x}_n\in A_n\}=o(n^{-1/2})$. We put

$$S_n^*=c_1(F, k)U_n+(c_2(F, k)Q_n+\tilde{c}_2(F, k)\tilde{Q}_n)/(2n^{1/2}) \\ +(c_3(F, k)V_n+\tilde{c}_3(F, k)\tilde{V}_n)/n^{1/2} ,$$

then we have

$$E_F(S_n^*)=(c_2(F, k)\gamma(F)+\tilde{c}_2(F, k)\tilde{\gamma}(F))/(2n^{1/2}) ,$$

$$V_F(S_n^*)=J(F, k)+2c_1(F, k)(c_3(F, k)\alpha(F)+\tilde{c}_3(F, k)\tilde{\alpha}(F))/n^{1/2} ,$$

$$E_F((S_n^*-E_F(S_n^*))^3)=[c_1^3(F, k)\beta(F)+3c_1^2(F, k)(c_2(F, k)\delta(F) \\ +\tilde{c}_2(F, k)\tilde{\delta}(F))]/n^{1/2}+o(n^{-1/2}) .$$

To prove this we use the fact that

$$E_F(U_n^2)=c_1^{-2}(F, k)J(F, k) ,\qquad E_F(Q_n)=\gamma(F) ,$$

$$E_F(U_n^2Q_n)=2\delta(F)+c_1^{-2}(F, k)J(F, k)\gamma(F)+O(n^{-1}) ,$$

$$E_F(U_n^2\tilde{Q}_n)=2\tilde{\delta}(F)+c_1^{-2}(F, k)J(F, k)\tilde{\gamma}(F)+O(n^{-1}) ,$$

$$E_F(U_n^2V_n)=O(n^{-1/2}) ,\qquad E_F(U_n^2\tilde{V}_n)=O(n^{-1/2}) ,\qquad E_F(U_n^3)=\beta(F)/n^{1/2} ,$$

$$E_F(U_nQ_n)=O(n^{-1/2}) ,\qquad E_F(U_n\tilde{Q}_n)=O(n^{-1/2}) ,\qquad E_F(V_nQ_n)=O(n^{-1/2}) ,$$

$$E_F(V_n\tilde{Q}_n)=O(n^{-1/2}) ,\qquad E_F(\tilde{V}_nQ_n)=O(n^{-1/2}) \quad\text{and}\quad E_F(\tilde{V}_n\tilde{Q}_n)=O(n^{-1/2}) .$$

Hence according to the Gram-Charlier (Edgeworth) expansion, we have

(4.3) $$P_{F,n}\{S_n^*\leqq t\}=\Phi(tJ^{-1/2}(F,k))-n^{-1/2}\phi(tJ^{-1/2}(F,k))\Psi^*(t,F,k)+o(n^{-1/2}) .$$

In fact, this can be shown by Esseen's smoothing lemma as follows (cf. Feller [7], XVI. 3, Lemma 2). Let $\tilde{S}_n=(S_n^*-\mu_n^*)/\sigma_n^*$, $\mu_n^*=E_F(S_n^*)$, $\sigma_n^{*2}=V_F(S_n^*)$ and $\kappa_{3,n}=E_F(\tilde{S}_n^3)$. Define $\tilde{F}_n(x)=P_{F,n}\{\tilde{S}_n\leqq x\}$ and $K_n(x)=\Phi(x)-\phi(x)(x^2-1)\kappa_{3,n}/6$, and denote by $\rho_n(u)$, $\psi_n(u)$ the Fourier transforms of $\tilde{F}_n$, K_n respectively. We note that $\psi_n(u)=e^{-u^2/2}(1+\kappa_{3,n}(iu)^3/6)$. By Esseen's lemma, for any $M>0$

(4.4) $$\sup_{x\in R}|\tilde{F}_n(x)-K_n(x)|\leqq\pi^{-1}\int_{-Mn^{1/2}}^{Mn^{1/2}}(|\rho_n(u)-\psi_n(u)|/|u|)du+K_0/(n^{1/2}M) ,$$

where K_0 is a constant not depending on n and M. For $\delta>0$ ($\delta<M$) let

$$J_{1,n}=\int_{|u|\leq\delta n^{1/2}}(|\rho_n(u)-\psi_n(u)|/|u|)du ,$$

$$J_{2,n}=\int_{\delta n^{1/2}\leq|u|\leq Mn^{1/2}}(|\rho_n(u)-\psi_n(u)|/|u|)du .$$

We note that S_n^* can be rewritten as follows:

$$S_n^*=n^{-1/2}\sum_{i=1}^n U_{i,n}^*+n^{-1}\sum_{i=1}^n V_{i,n}^*+2^{-1}n^{-3/2}\sum_{i=1}^n\sum_{j=1}^n Q_{ij,n}^* ,$$

where

$$U_{i,n}^*=\langle c_1(F,k)u_F,\ y_{F,i}\rangle/\sigma_n^* ,$$

$$V_{i,n}^*=\langle c_1(F,k)V_F+c_2(F,k)\tilde{V}_F,\ y_{F,i}\rangle/\sigma_n^* ,$$

$$Q_{ij,n}^*=(\langle(c_2(F,k)q_F+\tilde{c}_2(F,k)\tilde{q}_F)y_{F,i},\ y_{F,j}\rangle-\mu_{ij})/\sigma_n^*$$

and

$$\mu_{ij}=E_F(\langle(c_2(F,k)q_F+\tilde{c}_2(F,k)\tilde{q}_F)y_{F,i},\ y_{F,j}\rangle) .$$

Taking account of the condition (c) of Assumption 2, since the random variables $\{U_{i,n}^*, V_{i,n}^*, Q_{ij,n}^*\}$ are bounded, we can verify $J_{1,n}=o(n^{-1/2})$ with argument similar to Callaert et al. [5], Section 3.

Let

$$J'_{2,n}=\int_{\delta n^{1/2}<|u|<Mn^{1/2}}(|\rho_n(u)|/|u|)du ,$$

$$J''_{2,n}=\int_{\delta n^{1/2}<|u|<Mn^{1/2}}(|\psi_n(u)|/|u|)du .$$

It is clear that $J''_{2,n}=o(n^{-1/2})$. By a similar method to the one used in Callaert et al. [5], Section 4, we can evaluate $J'_{2,n}$ as follows. Let

$$\tilde{g}_{in}=U^*_{i,n}+n^{-1/2}V^*_{i,n}+2^{-1}n^{-1}Q^*_{ii,n}\ ,$$

$$A_{m,n}=\sum_{i=1}^{m}\tilde{g}_{in} \quad\text{and}\quad B_{r,s,n}=\sum_{k=1}^{r}\sum_{l=k+1}^{s}Q^*_{k,l,n}\ .$$

We have by Taylor expansion

$$\begin{aligned}(4.5)\qquad \gamma_n(u):&=E_F(\exp(iuS^*_n))\\ &=E_F[\alpha\{1+i(2u)n^{-3/2}B_{m,n}+2^{-1}(2iun^{-3/2}B_{m,2})^2\exp(2i\theta un^{-3/2}B_{m,n})\}]\ ,\end{aligned}$$

where $\alpha=\exp(iun^{-1/2}A_n)\exp(2iun^{-3/2}(B_{n-1,n}-B_{m,n}))$ and $0<\theta<1$. We put

$$\begin{aligned}I_1&=E_F(\alpha)\ ,\\ I_2&=E_F(2i\alpha un^{-3/2}B_{m,n})\ ,\\ I_3&=E_F[2^{-1}(2iun^{-3/2}B_{m,n})^2\exp(2i\theta un^{-3/2}B_{m,n})]\ .\end{aligned}$$

From Lemma 4 in Feller [7], XV. 1, we have

$$(4.6)\qquad \begin{aligned}&|I_1|\leqq|E_F(\exp(iun^{-1/2}A_m))|\leqq(1-\eta_0)^m\\ &\qquad\text{for some constant } \eta_0,\ 0<\eta_0<1, \text{ and also}\\ &|I_2|\leqq 2|u|n^{-3/2}\sum_{j=1}^{m}\sum_{k=j+1}^{n}\{|E_F(\exp(iun^{-1/2}\tilde{g}_{1,n}(X_1))|^{m-2}\cdot E_F(|Q^*_{j,,k,n}|)\}\\ &\qquad\leqq K_1|u|n^{-1/2}m(1-\eta_0)^{m-2}\\ &\qquad(K_1 \text{ is a constant not depending on } m,\ n \text{ and } u)\ .\end{aligned}$$

From the martingale property of $\{B_{m,n}\}$ (cf. Lemma 1 in Callaert et al. [5]) we have

$$(4.7)\qquad |I_3|\leqq u^2n^{-3}E_F(|B_{m,n}|^2)\leqq K_2n^{-2}mu^2 \quad (K_2 \text{ is a constant independent of } u,\ n,\ m)\ .$$

Taking m to be $m=n^{1/4}$ it follows from (4.5), (4.6) and (4.7) that

$$(4.8)\qquad \sup\{|\gamma_n(u)|;\ n^{1/2}\delta<|u|<n^{1/2}M\}=o(n^{-1/2})\ .$$

Thus we have $J_{2,n}=o(n^{-1/2})$. Hence it follows from (4.4) that

$$(4.9)\qquad \sup_{x\in R}|\tilde{F}_n(x)-K_n(x)|=o(n^{-1/2})\ .$$

We note that from (4.2) it follows

(4.10) $$P_{F,n}\{S_n \leqq t\} = P_{F,n}\{S_n^* \leqq t\} + o(n^{-1/2}) .$$

From (4.9) and (4.10) we have the desired expansion (4.1).

For $F \in \Theta^*$ let $\tau(F) = E_F(\langle u_F, y_{F,1}\rangle^2)$, and define

$$h(x, F) = w_F^{(2)}(x)\{-(\gamma(F)/2) + \delta(F)/(2\tau(F))\} + \tilde{w}_F^{(2)}(x)\{-(\tilde{\gamma}(F)/2) + \tilde{\delta}(F)/(2\tau(F))\} + w_F^{(1)}(x)\beta(F)/(6\tau(F)) .$$

We define $g_{n,B}^*(x) = \hat{g}_{n,B}(x) + n^{-1}h(x, \hat{F}_n)$. From Theorems 1 and 2 we have the following result which asserts that the corrected bootstrap estimator $\{g_{n,B}^*\}$ is a second order asymptotic efficient estimator of $\{g_n\}$.

THEOREM 3. *Suppose that Assumption 2 is satisfied. We assume that β, γ, $\tilde{\gamma}$, δ, $\tilde{\delta}$ and τ are continuous with respect to F on Θ^* and that $w_F^{(1)}(x)$, $w_F^{(2)}(x)$, $\tilde{w}_F^{(2)}(x)$ are continuous with respect to F as functions from Θ^* to $\mathscr{B}$. Then we have*

(a) *$\{g_{n,B}^*\}$ is a second order AMU estimator of $\{g_n\}$.*

(b) *$\{g_{n,B}^*\}$ is a second order asymptotically efficient AMU estimator of $\{g_n\}$.*

PROOF. Take any $k \in K$, any $F \in \Theta$ and any sequence $\{G_n\}$, $G_n \in B_n^*(F, c_n)$ and let $\hat{S}_n = n^{1/2}\langle k, g_{n,B}^* - g_n(\cdot, G_n)\rangle$. By a similar argument to that developed for $\{S_n\}$ in the proof of Theorem 2 we have the following expansion for $\{\hat{S}_n\}$:

(4.11) $$P_{G_n,n}\{\hat{S}_n \leqq t\} = \Phi(tJ^{-1/2}(G_n, k)) - n^{-1/2}\phi(tJ^{-1/2}(G_n, k))\Psi(t, G_n, k) + o(n^{-1/2}) .$$

From this we have $\{g_{n,B}^*\} \in \mathscr{C}$. In particular if we take G_n to be F in (4.11) then the R.H.S. of (4.11) equals the R.H.S. of (3.2) up to the order $o(n^{-1/2})$. Therefore $\{g_{n,B}^*\}$ is a second order efficient AMU estimator of $\{g_n\}$.

§ 5. An example satisfying Assumption 2.

In this section we give a typical example satisfying Assumption 2. Following Beran [2], Section 3, let $\hat{T}_n = 2n^{-1}(n-1)^{-1}\sum_{1\leq i<j\leq n} t(X_i, X_j)$ be the second degree U-statistic where t is symmetric in its arguments. We assume that, as in Beran [2], t is absolutely continuous, vanishes outside a square $[-B, B]$ and has essentially bounded derivative. Let μ be the Lebesgue measure on $[-B, B]$. Let $m(G) = E_G(t(X_1, X_2))$, $g_G(X_i) = E_G(h_G(X_i, X_j) \mid X_i)$, $h_G(X_i, X_j) = t(X_i, X_j) - m(G)$, $d_G(X_i, X_j) = h_G(X_i, X_j) -$

$g_G(X_i)-g_G(X_j)$, $s_G^2=E_G(g_G^2(X_1))$, $s_n^2(G)=V_G(n^{1/2}\hat{T}_n)$ and $s_0(G)=2s_G$. Let $\mathscr{V}_0$ be the class of functions v on $\boldsymbol{R}$ such that $v(t)=a^{-1}(1-a^{-1}|t|)^+$ for some $a>0$. Define

$$k_3(G)=s_G^{-3}[E_G g_G^3(X_1)+3E_G\{g_G(X_1)g_G(X_2)d_G(X_1, X_2)\}] ,$$

$$k_4(G)=s_G^{-4}[E_G g_G^4(X_1)-3s_G^4+12E_G\{g_G^2(X_1)g_G(X_2)d_G(X_1, X_2)\} \\ +12E_G\{g_G(X_2)g_G(X_3)d_G(X_1, X_2)d_G(X_1, X_3)\}] ,$$

$$t_1(x)=\phi(x)(x^2-1)/6 ,$$

$$t_2(x)=\phi(x)(x^3-3x)/24$$

and

$$t_3(x)=\phi(x)(x^5-10x^3+15x)/72 .$$

Let Θ^* be the set of d.f. F on $\boldsymbol{R}$ such that $s_0(F)\neq 0$. Let $J_n(x, G)=P_{G,n}\{n^{1/2}(\hat{T}_n-m(G))/s_n(G)\leqq x\}$ and define for $v\in\mathscr{V}_0$ $J_{n,v}(\cdot, G)=J_n(\cdot, G)*v$, which means the convolution of J_n and v. Let $\{c_n\}$ be any sequence of positive numbers satisfying $n^{-1/2}c_n^3$ tending to 0 as $n\to\infty$. Using essentially the same argument as in Beran [2], Section 3, we have

$$\sup_{G\in B_n(F,c_n)}|s_n(G)-s_n(F)-\langle h_F^*, G-F\rangle-2^{-1}\langle q_F(G-F), G-F\rangle|=o(n^{-1}) \tag{5.1}$$

and

$$\sup_{G\in B_n(F,c_n)}|k_3(G)-k_3(F)-\langle v_F^*, G-F\rangle|=o(n^{-1/2}) , \tag{5.2}$$

where $h_F^*(x)=4(g_F(x)+2m(F))e_F(x)/s_0(F)$, $e_F(x)=\int_R F(y)t_{11}(x, y)dy$ ($t_{11}(x, y)$ denotes the derivative of $t(x, y)$), and q_F^*, v_F^* are some functions contained in $L^1(\mu)\times L^1(\mu)$ and $L^1(\mu)$ respectively. We can also verify

$$\sup_{G\in B_n(F,c_n)}\|J_{n,v}(\cdot, G)-\Phi_v(\cdot)+n^{-1/2}k_3(G)t_{1,v}(\cdot) \\ +n^{-1}(k_4(G)t_{2,v}(\cdot)+k_3^2(G)t_{3,v}(\cdot))\|=o(n^{-1}) . \tag{5.3}$$

Here Φ_v, $t_{i,v}$ ($i=1, 2, 3$) mean the convolutions of Φ, t_i ($i=1, 2, 3$) and v respectively.

Define $g_n(x, G)=J_{n,v}(x/s_n(G), G)$ for some $v\in\mathscr{V}_0$ and let Θ be the set of all $F\in\Theta^*$ such that the d.f. of $\langle h_F^*, y_{F,1}\rangle$ is non-lattice. From (5.1), (5.2) and (5.3) we can verify that $\{g_n\}$ satisfies Assumption 2 with

$$g_{n,0}(x, G)=\Phi_v(x/s_n(G)) , \qquad g_{n,1}(x, G)=-k_3(G)t_{1,v}(x/s_n(G)) ,$$

$$g_{n,2}(x, G)=-k_4(G)t_{2,v}(x/s_n(G))-k_3^2(G)t_{3,v}(x/s_n(G)) , \qquad u_F(x)=h_F^*(x) ,$$

$$w_F^{(1)}(x)=w_F^{(2)}(x)=-x\phi_v(x/s_0(F))/s_0^2(F)\ , \qquad \tilde{w}_F^{(2)}(x)=-x\cdot\frac{\partial}{\partial s}[\phi_v(x/s)/s^2]_{s_0(F)}\ ,$$

$$w_F^{(3)}(x)=k_3(F)xt'_{1,v}(x/s_0(F))/s_0^2(F)\ , \qquad \tilde{w}_F^{(3)}(x)=-t_{1,v}(x/s_0(F))\ ,$$

$$v_F(x)=h_F^*(x)\ , \qquad \tilde{v}_F(x)=v_F^*(x)\ ,$$

$$q_F(x,\ y)=q^*(x,\ y) \quad \text{and} \quad \tilde{q}_F(x,\ y)=h_F^*(x)h_F^*(y)\ .$$

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Present Address:
DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCE AND ENGINEERING, WASEDA UNIVERSITY
OKUBO, SHINJUKU-KU, TOKYO 160, JAPAN