TOKYO J. MATH.
VOL. 11, NO. 1, 1988

Flow-Spines and Seifert Fibred Structure of 3-Manifolds

Ippei ISHII

Keio University

The concept of a flow-spine of a closed 3-manifold M was introduced in [5]. In this paper, we shall give a sufficient condition for M represented by a flow-spine to be Seifert fibred (Theorem 1 in §2). In §3 the orbit manifold and exceptional fibres are completely determined by a DS-diagram which is induced by a flow-spine. We give an example in §4. And in §5, we study Seifert fibred submanifolds and embedded tori determined by a DS-diagram.

§1. A flow-spine and a DS-diagram with E-cycle.

A *normal pair* (ψ_t, Σ) on a closed 3-manifold M is a pair of a non-singular flow ψ_t on M and its compact local section Σ satisfying that

(i) Σ is homeomorphic to a compact 2-disk,

(ii) $T_-(x)=\sup\{t<0 \mid \psi_t(x)\in\Sigma\}$ and $T_+(x)=\inf\{t>0 \mid \psi_t(x)\in\Sigma\}$ are both finite for any $x\in M$, and

(iii) $\partial\Sigma$ is ψ_t-transversal at $(x, T_+(x))\in\partial\Sigma\times\boldsymbol{R}$ for any $x\in\partial\Sigma$,

(for the precise definition, see [5]). Flow-spines $P_\pm=P_\pm(\psi_t, \Sigma)$ are defined by

$$P_-=\Sigma\cup\{\psi_t(x) \mid x\in\partial\Sigma,\ T_-(x)\leqq\psi_t(x)\leqq 0\}\ ,$$
$$P_+=\Sigma\cup\{\psi_t(x) \mid x\in\partial\Sigma,\ 0\leqq\psi_t(x)\leqq T_+(x)\}\ ,$$

each of which forms a standard spine of M. It was shown in [5] that any closed 3-manifold admits a normal pair on it.

On the other hand, the notion of a *closed fake surface* and a *DS-diagram* was introduced in [2] and [3]. For a closed fake surface P, we denote by $\mathfrak{S}_j(P)$ $(j=1, 2, 3)$ the set of the j-th singularities of P (see [2] for the definition). Let P be a closed fake surface which admits a local homeomorphism $f: S^2\to P$ (S^2 is the 2-sphere) such that $\# f^{-1}(x)=j+1$ for $x\in\mathfrak{S}_j(P)$ $(j=1, 2, 3)$. Such an f is called an *identification map*. Then $G=f^{-1}(\mathfrak{S}_2(P))$ is a 3-regular graph on S^2. We call (S^2, P, G, f) a DS-

Received April 8, 1987

diagram (see [3], for the precise), which we simply write (G, f) in what follows. Considering S^2 as the boundary of a 3-ball B^3 and identifying the points on $S^2=\partial B^3$ by f (i.e., $x\sim y$ iff $f(x)=f(y)$), we get a closed 3-manifold $M=B^3/f$ which has a closed fake surface $P=S^2/f$ as its standard spine. Conversely every closed 3-manifold and its standard spine are represented by a DS-diagram (see [3]). For a graph G on S^2, we denote by $V(G)$, $E(G)$ and $F(G)$ the set of vertexes, edges and faces of G respectively (see [3] for the precise definition). Also the underlying space of G is denoted by the same letter G.

A special class of DS-diagrams, *DS-diagrams with E-cycle*, was introduced in [4], and it was shown in [6] that DS-diagrams of this class exactly correspond to those determined by flow-spines. A simple closed curve $e\subset G$ is called an *E-cycle* of a DS-diagram (G, f), if there are a cycle $(\varepsilon_1, \cdots, \varepsilon_n)$ $(\varepsilon_j\in E(G))$ of G and faces $X_1, \cdots, X_m\in F(G)$ such that

(i) $e=\cup_{j=1}^n \bar{\varepsilon}_j=\partial(\cup_{k=1}^m \overline{X}_k)$,

(ii) $f(\varepsilon_j)\neq f(\varepsilon_{j'})$ for $1\leqq j<j'\leqq n$,

(iii) $f(X_k)\neq f(X_{k'})$ for $1\leqq k<k'\leqq m$, and

(iv) $f(\cup_{k=1}^m \overline{X}_k)=S^2/f$.

The next lemma immediately follows from the method for constructing a normal pair generating a given DS-diagram with E-cycle (see [6]).

LEMMA 1. *Let (G, f) be a DS-diagram with an E-cycle e, Σ_j $(j=1, 2)$ be the closure of the components of S^2-e, and h be a homeomorphism from Σ_2 onto Σ_1 such that $h|_{\partial\Sigma_2}$ is the identity map of e. Then we can construct a normal pair (ψ_t, Σ) on $M=B^3/f$ such that there is an embedding $\iota\colon \Sigma_1\to M$ satisfying that $\iota(G\cap\Sigma_1)=\partial\Sigma\cup\hat{T}_-(\partial\Sigma)$ and $\iota(h(G\cap\Sigma_2))=\partial\Sigma\cup\hat{T}_+(\partial\Sigma)$, where $\hat{T}_\pm(x)=\psi_\sigma(x)$ $(\sigma=T_\pm(x))$.*

§2. A characterization of a DS-diagram of a Seifert fibred manifold.

Let (G, f) be a DS-diagram with an E-cycle e, Σ'_j $(j=1, 2)$ be the components of S^2-e, and Σ_j be the closure of Σ'_j. Let $a\in\mathfrak{S}_3(P)$ be one of the third singularities of $P=S^2/f$. Then $f^{-1}(a)\subset V(G)$ consists of four points, two of which are on e. For a point $x\in V(G)\cap e$, we denote by A_x an edge of G such that $x\in\partial A_x$ and $A_x\not\subset e$. For convenience, we divide $V(G)\cap e$ into two sets $V_1(G)$ and $V_2(G)$ as follows: $x\in V_j(G)$ iff $A_x\subset\Sigma_j$ $(j=1, 2)$. As is shown in [4], for any $x\in V(G)\cap\Sigma'_1$ there are uniquely determined points x_1, x_2 and x_3 such that $f(x)=f(x_j)$ $(j=1, 2, 3)$, $x_j\in V_j(G)$ $(j=1, 2)$ and $x_3\in\Sigma'_2$. And, for a point $x\in G-V(G)$, there are points x_1 and x_2 such that $f(x)=f(x_j)$ $(j=1, 2)$, $x_1\in e$ and $x_2\in\Sigma'_2$. Hence we can define a map $g: \Sigma_1 \to \Sigma_2$ as follows:

(i) for $x \in V(G) \cap \Sigma_1'$,

$$y = g(x) \quad \text{iff} \quad y \in V_1(G) \text{ and } f(y) = f(x) ,$$

(ii) for $x \in V_1(G)$,

$$y = g(x) \quad \text{iff} \quad y \in V_2(G) \text{ and } f(y) = f(x) ,$$

(iii) for $x \in V_2(G)$,

$$y = g(x) \quad \text{iff} \quad y \in V(G) \cap \Sigma_2' \text{ and } f(y) = f(x) ,$$

(iv) for $x \in (G - V(G)) \cap \Sigma_1'$,

$$y = g(x) \quad \text{iff} \quad y \in e - V(G) \text{ and } f(y) = f(x) ,$$

(v) for $x \in e - V(G)$,

$$y = f(x) \quad \text{iff} \quad y \in (G - V(G)) \cap \Sigma_2' \text{ and } f(y) = f(x) ,$$

(vi) for $x \in \Sigma_1' - G$,

$$y = f(x) \quad \text{iff} \quad y \in \Sigma_2' - G \text{ and } f(y) = f(x) .$$

We call such a g a *reversing map* of (G, f). A reversing map is *not* continuous. Using this notion of a reversing map, we can state a condition for $M = B^3/f$ to be Seifert fibred as follows.

THEOREM 1. *Let (G, f) be a DS-diagram with an E-cycle e, Σ_1 and Σ_2 be as above, and $g: \Sigma_1 \to \Sigma_2$ be the reversing map. Then B^3/f is a Seifert fibred manifold if there are a graph $\widetilde{G}$ in Σ_1 and a homeomorphism h from Σ_2 onto Σ_1 such that*

(a) *$G \cap \Sigma_1 \subset \widetilde{G}$ and $V(G) \cap \Sigma_1 \subset V(\widetilde{G})$,*
(b) *$h|_{\partial \Sigma_2} = \text{identity}$,*
(c) *$h(g(\widetilde{G})) = \widetilde{G}$,*
(d) *$h(g(V(\widetilde{G}))) = V(\widetilde{G})$.*

PROOF. By Lemma 1 there is a normal pair (ψ_t, Σ) on $M = B^3/f$ satisfying the conditions in the lemma. Let $\iota: \Sigma_1 \to \Sigma$ be an embedding as in Lemma 1. By this ι we identify Σ with Σ_1. Then it also follows from the way for constructing (ψ_t, Σ) that $\hat{T}_+$ is continuous on $\widetilde{G} - V(\widetilde{G})$ and on $\Sigma - \widetilde{G}$, because $G \cap \Sigma_1$ is included in $\widetilde{G}$ and g is continuous both on $(G - V(G)) \cap \Sigma_1$ and on $\Sigma_1 - G$.

The condition (d) implies that for any $x \in V(\widetilde{G})$ there is a positive integer n such that $\hat{T}_+^n(x) = x$. And by the condition (c), deforming h if necessary, we can see that $\hat{T}_+$ is periodic on $\tilde{G}$. Therefore, using this property and the continuity of $\hat{T}_+$ on $\Sigma-\tilde{G}$, we can deform h so that $\hat{T}_+$ is periodic whole on Σ. This shows that, as a non-singular flow of the normal pair generating the DS-diagram (G, f), we can take a flow whose orbits are all homeomorphic to a circle S^1. Hence $M=B^3/f$ admits an S^1-action, and so M is Seifert fibred (cf. [1]). This completes the proof.

The converse of this theorem is partially true, that is, we have

THEOREM 2. *If M is Seifert fibred closed 3-manifold, then there is a normal pair on M whose DS-diagram satisfies the conditions in Theorem* 1.

PROOF. Let ψ_t be a flow on M whose orbits are fibres of the Seifert fibration of M. Then, taking an adequate compact local section Σ, we get a normal pair (ψ_t, Σ) with the desired conditions. This proves the theorem.

§3. Orbit manifold and exceptional fibres.

In this section, we shall explain how we can decide the orbit manifold and the exceptional fibres of a Seifert fibred manifold represented by a DS-diagram with E-cycle satisfying the conditions in Theorem 1.

For simplicity, we consider the case where (G, f) satisfies the following additional conditions (e)-(g) besides (a)-(d).

(e) For any component Δ of $\Sigma_1-\tilde{G}$, $\bar{\Delta}$ is homeomorphic to a compact 2-disc.

(f) If $\varepsilon\in E(\tilde{G})$, $a\in\partial\varepsilon\cap V(\tilde{G})$ and $(h\circ g)^k(\varepsilon)=\varepsilon$, then $(h\circ g)^k(x_j)\to a$ $(j\to\infty)$ as $x_j\to a$ within ε.

(g) If Δ is a component of $\Sigma_1-\tilde{G}$, $a\in\partial\Delta\cap V(\tilde{G})$, $(h\circ g)^k(a)=a$ and $(h\circ g)^k(x_j)\to(h\circ g)^k(a)$ $(j\to\infty)$ as $x_j\to a$ within Δ, then $(h\circ g)^k(\Delta)=\Delta$.

These conditions assure that exceptional fibres do not meet with $\tilde{G}$. Moreover we can easily see that if (G, f) satisfies the conditions (a)-(d), then we can always take $\tilde{G}$ and h so that they satisfy also the additional conditions (e), (f) and (g).

Let Δ_j^k $(k=1, \cdots, s,\ j=1, \cdots, r_k)$ be components of $\Sigma_1-\tilde{G}$, where Δ_j^k are so classified that $(h\circ g)(\Delta_j^k)=\Delta_{j+1}^k$ $(j=1, \cdots, r_k$ and $\Delta_{r_k+1}^k=\Delta_1^k)$. Then the restriction of $(h\circ g)^{r_k+1}$ onto Δ_k^1 is a homeomorphism of Δ_k^1, and moreover by the condition (e) it can be extended to a homeomorphism of $\bar{\Delta}_k^1$. We denote by h_k this extended homeomorphism of $\bar{\Delta}_1^k$, and by a_j^k $(j=1, \cdots, q_k)$ the points on $V(\tilde{G})\cap\partial\Delta_1^k$. We assume that $a_1^k, \cdots, a_{q_k}^k$ are arranged in the cyclic order on $\partial\Delta_1^k$.

Now let $a_{p_k}=h_k(a_1^k)$. If $p_k=1$, then, deforming h if necessary, we may assume that h_k is the identity map. In the case where $p_k>1$, we may assume that h_k has only one fixed point $w_k\in\Delta_1^k$ and is equivalent to a rotation around w_k by an angle $2\pi(p_k-1)/q_k$. Hence, considering the corresponding normal pair (ψ_t, Σ), we can see that the orbit of ψ_t through w_k is an exceptional fibre of index $(p_k-1)/q_k$, and the other orbits through the points on Δ_1^k are regular fibres. As is stated above, the conditions (f) and (g) implies that there is no other exceptional fibre.

The orbit manifold is quite easily determined. Define an equivalence relation "$\sim$" on Σ_1 as follows:

$$x\sim y \quad \text{iff} \quad (h\circ g)^k(x)=y \text{ for some } k\,.$$

Then it is evident that the orbit manifold is $\Sigma_1/\sim$.

§4. An example.

In this section, we shall apply the results in the previous sections to the DS-diagram indicated in Fig. 1. In Fig. 1, the vertexes and edges with the same names are identified by the identification map in the indicated direction. And faces X_k^+ and X_k^- are identified in the direction naturally determined by the direction of $\partial X_k^\pm$ (see [3] for the precise usage of such a representation of a DS-diagram).

This DS-diagram has an E-cycle $e=\partial(\cup_{k=1}^6\overline{X_k^+})$. We define Σ_1 and Σ_2 by $\Sigma_1=\cup\overline{X_k^+}$ and $\Sigma_2=\cup\overline{X_k^-}$. Now define the graph $\tilde{G}$ to be one obtained

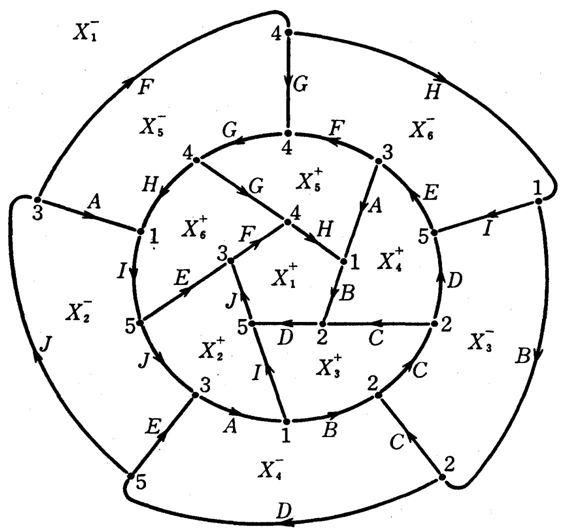


FIGURE 1

by adding to $G \cap \Sigma_1$ the edges drown by broken lines in Fig. 2, and name the components of $\Sigma_1 - \tilde{G}$ as indicated in Fig. 2.

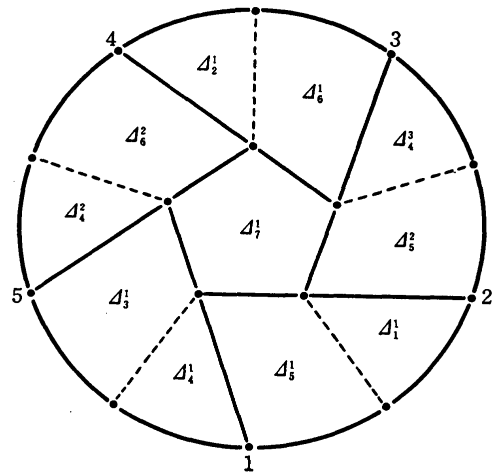
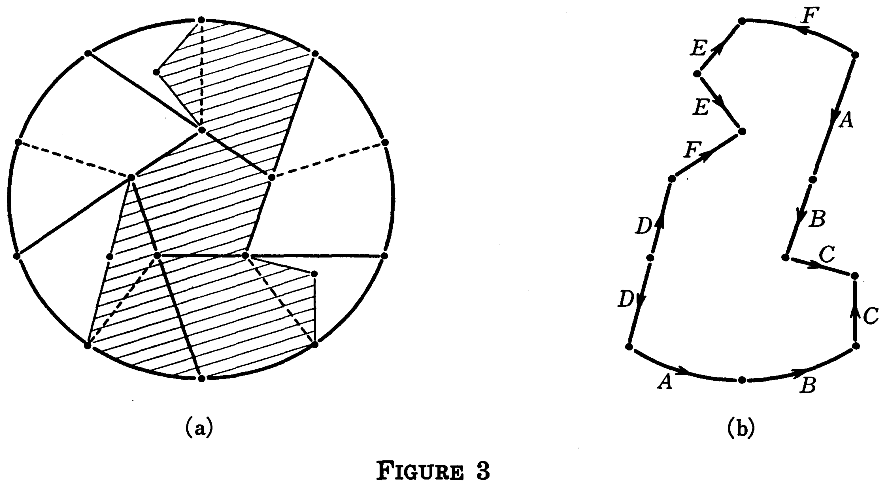


FIGURE 2

Then it is easy to see that for this $\tilde{G}$ we can take a homeomorphism $h: \Sigma_2 \to \Sigma_1$ satisfying the conditions (a)-(g) in §§ 2, 3. Hence the manifold represented by this DS-diagram is Seifert fibred. Moreover $h \circ g(\Delta_j^k) = \Delta_{j+1}^k$ for any $k = 1, \cdots, 7$ and $j = 1, \cdots, r_k$, where $r_1 = r_2 = r_3 = r_7 = 1$, $r_4 = 3$, $r_5 = r_6 = 2$. The integers p_k and q_k defined in § 3 are given by $q_1 = q_2 = q_4 = 3$, $q_3 = q_5 = q_6 = 4$, $q_7 = 5$, $p_1 = p_2 = 2$, $p_3 = 3$ and $p_4 = p_5 = p_6 = p_7 = 1$. Thus there are two exceptional fibres of index 1/3, and one exceptional fibre of index 1/2. The orbit manifold $\Sigma_1/\sim$ is obtained from the shaded domain in Fig. 3 (a) by identifying its boundary as indicated in Fig. 3 (b). Hence the orbit manifold is a 2-sphere.



(a) (b)

FIGURE 3

§5. Seifert fibred submanifold and embedded tori.

In this section, using the same technique as in the previous sections, we shall investigate a method for finding Seifert fibred submanifolds and embedded tori. We use the same notation as in §§1-3.

Let (G, f) be a DS-diagram with an E-cycle e, and let a graph $\tilde{G}$ and $h: \Sigma_2 \to \Sigma_1$ be those satisfying the conditions (a) and (b) in Theorem 1. And moreover, suppose that there is a compact subset Δ of Σ_1 such that

(c′) $\partial\Delta \subset \tilde{G}$ and $h(g(\Delta \cap \tilde{G})) = \Delta \cap \tilde{G}$, and

(d′) $h(g(\Delta \cap V(\tilde{G}))) = \Delta \cap V(\tilde{G})$,

where g is the reversing map. Now let (ψ_t, Σ) be a normal pair as in Lemma 1 for which $\hat{T}_+ = h \circ g$, and let $X = X(\Delta)$ be the set of points on orbits of ψ_t through Δ. If $\partial\Delta$ forms a 1-manifold, then X is a Seifert fibred submanifold of $M = B^3/f$.

Let X be a Seifert fibred submanifold obtained as above, and let Y be a component of ∂X. Then Y is an embedded torus or a Klein bottle, and $Y \cap \Sigma$ consists of circles in Σ and arcs in Σ connecting two points on $\partial\Sigma$. If $Y \cap \Sigma$ includes a circle, then obviously Y is compressible. Therefore $Y \cap \Sigma$ consists of mutually disjoint arcs if Y is incompressible.

In what follows, we investigate embedded tori. First we shall show that

LEMMA 2. *Let $\alpha = \{\alpha_1, \cdots, \alpha_r\}$ be a set of mutually disjoint compact arcs α_j on Σ_1 such that* (i) *$\partial\alpha_j = \alpha_j \cap \partial\Sigma_1$ for any j,* (ii) *$h \circ g(\cup \alpha_j) = \cup \alpha_j$ and* (iii) *for any α_i and α_j there is a k with $\alpha_j \cap (h \circ g)^k(\alpha_i) \neq \emptyset$. Then the set of points on the orbits of ψ_t through $\cup \alpha_j$ is a torus or a Klein bottle, which we denote by $Y(\alpha)$.*

PROOF. Define Y to be

$$Y = \{\psi_t(x) \mid x \in \cup \alpha_j,\ t \in \boldsymbol{R}\}\ .$$

Then, according to the condition (i), Y is a two dimensional submanifold invariant under ψ_t. Moreover Y is closed by the condition (ii), and is connected by (iii). Hence Y must be a closed 2-manifold with the Euler characteristic number zero. This proves the lemma.

In the rest of this paper, for simplicity, we shall consider only the case where α in the above lemma consists of exactly one arc, which we denote by the same letter α. The next proposition gives a condition for an arc $\alpha \subset \Sigma_1$ with the condition (i) in the above lemma to admit a homeomorphism h for which α satisfies the other conditions.

PROPOSITION 1. *Let α be an arc in Σ_1 with $\partial\alpha\subset e=\partial\Sigma_1$. Then we can choose a homeomorphism h: $\Sigma_2\to\Sigma_1$ for which α satisfies the conditions* (i)-(iii) *in Lemma* 2 *if and only if α has the following properties* (i) *and* (ii):

(i) *$\partial\alpha\cap(G-e)$ consists of exactly two points x_0 and x_1, and they satisfy that $f(x_j)\in f(\partial\alpha)$ $(j=0, 1)$,*

(ii) *$g(\bar{\beta})\cap e\neq\emptyset$ for any component β of $\alpha-\{x_0, x_1\}$ with $\bar{\beta}\cap e\neq\emptyset$, where g is the reversing map.*

PROOF. Suppose that α satisfies the conditions in Lemma 2 for a suitable h, and (ψ_t, Σ) is the corresponding normal pair. Then α is a local section for ψ_t restricted on $Y(\alpha)$, and intersects with every orbit of this restricted flow. And each of $\hat{T}_-(\partial\alpha)$ and $\hat{T}_+(\partial\alpha)$ consists of exactly two points. Obviously the points on $\hat{T}_-(\partial\alpha)$ correspond to x_0 and x_1 (cf. Lemma 1). Let β be the component of $\alpha-\hat{T}_-(\partial\alpha)$ with $\bar{\beta}\subset\mathrm{Int}\,\Sigma$, and γ be the component of $\alpha-\hat{T}_+(\partial\alpha)$ with $\bar{\gamma}\subset\mathrm{Int}\,\Sigma$. Then it is easy to see that $\hat{T}_-(\gamma)=\beta$. This fact implies the condition (ii) of the proposition, and proves the "only if" part.

Now let α be an arc satisfying the above condition (i) and (ii), and let β_0, β_1 and β_2 be the three components of $\alpha-\{x_0, x_1\}$ such that $x_j\in\bar{\beta}_j$ $(j=0, 1)$ and $\bar{\beta}_2\subset\mathrm{Int}\,\Sigma_1$. We put $y_j=g(x_j)$ and $\{z_j\}=\beta_j\cap e$ for $j=0$ and 1. Notice that $\{y_0, y_1\}=\{z_0, z_1\}$ by the condition (i). Hence, using the condition (ii), we have that $h\circ g(\beta_0)$ is an arc in Σ_1 connecting y_0 to $h\circ g(z_0)$, $h\circ g(\beta_1)$ is one connecting y_1 to $h\circ g(z_1)$ and $h\circ g(\beta_2)$ is one connecting $h\circ g(z_0)$ to $h\circ g(z_1)$. Therefore $h\circ g(\alpha)$ is an arc in Σ_1 which connects the two end points of α. This shows that we can deform h so that $h\circ g(\alpha)=\alpha$, namely α satisfies the conditions in Lemma 2. This completes the proof.

Furthermore we have that

PROPOSITION 2. *Let α be an arc with the conditions in the above proposition, and h: $\Sigma_2\to\Sigma_1$ be a homeomorphism for which α satisfies the conditions in Lemma* 2. *Then it holds that*

(i) *$Y(\alpha)$ is a torus if and only if x_0 and $g(x_0)$ are separated on α by x_1, and*

(ii) *$M-Y(\alpha)$ is connected if and only if, for some $a\in V_1(G)$, a and $g(a)$ are separated in Σ_1 by α.*

PROOF. Let β be the subarc of α with the end points x_0 and x_1. For this β we can choose a continuous function τ: $\beta\to\boldsymbol{R}$ such that $\tau(x_j)=T_+(x_j)$ $(j=0, 1)$ and $0<\tau(x)<T_+(x)$ for any $x\in\beta-\{x_0, x_1\}$. Then $\alpha\cup\hat{\tau}(\beta)$ is a cross-section of ψ_t restricted on $Y(\alpha)$, where $\hat{\tau}(x)=\psi_{\tau(x)}(x)$.

It is easy to see that the returning map for this cross-section is orientation preserving if and only if x_0 and $g(x_0)$ are separated by x_1. This proves the first part.

Noticing that $M-Y(\alpha)$ is connected if and only if ψ_t has an orbit which meets with both components of $\Sigma-\alpha$, we can easily show the second part of the proposition. This completes the proof.

Let α and h be those in Proposition 2, and X be the closure of a connected component of $M-Y(\alpha)$ ($X=M$ if $M-Y(\alpha)$ is connected). Define a subset Δ of Σ_1 to be $\Delta=\{x\in\Sigma_1 \mid f(x)\in X\}$. Then, applying the same method as in Theorem 1 to the graph $G\cap\Delta$, we can get a sufficient condition for X being Seifert fibred. In what follows, we shall give a presentation of the fundamental group $\pi_1(X)$ and determine its subgroup $\iota_*(\pi_1(Y(\alpha)))$, where ι_* is the homomorphism induced by the inclusion.

We denote by $C_1, \cdots, C_\lambda$ the components of $(G\cap\Delta)-V(G)$, and by $D_0, D_1, \cdots, D_\mu$ the components of $\Delta-G$. We assume that $g(x_j)\in D_j$ for $j=0, 1$, where x_0 and x_1 are those in Proposition 1. And, for each $m=1,\cdots,\lambda$, we define three integers $m\langle j\rangle$ ($j=1, 2, 3$, $0\leqq m\langle j\rangle\leqq\mu$) as follows: $n_j=m\langle j\rangle$ iff $C_m\subset\partial D_{n_j}$ for $j=1, 2$, $g(C_m)\subset\partial D_{n_3}$, and $g(C_m)$ is included in the boundary of $g(D_{n_1})$. Then, by the same method as in Theorem 4.2 of [5], we get the following presentation of $\pi_1(X)$:

$$\pi_1(X)=\langle y_0, y_1, \cdots, y_\mu; r_1, \cdots, r_\lambda\rangle$$
$$r_m=y_{m\langle 1\rangle}y_{m\langle 3\rangle}y_{m\langle 2\rangle}^{-1} \qquad (m=1, \cdots, \lambda)\,.$$

Moreover from the proof of Theorem 4.2 of [5] it follows that $\iota_*(\pi_1(Y(\alpha)))$ is generated by $[y_0]$ and $[y_1]$ in the above presentation.

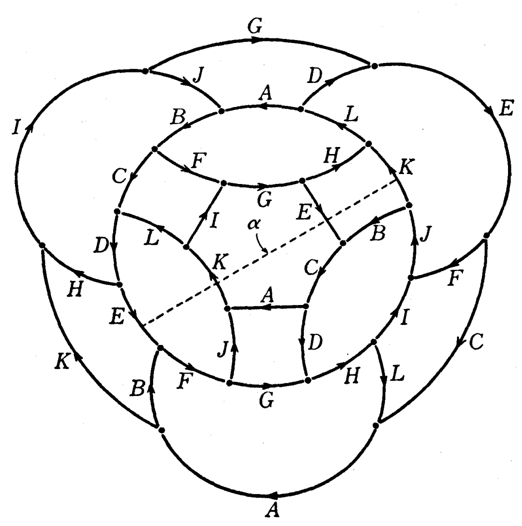


FIGURE 4

EXAMPLE I. The DS-diagram given by Fig. 4 has an arc α with the conditions in Proposition 1. By Proposition 2, $Y(\alpha)$ is a torus and does not separate M.

EXAMPLE II. The DS-diagram given by Fig. 5 has arcs α and α' with the conditions in Proposition 1. Using Proposition 2, we can see that $Y(\alpha)$ is a torus and $Y(\alpha')$ is a Klein bottle. And moreover $Y(\alpha)$ divide M into two components.

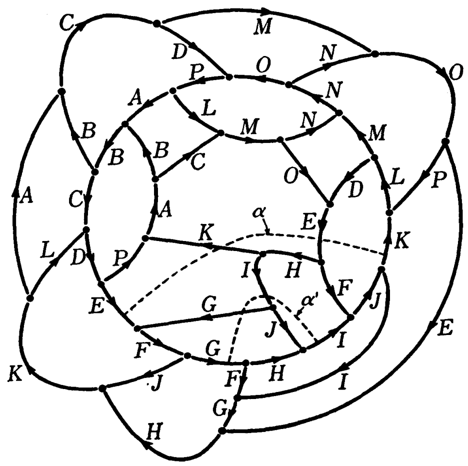


FIGURE 5

References

[1] D. B. A. EPSTEIN, Periodic flows on three-manifolds, Ann. of Math., **95** (1972), 66–82.
[2] H. IKEDA, Acyclic closed fake surfaces, Topology, **10** (1971), 9–36.
[3] H. IKEDA AND Y. INOUE, Invitation to DS-diagrams, Kobe J. Math., **2** (1985), 169–186.
[4] H. IKEDA, DS-diagrams with E-cycle, Kobe J. Math., **3** (1986), 104–112.
[5] I. ISHII, Flows and spines, Tokyo J. Math., **9** (1986), 505–525.
[6] I. ISHII, Combinatorial construction of a non-singular flow on a 3-manifold, Kobe J. Math., **3** (1986), 201–208.

Present Address:
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY, KEIO UNIVERSITY
HIYOSHI, KOHOKU-KU, YOKOHAMA 223, JAPAN