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On Compact Generalized Jordan Triple Systems of the Second Kind

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Dedicated to Professor Nagayoshi Iwahori on his sixtieth birthday

Introduction.

A finite dimensional graded Lie algebra $\mathscr{G}=\sum \mathscr{G}_k$ over a field F of characteristic zero is said to be of the ν-th kind, if $\mathscr{G}_{\pm k}=\{0\}$ for $k>\nu$. Let B: $(x, y, z)\mapsto(xyz)$ be a triple operation on a vector space U over F. The operation B is called a generalized Jordan triple system, if the equality $(uv(xyz))=((uvx)yz)-(x(vuy)z)+(xy(uvz))$ is valid for $u, v, x, y, z\in U$. If, in addition, the relation $(xyz)=(zyx)$ holds for $x, y, z\in U$, then B is said to be a Jordan triple system. Koecher [5] and Meyberg [7] studied interesting relationship between Jordan triple systems with nondegenerate trace forms and symmetric Lie algebras $(\mathscr{G}, \tau)$; here $\mathscr{G}$ is a semisimple graded Lie algebra of the 1st kind with $\mathscr{G}_0=[\mathscr{G}_{-1}, \mathscr{G}_1]$, and τ is a grade-reversing involution of $\mathscr{G}$. Our main concern is to generalize this connection to the case of generalized Jordan triple systems. It is known (Kantor [3]) that to a generalized Jordan triple system B on U there corresponds a graded Lie algebra $\mathscr{L}(B)=\sum U_i$ with $U_{-1}=U$. The triple system B is called of the ν-th kind, if the graded Lie algebra $\mathscr{L}(B)$ is of the ν-th kind. Under a certain condition (A) for B (cf. § 1), $\mathscr{L}(B)$ admits a grade-reversing involution τ_B. The pair $(\mathscr{L}(B), \tau_B)$ is considered to be a generalization of the symmetric Lie algebra corresponding to a Jordan triple system. On the other hand, K. Yamaguti [8] introduced the bilinear forms γ_B for a wider class of triple systems. For a generalized Jordan triple system B, the form γ_B is symmetric, and, as is seen in the present paper, it plays the same role as the trace form for a Jordan triple system does. Now suppose B is of the 2nd kind. The first aim of this paper is to prove the following implications (Propositions 2.4, 2.5, 2.10 and Theorem 2.8):

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B satisfies the condition (A)

B is simple $\nearrow$ $\searrow$ $\uparrow$ $\nearrow$ $\mathscr{L}(B)$ is semisimple.

γ_B is nondegenerate

Under the assumption that γ_B is nondegenerate, we will next give a formula which describes a relationship between the Killing form of $\mathscr{L}(B)$ and the symmetric bilinear form γ_B (Theorem 2.13). For the case where F is the field of real numbers, B is said to be compact if γ_B is positive definite. We will prove that B is compact if and only if the grade-reversing involution τ_B is a Cartan involution (Theorem 3.3). In Theorem 3.7 we will show that, under the assumption of compactness for B, $\mathscr{L}(B)$ is simple if and only if B is simple.

Finally we should remark that compact real simple generalized Jordan triple systems B of the 2nd kind with $\mathscr{L}(B)$ classical can be classified (see [2]).

§ 1. Basic facts on the generalized Jordan triple systems of the second kind.

Let U be a finite dimensional vector space over a field F of characteristic zero and B: $U\times U\times U \rightarrow U$ be a trilinear mapping. Then the pair (U, B) is called a *triple system* over F. We shall often write (xyz) instead of $B(x, y, z)$. For subspaces V_i $(1\leqq i\leqq 3)$ of U, we denote by $(V_1V_2V_3)$ the subspace spanned by all elements of the form $(x_1x_2x_3)$ for $x_i\in V_i$. A triple system (U, B) is called a *generalized Jordan triple system* (abbreviated as GJTS) if the following equality is valid:

$$(1.1)\qquad (uv(xyz))=((uvx)yz)-(x(vuy)z)+(xy(uvz))$$

for $u, v, x, y, z\in U$. Furthermore, if the additional condition

$$(xyz)=(zyx) \qquad x, y, z\in U$$

is satisfied, then (U, B) is called a *Jordan triple system* (abbreviated as JTS). For a GJTS which is not a JTS, see Example 2.1. Starting from a given GJTS (U, B), Kantor [3] constructed a certain graded Lie algebra $\mathscr{L}(B)=\sum U_i$ such that $U_{-1}=U$. We call this Lie algebra $\mathscr{L}(B)$ the *Kantor algebra* for (U, B). We say that (U, B) is of the *i-th kind* if $U_{\pm k}=\{0\}$ for all $k>i$. Note that in our conventions every GJTS of the 1st kind is considered as a GJTS of the 2nd kind satisfying $U_{\pm 2}=\{0\}$. It is known [3] that a GJTS is of the 1st kind if and only if it is a JTS. For an element $a\in U$, let us define a bilinear map B_a on U by putting

$$B_a(x, y) = B(x, a, y) \qquad \text{for} \quad x, y \in U .$$

We say that (U, B) *satisfies the condition* (A) if $B_a = 0$ implies $a = 0$. In this case there exists a grade-reversing involutive automorphism τ_B of $\mathscr{L}(B)$ such that $\tau_B(a) = B_a$ for $a \in U$ (see [2] Proposition 3.8). The automorphism τ_B is called the *grade-reversing canonical involution* of $\mathscr{L}(B)$. Let us define the two linear endomorphisms L_{ab} and S_{ab} on U $(a, b \in U)$ by

$$L_{ab}(x) = (abx) , \qquad S_{ab}(x) = (axb) - (bxa) .$$

Let $\mathscr{S}$ be the subspace of $\mathrm{End}(U)$ spanned by operators S_{ab}. Following the arguments in Kantor [3], one can prove that if (U, B) satisfies the condition (A), then there exists a linear isomorphism of U_{-2} onto $\mathscr{S}$. We can thus identify U_{-2} with $\mathscr{S}$. We restate a result of Kantor [3] as follows, in which the condition (A) should be added as an assumption; a bracket relation there should be also corrected.

THEOREM 1.1 ([3]). *Let (U, B) be a GJTS of the 2nd kind satisfying the condition* (A) *and let τ_B be the grade-reversing canonical involution of the Kantor algebra $\mathscr{L}(B) = \sum U_i$ for (U, B). Then,*

(i) $U_{-2} = \mathscr{S}$, $U_{-1} = U$, $U_1 = \tau_B(U_{-1})$, $U_2 = \tau_B(U_{-2})$; U_0 *is the subspace of* $\mathrm{End}(U)$ *spanned by operators* L_{ab}.

(ii) *If we denote $\tau_B(X)$ by $\bar{X}$, then we have the following bracket relations in $\mathscr{L}(B)$:*

$$\begin{aligned} &[a, b] = S_{ba} , \quad [\bar{a}, b] = L_{ba} , \quad [L_{ab}, c] = (abc) , \quad [L_{ab}, \bar{c}] = -\overline{(bac)} , \\ &[\bar{S}_{ab}, c] = \overline{S_{ab}(c)} , \quad [L_{ab}, S_{cd}] = S_{(abc)d} + S_{c(abd)} , \\ &[S_{ab}, \bar{S}_{cd}] = L_{(acb)d} - L_{(bca)d} - L_{(adb)c} + L_{(bda)c} , \\ &[L_{ab}, L_{cd}] = L_{(abc)d} - L_{c(bad)} , \end{aligned} \tag{1.2}$$

where $a, b, c, d \in U$.

Let (U, B) be a GJTS of the 2nd kind over F. Put

$$W = U_{-1} + U_1 , \qquad V = U_{-2} + U_0 + U_2 .$$

Then, since $\mathscr{L}(B) = \sum U_i$ is a graded Lie algebra, the following relations are obviously valid:

$$\mathscr{L}(B) = V + W , \quad [V, V] \subset V , \quad [V, W] \subset W , \quad [W, W] \subset V . \tag{1.3}$$

Therefore the space W becomes a Lie triple system (abbreviated as LTS) with triple product $\{XYZ\} = [[X, Y], Z]$. By $L(X, Y)$ we denote the linear endomorphism $Z \mapsto \{XYZ\}$ on W. Let $L(W, W)$ be the space spanned by operators $L(X, Y)$ and let

$$\mathscr{L}(W)=L(W, W)+W$$

be the standard imbedding Lie algebra of the LTS W (see [6]). Note that $L(W, W)$ is a subalgebra of $\mathscr{L}(W)$. We define the linear mapping φ of V into $\mathrm{End}(W)$ by

$$\varphi(X)=\mathrm{ad}_W(X) \quad \text{(the restriction of } \mathrm{ad}(X) \text{ on } W\text{)} . \tag{1.4}$$

Note that $\varphi([X, Y])=\mathrm{ad}_W([X, Y])=L(X, Y)$ for $X, Y \in W$.

LEMMA 1.2. *If* (U, B) *satisfies the condition* (A), *then* φ *is a Lie isomorphism of* V *onto* $L(W, W)$.

PROOF. It follows from Theorem 1.1 that $[U_{-1}, U_{-1}]=U_{-2}$, $[U_{-1}, U_1]=U_0$, $[U_1, U_1]=U_2$, and consequently $[W, W]=V$. Hence, we get $\varphi(V)=\varphi([W, W])=L(W, W)$. Therefore φ is surjective. Since φ is obviously a Lie homomorphism, it is enough to prove that φ is injective. Suppose that $\varphi(X)=0$ for $X \in V$. Denoting X by $X=S_1+T+\bar{S}_2$ ($S_i \in U_{-2}$, $T \in U_0$), we have $\{0\}=\varphi(X)W=[X, W]=[S_1+T+\bar{S}_2, U_{-1}+U_1]=([S_1, U_1]+[T, U_{-1}])+([T, U_1]+[\bar{S}_2, U_{-1}])$. Since $U_{-1}+U_1$ is a direct sum, this means that $[S_1, U_1]+[T, U_{-1}]=\{0\}$ and $[T, U_1]+[\bar{S}_2, U_{-1}]=\{0\}$. Hence, for any elements $x, y \in U_{-1}=U$, we have

$$[S_1, \bar{x}]+[T, y]=0 , \qquad [T, \bar{x}]+[\bar{S}_2, y]=0 . \tag{1.5}$$

Putting $x=0$ in (1.5), we get $[T, y]=0$ and $[\bar{S}_2, y]=0$. By (1.2), we have $T(y)=0$ and $\overline{S_2(y)}=0$. Since y is an arbitrary element in U, and since τ_B is an isomorphism, it follows that $T=S_2=0$. Similarly, putting $y=0$ in (1.5), we can show that $S_1=0$. Therefore we have $X=0$.

PROPOSITION 1.3. *Let* (U, B) *be a GJTS of the 2nd kind and* $\mathscr{L}(B)$ *be the Kantor algebra for* (U, B). *Let* $\mathscr{L}(W)$ *be the standard imbedding Lie algebra of the LTS* W. *If* (U, B) *satisfies the condition* (A), *then* $\mathscr{L}(B)$ *is isomorphic to* $\mathscr{L}(W)$.

PROOF. We define the map ψ: $\mathscr{L}(B)\to\mathscr{L}(W)$ by $\psi(X+Y)=\varphi(X)+Y$ ($X \in V$, $Y \in W$). Since φ is a Lie isomorphism by Lemma 1.2, it can be easily proved that ψ is also a Lie isomorphism.

By this proposition, the Kantor algebra for a GJTS of the 2nd kind satisfying the condition (A) may be viewed as the standard imbedding Lie algebra of a certain LTS.

§ 2. Nondegenerate generalized Jordan triple systems of the second kind.

Throughout this section, we will keep the notations in the previous section.

2.1. Let (U, B) be a GJTS of the 2nd kind over F. We denote the linear endomorphism $z \mapsto (zxy)$ on U by R_{xy}. Let us consider the symmetric bilinear form on U:

$$\gamma_B(x, y) = \frac{1}{2}\mathrm{Tr}(2R_{xy} + 2R_{yx} - L_{xy} - L_{yx}) ,$$

where $\mathrm{Tr}(f)$ means the trace of a linear endomorphism f. The form γ_B is a special case of the bilinear form considered by K. Yamaguti [8]. In the case of a JTS, this form coincides with the usual trace form γ defined by $\gamma(x, y) = (1/2)\mathrm{Tr}(L_{xy} + L_{yx})$. We call γ_B the *trace form* of the GJTS of the 2nd kind (U, B).

EXAMPLE 2.1. Let $M(p, q-p; \boldsymbol{C})$, $p<q$ be the real vector space of all $p \times (q-p)$ matrices with coefficients in the complex number field $\boldsymbol{C}$. For an element $X \in M(p, q-p; \boldsymbol{C})$ we denote by X^* the transposed conjugate matrix of X. We define a trilinear map B on $M(p, q-p; \boldsymbol{C})$ by

$$B(X, Y, Z) = XY^*Z + ZY^*X - ZX^*Y .$$

Then, by direct calculations, $(M(p, q-p; \boldsymbol{C}), B)$ is seen to be a real GJTS of the 2nd kind, which is not a JTS. In this case $\mathscr{L}(B)$ is isomorphic to the Lie algebra $\mathfrak{su}(p, q)$ (see [2]). We will compute the trace form γ_B. For given $X, Y \in M(p, q-p; \boldsymbol{C})$, let us first consider the real linear endomorphism T on $M(p, q-p; \boldsymbol{C})$ defined by $T(Z) = XZ^*Y$. Then, direct computations show that

$$\mathrm{Tr}_{\boldsymbol{R}}(T) = 0 . \tag{2.1}$$

Let A (resp. B) be a square matrix of degree p (resp. $q-p$), and let λ_A (resp. ρ_B) be the left (resp. right) multiplication by A (resp. B) on $M(p, q-p; \boldsymbol{C})$. By using (2.1), we have

$$\gamma_B(X, Y) = \frac{1}{2}\mathrm{Tr}_{\boldsymbol{R}}(2\rho_{X^*Y} + 2\rho_{Y^*X} + \lambda_{YX^*} + \lambda_{XY^*}) .$$

On the other hand, we see that

$$\mathrm{Tr}_{\boldsymbol{C}}(\rho_{X^*Y}) = \mathrm{Tr}(E_p \otimes X^*Y) = p\overline{(\mathrm{Tr}\, XY^*)} ,$$
$$\mathrm{Tr}_{\boldsymbol{C}}(\lambda_{XY^*}) = \mathrm{Tr}(XY^* \otimes E_{q-p}) = (q-p)(\mathrm{Tr}\, XY^*) ,$$

where E_p (resp. E_{q-p}) is the unit matrix of degree p (resp. $q-p$). By using these equalities we get

$$\gamma_B(X, Y)=2(p+q)\mathrm{Re}(\mathrm{Tr}\, XY^*) ,$$

where Re denotes the real part. γ_B is thus positive definite.

Let β be the Killing form of the Kantor algebra $\mathscr{L}(B)$. Since $\mathscr{L}(B)=\sum U_i$ is a graded Lie algebra, we have that $\beta(U_i, U_j)=0$ if $i+j\neq 0$. Hence we get

$$\beta(V, W)=0 . \tag{2.2}$$

From now on, we assume that (U, B) satisfies the condition (A). Then, since $\mathscr{L}(B)$ is isomorphic with $\mathscr{L}(W)$ by Proposition 1.3, β can be considered to be the Killing form of $\mathscr{L}(W)$. Let α be the Ricci (or Killing) form of the LTS W defined by

$$\alpha(X, Y)=\frac{1}{2}\mathrm{Tr}(R(X, Y)+R(Y, X)) ,$$

where $R(X, Y)$ is the linear endomorphism on W defined by $Z\mapsto\{ZXY\}$. It is well known (see [6]) that

$$\beta(X, Y)=2\alpha(X, Y) \qquad \text{for} \quad X, Y\in W .$$

The following lemma is essentially obtained by Yamaguti [8]. His result is different from ours only in the sign.

LEMMA 2.2. *For $x_i, y_i\in U$ $(i=1, 2)$, we have*

$$\beta(x_1+\bar{x}_2, y_1+\bar{y}_2)=-2\{\gamma_B(x_1, y_2)+\gamma_B(x_2, y_1)\} .$$

PROPOSITION 2.3. *Let (U, B) be a GJTS of the 2nd kind satisfying the condition (A). If the trace form γ_B is identically zero, then the Kantor algebra $\mathscr{L}(B)$ is solvable.*

PROOF. By Lemma 2.2 the Killing form β of $\mathscr{L}(B)$ is identically zero on W. Choose an element $X\in V$. Since $V=[W, W]$, X can be written as $X=\sum[Y_i, Z_i]$ $(Y_i, Z_i\in W)$. Then, for an arbitrary element $X'\in V$, we have $\beta(X, X')=\sum\beta([Y_i, Z_i], X')=\sum\beta(Y_i, [Z_i, X'])=0$, because Y_i and $[Z_i, X']$ are in W. Therefore β is also identically zero on V. In view of (2.2), we obtain that β is identically zero on $\mathscr{L}(B)$. Hence $\mathscr{L}(B)$ is solvable.

PROPOSITION 2.4. *Let (U, B) be a GJTS of the 2nd kind satisfying the condition (A). Let γ_B be the trace form of (U, B) and $\mathscr{L}(B)$ be the*

Kantor algebra for (U, B). *Then* γ_B *is nondegenerate if and only if* $\mathscr{L}(B)$ *is semisimple.*

PROOF. Kamiya [1] proved that γ_B is nondegenerate if and only if $\mathscr{L}(W)$ is semisimple. Combining this with Proposition 1.3, we obtain this proposition.

2.2. Let (U, B) be a GJTS over F. A subspace I of U is called an *ideal* (resp. *K-ideal*) if $(UUI)+(UIU)+(IUU)\subset I$ (resp. $(UUI)+(IUU)\subset I$) is valid. Obviously any ideal is a K-ideal. The whole space U and $\{0\}$ are called the trivial ideals. (U, B) is said to be *simple* (resp. *K-simple*) if B is not a zero map and U has no non-trivial ideal (resp. K-ideal). Hence every K-simple GJTS is simple.

PROPOSITION 2.5. *Every simple GJTS* (U,B) *satisfies the condition* (A).

PROOF. Put $I=\{a\in U\,|\,B_a=0\}$. Let $u, v, x, y\in U$ and $a\in I$. Using (1.1), we get $B_{(xya)}(u, v)=(u(xya)v)=-(yx(uav))+((yxu)av)+(ua(yxv))=0$. It follows that $B_{(xya)}=0$, that is, $(xya)\in I$. Hence we have $(UUI)\subset I$. Similarly we can obtain $(IUU)\subset I$. Obviously we have $(UIU)=\{0\}\subset I$. Therefore I is an ideal of U. From the assumption of simplicity, we have $I=\{0\}$ or $I=U$. If we suppose that $I=U$, then we have $(UUU)=\{0\}$, which contradicts the assumption that B is not a zero map. Hence we have to have $I=\{0\}$. This means that (U, B) satisfies the condition (A).

LEMMA 2.6. *Let* (U, B) *be a GJTS of the 2nd kind. If it is simple, then* $[V, W]=W$ *is valid.*

PROOF. Since (UUU) is an ideal of U, we have $(UUU)=U$ from the assumption of simplicity. By Proposition 2.5 and Theorem 1.1, we have $U_0=[U_1, U_{-1}]=[\tau_B(U_{-1}), U_{-1}]$ and $U_{-1}=U$. Hence, using the equality $[[\bar{x}, y], z]=[L_{yx}, z]=(yxz)$, we obtain that

$$[U_0, U_{-1}]=[[\tau_B(U_{-1}), U_{-1}], U_{-1}]=(UUU)=U_{-1}\,.$$

By applying τ_B to this equality, we have also that

$$[U_0, U_1]=\tau_B([U_0, U_{-1}])=\tau_B(U_{-1})=U_1\,.$$

From these two equalities, we get the relation

$$[V, W]\supset[U_0, U_{-1}+U_1]=U_{-1}+U_1=W\,.$$

Since the converse inclusion is known in (1.3), we obtain $[V, W]=W$.

LEMMA 2.7 ([1]). *For a GJTS* (U, B) *of the 2nd kind, the following* *relation is valid:*

$$\gamma_B((xyz), w)=\gamma_B(z, (yxw))=\gamma_B(x, (wzy)) .$$

THEOREM 2.8. *Let (U, B) be a GJTS of the 2nd kind. If it is simple, then the trace form γ_B is nondegenerate.*

PROOF. Put $U^\perp=\{a \in U \mid \gamma_B(a, U)=0\}$. Let $x, y, z \in U$ and $a \in U^\perp$. By Lemma 2.7, we have that

$$\gamma_B((xya), z)=\gamma_B(a, (yxz))=0 ,$$
$$\gamma_B((axy), z)=\gamma_B(a, (zyx))=0 ,$$
$$\gamma_B((xay), z)=\gamma_B(x, (zya))=\gamma_B((yzx), a)=0 .$$

It follows from these equalities that $U^\perp$ is an ideal of U. Hence we have $U^\perp=\{0\}$ or $U^\perp=U$, that is, γ_B is nondegenerate or identically zero. Now let us assume that γ_B is identically zero. Then, by Proposition 2.3, $\mathscr{L}(B)$ is a solvable Lie algebra. Consequently, we have

$$[\mathscr{L}(B), \mathscr{L}(B)]\neq\mathscr{L}(B) . \tag{2.3}$$

On the other hand, using Proposition 2.5, Lemma 2.6 and (1.3), we obtain that

$$\begin{aligned}[\mathscr{L}(B), \mathscr{L}(B)]&=[V+W, V+W]\\ &=[V, V]+[V, W]+[W, W]=V+W=\mathscr{L}(B) ,\end{aligned}$$

which contradicts (2.3). Therefore γ_B is nondegenerate.

Combining this theorem with Propositions 2.4 and 2.5, we obtain a Kantor's result [4], which was stated without proof.

COROLLARY 2.9. *Let (U, B) be a GJTS of the 2nd kind. If it is simple, then the Kantor algebra $\mathscr{L}(B)$ is semisimple.*

2.3. Let (U, B) be a GJTS of the 2nd kind over F. (U, B) is said to be *nondegenerate* if its trace form γ_B is nondegenerate. In this subsection, we assume that (U, B) is a nondegenerate GJTS of the 2nd kind. We denote by $X^\vee$ the adjoint operator of $X \in \mathrm{End}(U)$ relative to γ_B.

PROPOSITION 2.10. *A nondegenerate GJTS of the 2nd kind satisfies the condition* (A).

PROOF. Let a be an element satisfying $B_a=0$, that is, $(xay)=0$ for $x, y \in U$. It follows that $L_{xa}=R_{ax}=0$. Hence $\gamma_B(a, x)$ is expressed as follows:

$$\gamma_B(a, x) = \frac{1}{2}\mathrm{Tr}(2R_{xa} - L_{ax}) \ . \tag{2.4}$$

Since γ_B is nondegenerate, it follows from Lemma 2.7 that

$$L_{xy}{}^{\nu} = L_{yx} \ , \qquad R_{xy}{}^{\nu} = R_{yx} \ . \tag{2.5}$$

Hence we have $\mathrm{Tr}\, L_{yx} = \mathrm{Tr}\, L_{xy}$ and $\mathrm{Tr}\, R_{yx} = \mathrm{Tr}\, R_{xy}$. Substituting these into (2.4), we get $\gamma_B(a, x) = (1/2)\mathrm{Tr}(2R_{xa} - L_{ax}) = (1/2)\mathrm{Tr}(2R_{ax} - L_{xa}) = 0$. From the nondegeneracy of γ_B, it follows that $a = 0$. This completes the proof.

LEMMA 2.11. *In a nondegenerate GJTS (U, B) of the 2nd kind, we have*

$$T^{\nu} = -\bar{T} \qquad \text{for} \quad T \in U_0 \ , \tag{2.6}$$

$$S^{\nu} = -S \qquad \text{for} \quad S \in U_{-2} \ . \tag{2.7}$$

PROOF. Using (1.2), we have $\bar{L}_{xy} = \tau_B([\bar{y}, x]) = [y, \bar{x}] = -L_{yx}$. Combining this with (2.5), we get $L_{xy}{}^{\nu} = -\bar{L}_{xy}$. Since U_0 is the linear span of operators L_{xy}, (2.6) is valid. Using Lemma 2.7, we have

$$\begin{aligned}\gamma_B(S_{xy}(u), v) &= \gamma_B((xuy), v) - \gamma_B((yux), v) = \gamma_B(y, (uxv)) - \gamma_B(y, (vxu)) \\ &= \gamma_B((yvx), u) - \gamma_B((xvy), u) = -\gamma_B(S_{xy}(v), u) \ .\end{aligned}$$

It follows that $S_{xy}{}^{\nu} = -S_{xy}$. Since U_{-2} is the linear span of operators S_{xy}, (2.7) is also valid.

Let us recall the homomorphism φ in (1.4). Lemma 1.2 and Proposition 2.10 show that φ is a Lie isomorphism of V onto $L(W, W)$ if (U, B) is nondegenerate.

LEMMA 2.12. *For a nondegenerate GJTS (U, B) of the 2nd kind, we have*

$$\mathrm{Tr}_W\, \varphi(T_1)\varphi(T_2) = 2\, \mathrm{Tr}_U(T_1 T_2) \qquad \text{for} \quad T_i \in U_0 \ , \tag{2.8}$$

$$\mathrm{Tr}_W\, \varphi(S_1)\varphi(\bar{S}_2) = \mathrm{Tr}_U(S_1 S_2) \qquad \text{for} \quad S_i \in U_{-2} \ . \tag{2.9}$$

PROOF. For $x \in U$ and $T \in U_0$, we have that $[T, x] = T(x)$ and $[T, \bar{x}] = \tau_B([\bar{T}, x]) = \tau_B(\bar{T}(x))$. Let $x, y \in U$ and $T_i \in U_0$ $(i = 1, 2)$. Using those two relations, we get

$$\begin{aligned}\varphi(T_1)\varphi(T_2)(x + \tau_B(y)) &= [T_1, [T_2, x + \bar{y}]] = [T_1, T_2(x) + \tau_B(\bar{T}_2(y))] \\ &= T_1 T_2(x) + \tau_B(\bar{T}_1 \bar{T}_2(y)) \ .\end{aligned}$$

Since τ_B is an isomorphism, it follows that

$$\mathrm{Tr}_W\, \varphi(T_1)\varphi(T_2) = \mathrm{Tr}_U(T_1T_2) + \mathrm{Tr}_U(\bar{T}_1\bar{T}_2) . \tag{2.10}$$

By Lemma 2.11, we have

$$\mathrm{Tr}_U(\bar{T}_1\bar{T}_2) = \mathrm{Tr}_U(T_1^{\nu}T_2^{\nu}) = \mathrm{Tr}_U(T_2T_1)^{\nu} = \mathrm{Tr}_U(T_2T_1) = \mathrm{Tr}_U(T_1T_2) .$$

Substituting this into (2.10), we obtain (2.8). Similarly, from the relation

$$\varphi(S_1)\varphi(\bar{S}_2)(x+\tau_B(y)) = [S_1, [\bar{S}_2, x+\bar{y}]] = [S_1, \overline{S_2(x)}] = S_1S_2(x) ,$$

we get (2.9).

THEOREM 2.13. *Let (U, B) be a nondegenerate GJTS of the 2nd kind, and let β be the Killing form of the Kantor algebra $\mathscr{L}(B)$ for (U, B). Let $X_i = S_i + x_i + T_i + \bar{y}_i + \bar{S}'_i$ $(i=1, 2)$ be elements in $\mathscr{L}(B)$, where $S_i, S'_i \in U_{-2}$, $T_i \in U_0$, $x_i, y_i \in U$. Then we have*

$$\begin{aligned}\beta(X_1, X_2) &= \beta_V(S_1, \bar{S}'_2) + \beta_V(T_1, T_2) + \beta_V(\bar{S}'_1, S_2) \\ &\quad + \mathrm{Tr}_U(S_1S'_2 + 2T_1T_2 + S'_1S_2) - 2\{\gamma_B(x_1, y_2) + \gamma_B(y_1, x_2)\} ,\end{aligned} \tag{2.11}$$

where β_V is the Killing form of the subalgebra V of $\mathscr{L}(B)$.

PROOF. Since $\beta(U_i, U_j) = 0$ for i and j such that $i+j \neq 0$, we have

$$\beta(X_1, X_2) = \beta(S_1, \bar{S}'_2) + \beta(T_1, T_2) + \beta(\bar{S}'_1, S_2) + \beta(x_1, \bar{y}_2) + \beta(\bar{y}_1, x_2) . \tag{2.12}$$

It follows from Lemma 2.2 that

$$\beta(x_1, \bar{y}_2) + \beta(\bar{y}_1, x_2) = -2\{\gamma_B(x_1, y_2) + \gamma_B(y_1, x_2)\} . \tag{2.13}$$

Now let us assume that Y and Z are elements in V. Since the subspaces V and W are invariant under the map $\mathrm{ad}(Y)\mathrm{ad}(Z)$, we have

$$\begin{aligned}\beta(Y, Z) &= \mathrm{Tr}_V\, \mathrm{ad}(Y)\mathrm{ad}(Z) + \mathrm{Tr}_W\, \mathrm{ad}(Y)\mathrm{ad}(Z) \\ &= \beta_V(Y, Z) + \mathrm{Tr}_W\, \varphi(Y)\varphi(Z) .\end{aligned} \tag{2.14}$$

Hence, using Lemma 2.12, we have

$$\begin{aligned}\beta(S_1, \bar{S}'_2) &= \beta_V(S_1, \bar{S}'_2) + \mathrm{Tr}_U(S_1S'_2) , \\ \beta(T_1, T_2) &= \beta_V(T_1, T_2) + 2\,\mathrm{Tr}_U(T_1T_2) , \\ \beta(\bar{S}'_1, S_2) &= \beta_V(\bar{S}'_1, S_2) + \mathrm{Tr}_U(S'_1S_2) .\end{aligned} \tag{2.15}$$

Substituting (2.13) and (2.15) into (2.12), we obtain (2.11).

REMARK. The above theorem contains the corresponding result for JTS's by Koecher [5].

§ 3. Compact generalized Jordan triple systems of the second kind.

In this section, we restrict our attention to the case where F is the real number field $\boldsymbol{R}$. We keep the notations in the previous sections.

3.1. Let (U, B) be a real GJTS of the 2nd kind. (U, B) is said to be *compact* if its trace form γ_B is positive definite. Later on, let us assume that (U, B) is a compact GJTS of the 2nd kind. Since γ_B is non-degenerate, (U, B) satisfies the condition (A) by Proposition 2.10. Hence, by Proposition 2.4, $\mathscr{L}(B)$ is semisimple. We define an inner product $\langle\ ,\ \rangle$ on the subspace W of $\mathscr{L}(B)$ as follows:

$$\langle x_1+\tau_B(x_2),\ y_1+\tau_B(y_2)\rangle=\gamma_B(x_1, y_1)+\gamma_B(x_2, y_2)$$

for $x_i, y_i \in U$ $(i=1, 2)$. From Lemma 2.2 it follows that

$$\langle X, Y\rangle=-\frac{1}{2}\beta(X, \tau_B(Y)) \qquad \text{for} \quad X, Y\in W\,. \tag{3.1}$$

For $P\in \mathrm{End}(W)$, let us denote its adjoint operator relative to $\langle\ ,\ \rangle$ by P^*.

LEMMA 3.1. *We have*

$$\varphi(X)^*=-\varphi(\tau_B(X)) \qquad \textit{for} \quad X\in V\,, \tag{3.2}$$

$$L(Y, Z)^*=L(\tau_B(Z), \tau_B(Y)) \qquad \textit{for} \quad Y, Z\in W\,. \tag{3.3}$$

PROOF. For $Y, Z\in W$, using (3.1), we get

$$\begin{aligned}\langle\varphi(X)^*(Y), Z\rangle&=\langle Y, \varphi(X)(Z)\rangle=\langle Y, [X, Z]\rangle=-\frac{1}{2}\beta(Y, \tau_B([X, Z]))\\&=-\frac{1}{2}\beta(Y, [\tau_B(X), \tau_B(Z)])=-\frac{1}{2}\beta([Y, \tau_B(X)], \tau_B(Z))\\&=\frac{1}{2}\beta(\varphi(\tau_B(X))(Y), \tau_B(Z))=-\langle\varphi(\tau_B(X))(Y), Z\rangle\,,\end{aligned}$$

from which (3.2) follows. Moreover we have

$$L(Y, Z)^*=\varphi([Y, Z])^*=-\varphi(\tau_B([Y, Z]))=-\varphi([\tau_B(Y), \tau_B(Z)])=L(\tau_B(Z), \tau_B(Y))\,.$$

Hence (3.3) is also valid.

The relation (3.3) implies that $L(W, W)^*\subset L(W, W)$. Let us define an inner product $(\ ,\ )$ on the space $L(W, W)$ by

$$(P, Q)=\mathrm{Tr}_W\, PQ^* \qquad \text{for} \quad P, Q\in L(W, W)\,.$$

We denote by $\sigma^{\sim}$ the adjoint operator of $\sigma \in \mathrm{End}(L(W, W))$ relative to $(\ ,\ )$. For $P, Q, R \in L(W, W)$, we have

$$([P, Q], R) = \mathrm{Tr}_W(PQ-QP)R^* = \mathrm{Tr}_W\, Q(R^*P-PR^*)$$
$$= \mathrm{Tr}_W\, Q(P^*R-RP^*)^* = (Q, [P^*, R]) .$$

This means that

$$(3.4) \qquad (\mathrm{ad}(P))^{\sim} = \mathrm{ad}(P^*) \qquad \text{for} \quad P \in L(W, W) .$$

LEMMA 3.2. *$\beta_V(X, \tau_B(X)) \leqq 0$ for $X \in V$.*

PROOF. Let us denote by β_L the Killing form of the Lie algebra $L(W, W)$. Since the map φ is an isomorphism of V onto $L(W, W)$, we have

$$\beta_V(X, Y) = \beta_L(\varphi(X), \varphi(Y)) \qquad \text{for} \quad X, Y \in V .$$

Using this equality together with (3.2) and (3.4), we obtain

$$\beta_V(X, \tau_B(X)) = \beta_L(\varphi(X), \varphi(\tau_B(X))) = -\beta_L(\varphi(X), \varphi(X)^*)$$
$$= -\mathrm{Tr}_{L(W,W)}\, \mathrm{ad}(\varphi(X))\mathrm{ad}(\varphi(X)^*)$$
$$= -\mathrm{Tr}_{L(W,W)}\, \mathrm{ad}(\varphi(X))(\mathrm{ad}(\varphi(X)))^{\sim} \leqq 0 .$$

The following theorem gives a characterization for a GJTS to be compact.

THEOREM 3.3. *Let (U, B) be a real nondegenerate GJTS of the 2nd kind and τ_B be the grade-reversing canonical involution of the Kantor algebra $\mathscr{L}(B)$. Then (U, B) is compact if and only if τ_B is a Cartan involution of $\mathscr{L}(B)$.*

PROOF. Let us assume that (U, B) is compact. Since γ_B is nondegenerate in this case, it follows from Propositions 2.4 and 2.10 that $\mathscr{L}(B)$ is semisimple. For an element $X \in \mathscr{L}(B)$, we write it as $X = X_V + X_W$ $(X_V \in V,\ X_W \in W)$. Then, using (2.2), (2.14), (3.1) and (3.2), we have

$$\beta(X, \tau_B(X)) = \beta_V(X_V, \tau_B(X_V)) + \mathrm{Tr}_W\, \varphi(X_V)\varphi(\tau_B(X_V)) + \beta(X_W, \tau_B(X_W))$$
$$= \beta_V(X_V, \tau_B(X_V)) - \mathrm{Tr}_W\, \varphi(X_V)\varphi(X_V)^* - 2\langle X_W, X_W\rangle .$$

From Lemma 3.2, we obtain $\beta(X, \tau_B(X)) \leqq 0$. Now suppose that $\beta(X, \tau_B(X)) = 0$. Then, in view of Lemma 3.2, we have $\mathrm{Tr}_W\, \varphi(X_V)\varphi(X_V)^* = \langle X_W, X_W\rangle = 0$. It follows that $\varphi(X_V) = 0$ and $X_W = 0$. Since φ is an isomorphism, we obtain $X_V = 0$, and consequently $X = 0$. Thus we have shown that the bilinear form $\beta(X, \tau_B(Y))$ on $\mathscr{L}(B)$ is negative definite. Consequently τ_B is a Cartan involution.

Conversely, let us assume that τ_B is a Cartan involution of $\mathscr{L}(B)$. Then the bilinear form $\beta(X, \tau_B(Y))$ is negative definite. Since γ_B is non-degenerate, Proposition 2.10 and Lemma 2.2 give the relation

$$\gamma_B(x, y) = -\frac{1}{2}\beta(x, \tau_B(y)) .$$

Therefore γ_B is positive definite, that is, (U, B) is compact.

From Theorems 2.8 and 3.3, we obtain the following

COROLLARY 3.4. *Let (U, B) be a real simple GJTS of the 2nd kind and τ_B be the grade-reversing canonical involution of the Kantor algebra $\mathscr{L}(B)$. Then (U, B) is compact if and only if τ_B is a Cartan involution of $\mathscr{L}(B)$.*

3.2. Let (U, B) be a compact simple GJTS of the 2nd kind. By Corollary 2.9, the Kantor algebra $\mathscr{L}(B) = \sum U_i$ is a semisimple graded Lie algebra. Therefore there exists the unique element $E \in U_0$ such that

$$U_i = \{X \in \mathscr{L}(B) \mid [E, X] = iX\} .$$

Let $\mathscr{L}(B) = \sum \mathscr{L}^k$ be the decomposition into the direct sum of simple ideals. Considering the operator $\mathrm{ad}(E)$, we can prove that every ideal $\mathscr{L}^k$ is a graded ideal, that is,

$$\mathscr{L}^k = \sum(\mathscr{L}^k \cap U_i) . \tag{3.5}$$

LEMMA 3.5. *$\tau_B(\mathscr{L}^k) = \mathscr{L}^k$ for each k.*

PROOF. Assume that $\tau_B(\mathscr{L}^k) = \mathscr{L}^l$ $(k \neq l)$. Since the relation $\beta(\mathscr{L}^k, \mathscr{L}^l) = 0$ holds, we have $\beta(X, \tau_B(X)) = 0$ for $X \in \mathscr{L}^k$. This contradicts the fact that the bilinear form $\beta(X, \tau_B(Y))$ is negative definite. Hence every simple ideal $\mathscr{L}^k$ is τ_B-invariant.

From the above lemma, it follows that

$$\tau_B(\mathscr{L}^k \cap U_{-i}) = \mathscr{L}^k \cap U_i \quad (i = 1, 2) . \tag{3.6}$$

We put $U^k = \mathscr{L}^k \cap U = \mathscr{L}^k \cap U_{-1}$.

LEMMA 3.6. *U^k is a non-zero ideal of (U, B).*

PROOF. Let $x \in U^k$. By Lemma 3.5, we have $\bar{x} = \tau_B(x) \in \mathscr{L}^k \cap U_1 \subset \mathscr{L}^k$. It follows that

$$(yxz) = [[\bar{x}, y], z] \in U^k \qquad \text{for} \quad y, z \in U .$$

Furthermore we have

$$(xyz)=[[\bar{y}, x], z] \in U^k\,, \qquad (yzx)=[[\bar{z}, y], x] \in U^k\,.$$

These relations imply that U^k is an ideal of the GJTS (U, B). Now suppose that $U^k=\{0\}$. Then, from (3.6), we have $\mathscr{L}^k \cap U_1=\{0\}$. Furthermore, since $[U_{-1}, U_{-1}]=U_{-2}$ and $[U_{-1}, U_1]=U_0$, we obtain

$$\mathscr{L}^k \cap U_{-2}=\{0\}\,, \quad \mathscr{L}^k \cap U_2=\{0\}\,, \quad \mathscr{L}^k \cap U_0=\{0\}\,.$$

It follows from (3.5) that $\mathscr{L}^k=\{0\}$, which is a contradiction. Therefore U^k is not zero.

THEOREM 3.7. *Let (U, B) be a compact GJTS of the 2nd kind. Then the Kantor algebra $\mathscr{L}(B)$ is simple if and only if (U, B) is simple.*

PROOF. The "if" part follows from Propositions 2.4, 2.5 and Lemma 3.6. Suppose that $\mathscr{L}(B)$ is simple. Then, by a result of Kantor (Proposition 7′ in [3]), (U, B) is K-simple and hence it is simple.

From Theorem 3.7 and its proof we get

THEOREM 3.8. *Let (U, B) be a compact GJTS of the 2nd kind. Then (U, B) is simple if and only if it is K-simple.*

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